1. If $S$ is a nonempty set, $x_0 \in S$ is fixed, $F$ is a field, and $V$ is the space of functions from $S$ to $F$, show that the set $W = \{ f \in V : f(x_0) = 0 \}$ is a subspace of $V$.

2. If $W_1$ and $W_2$ are subspaces of $V$, show that $W_1 \cup W_2$ is a subspace of $V$ if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

3. Show that the set of convergent real-valued sequences is a subspace of the vector space of all real-valued sequences.

4. Show that $W$ is a subspace of $V$ if and only if $W = \text{span}(W)$.

5. If $\text{char}(F) \neq 3$ and $S = \{ v, w \}$ and $T = \{ v + w, v - 2w \}$, show that (a) if $S$ spans $V$ then $T$ spans $V$, (b) if $S$ is linearly independent then $T$ is linearly independent, and (c) if $S$ is a basis then $T$ is a basis.

6. Show that a vector space is infinite-dimensional if and only if it contains an infinite linearly-independent subset.

7. Show that any set of polynomials, no two of which have the same degree, is linearly independent.

8. Show that if $g \in P_n(F)$ has degree $n$ and $F$ has characteristic zero, then for any polynomial $f \in P_n(F)$ there exist scalars $a_0, \ldots, a_n$ such that $f = a_0g + a_1g' + a_2g'' + \cdots + a_ng^{(n)}$, where $g^{(n)}$ is the $n$th derivative of $g$.

9. Prove that if $W_1$ and $W_2$ are finite-dimensional then $\dim(W_1 \cap W_2) \leq \min(\dim W_1, \dim W_2)$ and $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2)$.

10. Let $T : V \rightarrow W$ be linear. If $V_1$ is a subspace of $V$, we define the image $T(V_1)$ as $T(V_1) = \{ T(v_1) : v_1 \in V_1 \}$. Show that $T(V_1)$ is a subspace of $W$. If $T$ is an isomorphism also show that $T$ restricts to an isomorphism of $V_1$ with $T(V_1)$, and deduce that $\dim(T(V_1)) = \dim(V_1)$.

11. Let $T : V \rightarrow W$ be linear. If $W_1$ is a subspace of $W$, we define the inverse image $T^{-1}(W_1)$ as $T^{-1}(W_1) = \{ v \in V : T(v) \in W_1 \}$. Show that $T^{-1}(W_1)$ is a subspace of $V$. (Note that $T^{-1}$ is not necessarily a function here.)

12. Let $S : V \rightarrow W$ and $T : U \rightarrow V$ be linear. Show that if $S$ and $T$ are both one-to-one then $ST$ is one-to-one, and also if $S$ and $T$ are both onto then $ST$ is also onto.

13. Let $T : V \rightarrow W$ be linear. If $T$ is one-to-one, show that $T$ maps linearly independent sets to linearly independent sets. If $T$ is onto, show that $T$ maps spanning sets to spanning sets.

14. Let $V = M_{n \times n}(F)$. If $B \in V$ is invertible, show that the map $\Phi_B : V \rightarrow V$ given by $\Phi_B(A) = B^{-1}AB$ is an isomorphism.