1. Let $\langle \cdot, \cdot \rangle$ be an inner product on $V$ with scalar field $F$. Identify each of the following statements as true or false:

(a) If $V$ is finite-dimensional and $W$ is any subspace of $V$, then $\dim(W) = \dim(W^\perp)$.

- **False** the correct formula is $\dim(W) + \dim(W^\perp) = \dim(V)$.

(b) If $V$ has an orthonormal basis $\{e_1, e_2, e_3\}$ and $W = \text{span}(e_1 + 2e_3)$, then $W^\perp = \text{span}(e_2, 2e_1 - e_3)$.

- **True** we have $\dim(W^\perp) = 2$ and the two given vectors are linearly independent and are both orthogonal to $e_1 + 2e_3$, so they do form a basis.

(c) If $\{v_1, \ldots, v_n\}$ is a basis of $V$, then $w = \langle w, v_1 \rangle v_1 + \cdots + \langle w, v_n \rangle v_n$ for any $w \in W$.

- **False** this is only true if $\{v_i\}$ is an orthonormal basis.

(d) If $\{w_1, \ldots, w_n\}$ is an orthonormal basis of $W$, then $w = \langle v, w_1 \rangle w_1 + \cdots + \langle v, w_n \rangle w_n$ is the orthogonal projection of $v$ into $W$.

- **True** this is the orthogonal projection formula we proved.

(e) If $V$ is finite-dimensional, $v \in V$, and $W$ is any subspace of $V$, the vector $w \in W$ minimizing $||v - w||$ is the orthogonal projection of $v$ into $W$.

- **True** this is the minimality property of the orthogonal projection.

(f) If $T: V \to V$ is linear, then the adjoint of $T$ exists and is unique.

- **False** the adjoint does not always necessarily exist over an arbitrary vector space. (It does exist, then it is unique.)

(g) If $T: V \to V$ is linear and $V$ is finite-dimensional, then the adjoint of $T$ exists and is unique.

- **True** we proved that the adjoint always exists over finite-dimensional vector spaces.

(h) If $T: V \to F$ is linear and $V$ is finite-dimensional, then there exists $w \in V$ such that $T(v) = \langle v, w \rangle$ for all $v \in V$.

- **True** this is the version of the Riesz representation theorem we established.

(i) For any $S, T: V \to V$ such that $S^*$ and $T^*$ exist, we have $(S + 2T)^* = S^* + 2T^*$.

- **True** this is a correct application of the properties of adjoints.

(j) For any $S, T: V \to V$ such that $S^*$ and $T^*$ exist, we have $(S + iT)^* = S^* + iT^*$.

- **False** the correct formula is $(S + iT)^* = S^* - iT^*$.

(k) For any $S, T: V \to V$ such that $S^*$ and $T^*$ exist, we have $(ST)^* = S^* T^*$.

- **False** the correct formula is $(ST)^* = T^* S^*$.

2. Calculate the following things (assume any unspecified inner product is the standard one):

(a) The angle between the vectors $v = (3, 1, 2)$ and $w = (1, -3, \sqrt{70})$ in $\mathbb{R}^3$.

- We compute $||v|| = \sqrt{14}$, $w = \sqrt{80}$, $v \cdot w = 2\sqrt{70}$, so $\theta = \cos^{-1}\left(\frac{v \cdot w}{||v|| ||w||}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(b) The “angle” between the functions $f(x) = 1 + \sin x$ and $g(x) = 1 - \sin x$ in $C[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x) \, dx$.

- We compute $||f|| = \sqrt{\int_0^{2\pi} (1 + \sin x)^2 \, dx} = \sqrt{3\pi}$, $||g|| = \sqrt{\int_0^{2\pi} (1 - \sin x)^2 \, dx} = \sqrt{3\pi}$, $\langle f, g \rangle = \int_0^{2\pi} (1 + \sin x)(1 - \sin x) \, dx = \pi$, so $\theta = \cos^{-1}\left(\frac{\langle f, g \rangle}{||f|| ||g||}\right) = \cos^{-1}\left(\frac{1}{3}\right)$
(c) Write \( \mathbf{v} = (9, 7, -8) \) as a linear combination of the orthogonal basis \((-1, 1, 2), (2, 0, 1), (1, 5, -2)\) of \( \mathbb{R}^3 \).

- Using the projection formula, we have \( \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 + \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \mathbf{e}_2 + \frac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle} \mathbf{e}_3 = \frac{-3}{2} \mathbf{e}_1 + 2 \mathbf{e}_2 + 2 \mathbf{e}_3 \).

(d) Write \( \mathbf{v} = (-5, 5, -6) \) as a linear combination of the orthogonal basis \((-i, 0), (1, 1, 2i), (i, 1)\) of \( \mathbb{C}^3 \).

- Using the projection formula, we have \( \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 + \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \mathbf{e}_2 + \frac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle} \mathbf{e}_3 = \frac{-5}{4} \mathbf{e}_1 + 2 \mathbf{e}_2 - 2 \mathbf{e}_3 \).

(e) A basis for \( W^\perp \), if \( W = \text{span}((1,1,1),(2,3,4,1)) \) inside \( \mathbb{R}^4 \).

- The orthogonal complement corresponds to the nullspace of the matrix whose rows are the given vectors.
- Row-reducing \( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \end{bmatrix} \) yields the reduced row-echelon form \( \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix} \).
- From the reduced row-echelon form, we see that \( \{( -2,1,0,1), (1, -2,1,0) \} \) is a basis for the nullspace and hence of \( W^\perp \).

(f) A basis for \( W^\perp \), if \( W = \text{span}((1,1,2i),(1,-i,4)) \) inside \( \mathbb{C}^3 \). [Hint: Over \( \mathbb{C} \), compute the complex conjugate of the nullspace.]

- The orthogonal complement corresponds to the complex conjugate of the nullspace of the matrix whose rows are the given vectors.
- Row-reducing \( \begin{bmatrix} 1 & 1 & 2i \\ 1 & -i & 4 \end{bmatrix} \) yields the reduced row-echelon form \( \begin{bmatrix} 1 & 0 & 1 - i \\ 0 & 1 & -1 + 3i \end{bmatrix} \).
- From the reduced row-echelon form, we see that \( \{ (1-1,i,1-3i,1) \} \) is a basis for the nullspace, so a basis of the orthogonal complement is \( \{( -1-i,1+3i,1) \} \).

(g) The orthogonal projection of \( \mathbf{v} = (2,0,11) \) into \( W = \text{span}\{ (1,2,2), (2,-2,1) \} \) inside \( \mathbb{R}^3 \), and also verify the relation \( ||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2 \).

- Notice that the vectors \( \mathbf{e}_1 = \frac{1}{2}(1,2,2) \) and \( \mathbf{e}_2 = \frac{1}{2}(2,-2,1) \) form an orthonormal basis for \( W \).
- Thus, the orthogonal projection is \( \mathbf{w} = \text{proj}_W(\mathbf{v}) = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{v}, \mathbf{e}_2 \rangle \mathbf{e}_2 = 8\mathbf{e}_1 + 5\mathbf{e}_2 = (6,2,7) \).
- We see that \( \mathbf{w}^\perp = \mathbf{v} - \mathbf{w} = (-4,-2,4) \) is orthogonal to both \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) so it is indeed in \( W^\perp \).
- Furthermore, \( ||\mathbf{v}||^2 = 125 \), while \( ||\mathbf{w}||^2 = 125 \) and \( ||\mathbf{w}^\perp||^2 = 36 \), so indeed \( ||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2 \).

(h) The orthogonal projection of \( \mathbf{v} = 1 + 2x^2 \) into \( W = \text{span}\{x,x^2,x^3\} \) with inner product \( \langle f,g \rangle = \int_{-1}^1 f(x)g(x) \, dx \).

- First we use Gram-Schmidt to find an orthogonal basis for \( W \), which yields \( \mathbf{e}_1 = x, \mathbf{e}_2 = x^2, \mathbf{e}_3 = x^3 - \frac{3}{5} x \).
- Then the projection is \( \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 + \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle} \mathbf{e}_2 + \frac{\langle \mathbf{v}, \mathbf{e}_3 \rangle}{\langle \mathbf{e}_3, \mathbf{e}_3 \rangle} \mathbf{e}_3 = 0\mathbf{e}_1 - \frac{11}{3}\mathbf{e}_2 + 0\mathbf{e}_3 = \frac{11}{3} x^2 \).

(i) The least-squares solution to the inconsistent system \( x + 3y = 9, 3x + y = 5, x + y = 2 \).

- We have \( A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \) and \( \mathbf{c} = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix} \). Since \( A \) clearly has rank 2, \( A^T A \) will be invertible and there will be a unique least-squares solution.

- We compute \( A^T A = \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix} \), which is indeed invertible and has inverse \( (A^T A)^{-1} = \frac{1}{72} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix} \).

- The least-squares solution is therefore \( \hat{x} = (A^T A)^{-1} A^T \mathbf{c} = \begin{bmatrix} 2/3 \\ 8/3 \end{bmatrix} \).

(j) The line \( y = mx+b \) that is the best model, using least squares, for the data points \( \{(4,7), (11,21), (15,29), (19,35), (30,49)\} \). (Give three decimal places.)

- We seek the least-squares solution for \( A \mathbf{x} = \mathbf{c} \), where \( A = \begin{bmatrix} 1 & 4 \\ 1 & 11 \\ 1 & 15 \\ 1 & 19 \\ 1 & 30 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} b \\ m \end{bmatrix} \), and \( \mathbf{c} = \begin{bmatrix} 7 \\ 21 \\ 29 \\ 35 \\ 49 \end{bmatrix} \).
• We compute \( A^T A = \begin{bmatrix} 5 & 79 \\ 79 & 1623 \end{bmatrix} \), so the least-squares solution is \( \hat{x} = (A^T A)^{-1} A^T c \approx \begin{bmatrix} 2.856 \\ 1.604 \end{bmatrix} \).

• Thus, to three decimal places, the desired line is \( y = 1.604x + 2.856 \).

(k) The quadratic \( y = ax^2 + bx + c \) that is the best model, using least squares, for the data points \( \{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\} \). (Give three decimal places.)

• We seek the least-squares solution for \( Ax = c \), with \( A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \), \( x = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \), \( c = \begin{bmatrix} 22 \\ 11 \\ 4 \end{bmatrix} \).

• We compute \( A^T A = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \), so the least-squares solution is \( \hat{x} = (A^T A)^{-1} A^T c \approx \begin{bmatrix} 3.829 \\ -2.4 \\ 3.286 \end{bmatrix} \).

• Thus, the desired quadratic polynomial is \( y = 3.829 - 2.4x + 3.286x^2 \).

3. Let \( V \) be an inner product space with scalar field \( F \). The goal of this problem is to prove the so-called “polarization identities”.

(a) If \( F = \mathbb{R} \), prove that \( \langle v, w \rangle = \frac{1}{4} ||v + w||^2 - \frac{1}{4} ||v - w||^2 \).

• We just expand the norms on the right-hand side: \( ||v + w||^2 - ||v - w||^2 = \langle v + w, v + w \rangle - \langle v - w, v - w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle - (\langle v, v \rangle - 2\langle v, w \rangle + \langle w, w \rangle) = 4\langle v, w \rangle \) so dividing by 4 yields the claimed result.

(b) If \( F = \mathbb{C} \), prove that \( \langle v, w \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k ||v + i^k w||^2 \).

• As in part (a) we just expand the norms on the right-hand side:

\[
\sum_{k=1}^{4} i^k ||v + i^k w||^2 = ||v + w||^2 + i||v + iw||^2 - ||v - w||^2 - i||v - iw||^2
\]

\[
= \langle v + w, v + w \rangle + i\langle v + iw, v + iw \rangle - \langle v - w, v - w \rangle - i\langle v - iw, v - iw \rangle
\]

\[
= [(\langle v, v \rangle + \langle v, w \rangle + \langle v, w \rangle + \langle w, w \rangle) + i(\langle v, v \rangle - i\langle v, w \rangle + i\langle v, w \rangle + \langle w, w \rangle)]
\]

\[
- [(\langle v, v \rangle - \langle v, w \rangle - \langle w, v \rangle + \langle w, w \rangle) - i(\langle v, v \rangle + i\langle v, w \rangle - i\langle w, v \rangle + \langle w, w \rangle)]
\]

\[
= 4\langle v, w \rangle
\]

and so dividing by 4 yields the claimed result.

4. Let \( V \) be a finite-dimensional inner product space with orthonormal basis \( \{e_1, e_2, \ldots, e_n\} \).

(a) For any \( x \in V \) show that \( x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \cdots + \langle x, e_n \rangle e_n \).

• This is the orthogonal decomposition theorem proven in class.

• To summarize: since \( \{e_1, \ldots, e_n\} \) is a basis, we can write \( x = a_1 e_1 + \cdots + a_n e_n \) for unique scalars \( a_i \).

• Then \( \langle x, e_i \rangle = a_i \langle e_1, e_i \rangle + \cdots + a_n \langle e_n, e_i \rangle = a_i \) because all of the inner products \( \langle e_j, e_i \rangle \) are zero except for \( \langle e_i, e_i \rangle = 1 \), by the hypothesis that \( \{e_1, \ldots, e_n\} \) is orthonormal.

• Thus, \( x = \langle x, v_1 \rangle v_1 + \cdots + \langle x, v_n \rangle v_n \) as claimed.

(b) For any \( x, y \in V \) show that \( \langle x, y \rangle = \langle x, e_1 \rangle \overline{\langle y, e_1 \rangle} + \langle x, e_2 \rangle \overline{\langle y, e_2 \rangle} + \cdots + \langle x, e_n \rangle \overline{\langle y, e_n \rangle} \).

• Using the decomposition in part (a), write \( x = a_1 e_1 + \cdots + a_n e_n \) and \( y = b_1 e_1 + \cdots + b_n e_n \), with \( a_i = \langle x, e_i \rangle \) and \( b_i = \langle y, e_i \rangle \).
• Then \( \langle x, y \rangle = \langle a_1e_1 + \cdots + a_ne_n, b_1e_1 + \cdots + b_ne_n \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_ib_j \langle e_i, e_j \rangle \).

• But the only nonzero terms in this sum are the terms with \( i = j \), since \( \langle e_i, e_j \rangle = 0 \) unless \( i = j \).

• So \( \langle x, y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_ib_j \langle e_i, e_j \rangle = \sum_{i=1}^{n} \langle x, e_i \rangle \langle y, e_i \rangle \) as claimed.

(c) For any \( x \in V \) show that \( ||x||^2 = ||(x, e_1)^2 + ||(x, e_2)||^2 + \cdots + ||(x, e_n)||^2 = \sum_{i=1}^{n} ||x, e_i||^2 \). What theorem from classical geometry does this generalize?

• Simply set \( y = x \) in the formula from part (b), immediately giving \( ||x||^2 = \langle x, x \rangle = \sum_{i=1}^{n} ||x, e_i||^2 \).

• This statement is a generalization of the Pythagorean Theorem.

5. Suppose \( T : V \rightarrow W \) is a linear transformation and \( \langle \cdot, \cdot \rangle_W \) is an inner product on \( W \).

(a) If \( T \) is one-to-one, show that \( \langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W \) is an inner product on \( V \).

• We check the properties of an inner product.

  • [I1]: We have
    \[
    \langle x + ry, v \rangle_V = \langle T(x + ry), T(v) \rangle_W = \langle T(x) + rT(y), T(v) \rangle_W \\
    = \langle T(x), T(v) \rangle_W + r \langle T(y), T(v) \rangle_W \\
    = \langle x, v \rangle_V + r \langle y, v \rangle_V.
    \]

  • [I2]: We have \( \langle v_2, v_1 \rangle_V = \langle T(v_2), T(v_1) \rangle_W = \langle T(v_1), T(v_2) \rangle_W = \langle v_1, v_2 \rangle_V \).

  • [I3]: We have \( \langle v, v \rangle_V = \langle T(v), T(v) \rangle_W \geq 0 \) by the corresponding hypothesis for \( \langle \cdot, \cdot \rangle_W \). Furthermore, \( 0 = \langle v, v \rangle_V = \langle T(v), T(v) \rangle_W \) only when \( T(v) = 0 \); since \( T \) is one-to-one, this is equivalent to saying \( v = 0 \).

(b) Is the map defined in part (a) necessarily an inner product if the assumption that \( T \) is one-to-one is dropped? Explain why or why not.

• The issue is that [I3] may not hold (in fact, it will not hold!) if \( T \) is not one-to-one.

• As a dramatic example, if \( T \) is the zero transformation, then \( \langle v_1, v_2 \rangle_V = 0 \) for every \( v_1, v_2 \) in \( V \); this pairing does not only have \( \langle v, v \rangle = 0 \) when \( v = 0 \) so it fails [I3].

6. Let \( V = C[0, 2\pi] \) with inner product \( \langle f, g \rangle = \int_{0}^{2\pi} f(x)g(x) \, dx \). Also define

\[
\varphi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \text{and for positive integers } k \text{ set } \varphi_{2k-1}(x) = \frac{1}{\sqrt{\pi}} \cos(kx) \text{ and } \varphi_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin(kx).
\]

(a) Show that \( \{\varphi_0, \varphi_1, \varphi_2, \ldots\} \) is an orthonormal set in \( V \).

• First observe that \( \int_{0}^{2\pi} \sin(nx) \, dx = 0 = \int_{0}^{2\pi} \cos(nx) \, dx \) for any integer \( n \neq 0 \). Thus, \( \langle \varphi_0, \varphi_k \rangle = 0 \) for any \( k > 0 \).

• Furthermore, using the product-to-sum identities, we can write

\[
\varphi_{2a}\varphi_{2b} = \frac{1}{\pi} \sin(ax) \sin(bx) = \frac{1}{2\pi} [\cos(a-b)x - \cos(a+b)x] \\
\varphi_{2a-1}\varphi_{2b} = \frac{1}{\pi} \cos(ax) \sin(bx) = \frac{1}{2\pi} [\sin(a-b)x - \sin(a+b)x] \\
\varphi_{2a-1}\varphi_{2b-1} = \frac{1}{\pi} \cos(ax) \cos(bx) = \frac{1}{2\pi} [\cos(a-b)x + \cos(a+b)x]
\]

and so when \( a \neq b \), each inner product \( \langle \varphi_{2a}, \varphi_{2b} \rangle, \langle \varphi_{2a-1}, \varphi_{2b} \rangle, \) and \( \langle \varphi_{2a-1}, \varphi_{2b-1} \rangle \) is zero because both terms integrate to zero (the second also integrates to zero when \( a = b \)). Thus, the set is orthogonal.
Furthermore, we have \( \langle \varphi_0, \varphi_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} 1 \, dx = 1 \),  
\( \langle \varphi_{2k-1}, \varphi_{2k-1} \rangle = \frac{1}{\pi} \int_0^{2\pi} \cos^2(kx) \, dx = 1 \), and  
\( \langle \varphi_{2k}, \varphi_{2k} \rangle = \frac{1}{\pi} \int_0^{2\pi} \sin^2(kx) \, dx = 1 \). Thus, the set is orthonormal.

(b) Let \( f(x) = x \). Find \( ||f|| \) and \( \langle f, \varphi_n \rangle \) for each \( n \geq 0 \).

- We compute (using integration by parts as necessary)
  
  \[
  ||f|| = \sqrt{\int_0^{2\pi} x^2 \, dx} = \sqrt{\frac{8\pi^3}{3}}.
  \]
  
  \( \langle \varphi_0, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{2\pi}} \, dx = \sqrt{2\pi^3}. \)
  
  \( \langle \varphi_{2k-1}, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{\pi}} \cos(kx) \, dx = 0. \)
  
  \( \langle \varphi_{2k}, f \rangle = \int_0^{2\pi} \frac{x}{\sqrt{\pi}} \sin(kx) \, dx = -\frac{2\sqrt{\pi}}{k}. \)

(c) With \( f(x) = x \), assuming that \( ||f||^2 = \sum_{n=0}^{\infty} (\langle f, \varphi_n \rangle)^2 \) (see problem 5(c) for why this is a reasonable statement), find the exact value of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

- By part (b), the formula yields
  
  \( 2\pi^3 + \sum_{k=1}^{\infty} \left( -\frac{2\sqrt{\pi}}{k} \right)^2 = \frac{8\pi^3}{3}, \)
  
  so that \( \sum_{k=1}^{\infty} \frac{4\pi}{k^2} = \frac{2\pi^3}{3}. \)

- Dividing by \( 4\pi \) yields \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \) (This is, in fact, the actual value of this sum!)

**Remarks:** The identity \( ||f||^2 = \sum_{n=0}^{\infty} (\langle f, \varphi_n \rangle)^2 \) is known as Parseval’s identity. The problem of computing the value of the infinite sum \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is known as the Basel problem. The correct value was (famously) first found by Euler, who evaluated the sum by decomposing the function \( \frac{\sin(\pi x)}{\pi x} \) as the infinite product

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right)
\]

and then comparing the power series coefficients of both sides.

7. Suppose \( V \) is an inner product space (not necessarily finite-dimensional) and \( T : V \to V \) is a linear transformation possessing an adjoint \( T^* \). We say \( T \) is Hermitian (or self-adjoint) if \( T = T^* \), and that \( T \) is skew-Hermitian if \( T = -T^* \).

(a) Show that \( T \) is Hermitian if and only if \( iT \) is skew-Hermitian.

- Note \( (iT)^* = -iT \) so \( T = T^* \) if and only if \( (iT)^* = -iT \).

(b) Show that \( T + T^* \), \( T^*T \), and \( TT^* \) are all Hermitian, while \( T - T^* \) is skew-Hermitian.

- Note \( (T + T^*)^* = T^* + T = T^*T = T + T^* \), \( (T^*T)^* = T^*T^* = TT^* \), and \( (TT^*)^* = T^*T^* = TT^* \).

- Also, \( (T - T^*)^* = T - T^* = T^* - T \).

(c) Show that \( T \) can be written as \( T = S_1 + iS_2 \) for unique Hermitian transformations \( S_1 \) and \( S_2 \).

- In such a case we would necessarily have \( T^* = (S_1 + iS_2)^* = S_1^* - iS_2^* = S_1 - iS_2 \).

- Solving for \( S_1 \) and \( S_2 \) in terms of \( T \) and \( T^* \) then yields \( S_1 = \frac{1}{2}(T + T^*) \) and \( S_2 = \frac{1}{2i}(T - T^*) \), so these are the only possible choices.

- On the other hand, by (a) and (b), we see that these \( S_1 \) and \( S_2 \) are in fact Hermitian, so these are the unique choices.

(d) Suppose \( T \) is Hermitian. Prove that \( \langle T(v), v \rangle \) is a real number for any vector \( v \).

- If \( T^* = T \) then \( \langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, Tv \rangle = \langle T(v), v \rangle \), so \( \langle T(v), v \rangle \) equals its conjugate, hence real.