Math 4571: Advanced Linear Algebra

Midterm 1 (Instructor: Dummit)
February 13th, 2020

NAME (please print legibly): ________________________________
Your University ID Number: ________________________________

• Show all work and justify all answers. You MUST provide complete, clear responses for each problem, except when otherwise indicated. A correct answer without sufficient work may not receive full credit!

• You may appeal to any theorems, propositions, etc. covered at any point in the course, but please make clear what results you are using.

• In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.

• You are responsible for checking that this exam has all 8 pages.

Pledge of Honesty
I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________________________________

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1. (10 points) Let \( V = \mathbb{R}^5 \) and let \( S \) be the set of vectors \( \langle x_1, x_2, x_3, x_4, x_5 \rangle \in V \) such that \( x_5 = x_1 + x_2 \) and \( x_3 = x_4 \).

(a) Prove that \( S \) is a subspace of \( V \).

(b) Find a basis for \( S \) and the dimension of \( S \).
2. (12 points) In a vector space $V$, let

$$A = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$$
$$B = \{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n\}.$$

(a) Suppose $A$ is linearly independent. Prove that $B$ is also linearly independent.

(b) Suppose $A$ spans $V$. Prove that $B$ also spans $V$.

(c) Suppose $A$ is a basis for $V$. Prove that $B$ is a basis for $V$. 
3. **(12 points)** Let $S : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be the linear transformation $S(A) = A + A^T$.

(a) Show that $S$ is a linear transformation.

(b) Find $[S]_\beta$ for the standard basis $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(c) Find a basis for ker$(S)$ and for im$(S)$. 
4. (8 points) Suppose $T: V \to V$ is a linear transformation with the property that $T^3$ is the identity transformation. Prove that $T$ is one-to-one and onto.

5. (8 points) Suppose $T: V \to W$ is a linear transformation, where $\dim(V) = 300$ and $\dim(W) = 200$. Show that $\dim(\ker T) \geq 100$. 

6. (14 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T^2$ is the zero transformation, but $T$ is not.

(a) Show that the kernel of $T$ contains the image of $T$.

(b) Show that $\dim(\text{im}(T)) = 1$.

(c) Let $\mathbf{v}$ be a nonzero vector in $\text{im}(T)$, where $T(\mathbf{w}) = \mathbf{v}$. Prove that $\beta = \{\mathbf{v}, \mathbf{w}\}$ is a basis of $\mathbb{R}^2$. (Hint: Apply $T$ to a linear dependence.)

(d) With $\beta = \{\mathbf{v}, \mathbf{w}\}$ as in part (c), show that the matrix $[T]_{\beta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. 
7. (16 points) For each of the following, circle the correct response (there is no partial credit or penalty for wrong answers, and no work is required). Assume \( T : V \to W \) is a linear transformation, where \( V \) and \( W \) are not necessarily finite-dimensional.

True  False  The set \( \{1 + t^2, t - t^2 + t^3, 3 - 2t + t^3\} \) spans \( P_3(\mathbb{R}) \).

True  False  The dimension of a vector space is always positive.

True  False  If \( \dim(V) = 8 \), then no basis of \( V \) can have exactly 6 elements.

True  False  If \( \dim(V) = 8 \) and \( S \) contains 8 vectors, then \( S \) spans \( V \) if and only if \( S \) is linearly independent.

True  False  If \( \{v_1, \ldots, v_n\} \) is a basis of \( V \), then \( \{T(v_1), \ldots, T(v_n)\} \) is a basis for \( \text{im}(T) \).

True  False  There exists a linear map \( T : \mathbb{R}^5 \to \mathbb{R}^2 \) with nullity 2 and rank 3.

True  False  The map \( T : M_{2 \times 3}(\mathbb{R}) \to M_{3 \times 2}(\mathbb{R}) \) with \( T(A) = 2A^T \) is an isomorphism.

True  False  If \( T : V \to V \) is linear, then \( T \) is one-to-one if and only if \( T \) is onto.

Now assume that the vector spaces \( V \) and \( W \) are finite-dimensional, that \( \alpha, \beta, \) and \( \gamma \) are ordered bases of \( V, V, \) and \( W \) respectively, and that \( S \) and \( T \) are linear transformations.

True  False  \( \mathcal{L}(V, W) \) is isomorphic to \( \mathcal{L}(W, V) \).

True  False  If \( I : V \to V \) is the identity map, then \( [I]_{\alpha}^{\beta} \) is always the identity matrix.

True  False  If \( S : V \to W \) and \( T : W \to V \), then \( [ST]_{\gamma}^{\beta} = [S]_{\beta}^{\gamma}[T]_{\gamma}^{\beta} \).

True  False  For any \( T : V \to V \), there always exists an invertible matrix \( Q \) such that \( [T]_{\beta}^{\alpha} = Q^{-1}[T]_{\alpha}^{\alpha}Q \).
Blank page for scratch work.