1. Assume that the vector spaces $U, V, W$ are finite-dimensional over the field $F$, the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that $S, T$ are linear transformations. Identify each of the following statements as true or false:

(a) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ is an element of $M_{m \times n}(F)$.
   
   - False: if $\dim(V) = m$ and $\dim(W) = n$ then $[T]_\beta^\gamma$ is an $n \times m$ matrix. (Try it!)

(b) If $[S]_\alpha^\beta = [T]_\alpha^\beta$ then $S = T$.
   
   - True: the map associating a linear transformation with its associated matrix is an isomorphism, so two linear transformations have the same associated matrix if and only if they are equal.

(c) If $[T]_\alpha^\beta = [T]_\alpha^\gamma$ then $\alpha = \gamma$ and $\beta = \delta$.
   
   - False: for example if $T$ is the identity transformation on $\mathbb{R}^2$ and $\alpha, \beta$ are the standard basis and $\gamma, \delta$ are twice the standard basis, then the associated matrices are both the identity matrix but the bases are different.

(d) If $S : V \rightarrow W$ and $T : V \rightarrow W$ then $[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$.
   
   - True: this is the correct rule for computing the matrix associated to a sum.

(e) If $T : V \rightarrow W$ and $v \in V$, then $[T]_\alpha^\beta[v]_\beta = [Tv]_\alpha$.
   
   - False: the correct formula is $[T]_\alpha^\beta[v]_\alpha = [Tv]_\beta$.

(f) If $S : V \rightarrow W$ and $T : U \rightarrow V$, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma[T]_\alpha^\beta$.
   
   - True: this is the correct rule for computing the matrix associated to a composition.

(g) If $T : V \rightarrow V$ has an inverse $T^{-1}$, then $[T^{-1}]_\beta^\gamma = ([T]_\alpha^\beta)^{-1}$.
   
   - False: the correct formula is $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$, since we have the composition $[T^{-1}]_\gamma^\beta[T]_\beta^\gamma = [I]_\beta^\gamma = \frac{I_n}{n}$. (Try it!)

(h) If $T : V \rightarrow V$ has an inverse $T^{-1}$, then for any $v \in V$, $[T^{-1}v]_\gamma = ([T]_\gamma^\beta)^{-1}[v]_\beta$.
   
   - True: we have $([T]_\gamma^\beta)^{-1}[v]_\beta = [T^{-1}]_\gamma^\beta[v]_\beta = [T^{-1}v]_\gamma$ by the composition formula.

(i) If $T : V \rightarrow V$ and $[T]_\beta^\gamma$ is the identity matrix, then $T$ must be the identity transformation.
   
   - False: for example, if $T$ is the doubling map and $\gamma$ is obtained by doubling the vectors in $\beta$, then $[T]_\beta^\gamma$ is the identity matrix but $T$ is not the identity map.

(j) If $T : V \rightarrow V$ and $[T]_\beta^\gamma$ is the zero matrix, then $T$ must be the zero transformation.
   
   - False: if $[T]_\beta^\gamma$ is the zero matrix then $T(\beta_i) = 0$ for every vector $\beta_i \in \beta$. Then since $T$ is zero on a basis, it is zero on all of $V$.

(k) The space $L(V, W)$ of all linear transformations from $V$ to $W$ has dimension $\dim V \cdot \dim W$.
   
   - True: since $L(V, W)$ is isomorphic to $M_{\dim(V) \times \dim(W)}(F)$, their dimensions are also equal.

(l) If $A$ is an $m \times n$ matrix of rank $r$, then the solution space of $Ax = 0$ has dimension $r$.
   
   - False: the solution space is the nullspace, which has dimension $n - r$ if the matrix has rank $r$.

(m) If $A$ is an $m \times n$ matrix and the linear system $Ax = 0$ has infinitely many solutions, then the rank of $A$ is less than $n$.
   
   - True: in this case the nullspace must have dimension greater than 0, so by the nullity-rank theorem that means the rank must be less than $n$.

(n) If $A$ is an $n \times n$ matrix of rank $n$, then the equation $Ax = 0$ has only the solution $x = 0$.
   
   - True: by the nullity-rank theorem, this means that the nullspace of $A$ has dimension 0.
(o) If the columns of $A$ are all scalar multiples of some vector $v$, then rank($A$) ≤ 1.
    - True: the column space is spanned by $v$ so its dimension (which is the rank of $A$) is at most 1.

(p) For any $T : V \rightarrow V$, there always exists an invertible matrix $Q$ such that $[T]_\beta^\gamma = Q^{-1}[T]_\alpha^\beta Q$.
    - True: if we take $Q$ to be the change-of-basis matrix $[I]_\beta^\alpha$, then $Q^{-1} = [I]_\alpha^\beta$, so $Q^{-1}[T]_\alpha^\beta Q = [I]_\beta^\alpha[T]_\alpha^\beta[I]_\beta^\alpha = [T]_\beta^\beta$.

(q) For any $T : V \rightarrow V$, if $P = [I]_\beta^\beta$, then it is true that $[T]_\gamma^\gamma = P[T]_\beta^\beta P^{-1}$.
    - True: this is the change-of-basis formula. Explicitly, if $P = [I]_\gamma^\gamma$ then $P^{-1} = [I]_\gamma^\gamma$ and so $P[T]_\beta^\beta P^{-1} = [I]_\beta^\beta[T]_\beta^\beta[I]_\beta^\beta = [ITI]_\gamma^\gamma = [T]_\gamma^\gamma$.

2. For each linear transformation $T$ and given bases $\beta$ and $\gamma$, find $[T]_\beta^\gamma$:

(a) $T : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ given by $T(a,b) = (a-b, b-2a, 3b)$, with $\beta$ and $\gamma$ the standard bases.
    - We have $T(1,0) = (1,-2,0)$ and $T(0,1) = (-1,1,3)$ so the matrix is 
      $[T]_\beta^\gamma = \begin{bmatrix}
      1 & -1 \\
      -2 & 1 \\
      0 & 3
      \end{bmatrix}$.

(b) The trace map from $M_{2\times2}(\mathbb{R}) \rightarrow \mathbb{R}$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\}$ and $\gamma = \{1\}$.
    - We have $T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = 1$, $T(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}) = 0$, $T(\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}) = 0$, and $T(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = 4$.
    - So the matrix is $[T]_\beta^\gamma = \begin{bmatrix} 1 & 0 & 0 & 4 \end{bmatrix}$.

(c) $T : \mathbb{Q}^4 \rightarrow P_3(\mathbb{Q})$ given by $T(a,b,c,d) = a + (a+b)x + (a+3c)x^2 + (2a+d)x^3 + (b+5c+d)x^4$, with $\beta$ the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.
    - We have $T(1,0,0,0) = 1 + x + x^2 + 2x^3$, $T(0,1,0,0) = x + x^4$, $T(0,0,1,0) = 3x^2 + 5x^4$, and $T(0,0,0,1) = x^3 + x^4$.
    - So, reading off the coefficients in the proper order, the matrix is $[T]_\beta^\gamma = \begin{bmatrix}
      2 & 0 & 0 & 1 \\
      1 & 0 & 3 & 0 \\
      0 & 1 & 5 & 1 \\
      1 & 1 & 0 & 0 \\
      1 & 0 & 0 & 0
      \end{bmatrix}$.

(d) $T : M_{2\times2}(\mathbb{R}) \rightarrow M_{2\times2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} A$ with $\beta = \gamma$ the standard basis.
    - We have $T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$, $T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, and $T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$.
    - So, reading off the coefficients in the proper order, the matrix is $[T]_\beta^\gamma = \begin{bmatrix}
      1 & 0 & 2 & 0 \\
      0 & 1 & 0 & 2 \\
      3 & 0 & 4 & 0 \\
      0 & 3 & 0 & 4
      \end{bmatrix}$.

(e) The projection map (see problem 8 of homework 4) on $\mathbb{R}^3$ that maps the vectors $(1,2,1)$ and $(0,-3,1)$ to themselves and sends $(1,1,1)$ to the zero vector, with $\beta = \gamma = \{(1,2,1), (0,-3,1), (1,1,1)\}$.
    - We have $T(1,2,1) = (1,2,1)$, $T(0,-3,1) = (0,-3,1)$, and $T(1,1,1) = (0,0,0)$.
    - So the matrix is simply $[T]_\beta^\gamma = \begin{bmatrix}
      1 & 0 & 0 \\
      0 & 1 & 0 \\
      0 & 0 & 0
      \end{bmatrix}$. 

2
3. Let \( T : P_3(\mathbb{R}) \to P_4(\mathbb{R}) \) be given by \( T(p) = x^2 p''(x) \).

(a) With the bases \( \alpha = \{1, x, x^2, x^3\} \) and \( \gamma = \{1, x, x^2, x^3, x^4\} \), find \( [T]_\alpha^\gamma \).

- We have \( T(1) = 0, T(x) = 0, T(x^2) = 2x^2, \) and \( T(x^3) = 6x^3 \), so \( [T]_\alpha^\gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(b) If \( q(x) = 1 - x^2 + 2x^3 \), compute \( [q]_\alpha \) and \( [T(q)]_\gamma \) and verify that \( [T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha \).

- We have \( [q]_\alpha = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \) and \( T(q) = -2x^2 + 12x^3 \), so \( [T(q)]_\gamma = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 12 \\ 0 \end{bmatrix} \). Indeed, \( [T(q)]_\gamma = [T]_\alpha^\gamma [q]_\alpha \).

Notice that \( T = SU \) where \( U : P_3(\mathbb{R}) \to P_1(\mathbb{R}) \) has \( U(p) = p''(x) \) and \( S : P_1(\mathbb{R}) \to P_3(\mathbb{R}) \) has \( S(p) = x^2 p(x) \).

(c) With \( \beta = \{1, x\} \), compute the associated matrices \( [S]_\beta^\gamma \) and \( [U]_\beta^\alpha \) and then verify that \( [T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\beta^\alpha \).

- Since \( S(1) = x^2 \) and \( S(x) = x^3 \) we have \( [S]_\beta^\gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \), and likewise since \( U(1) = 0, U(x) = 0, \)

\( U(x^2) = 2, \) and \( U(x^3) = 6x, \) we see \( [U]_\beta^\alpha = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \). Then \( [T]_\alpha^\gamma = [S]_\beta^\gamma [U]_\beta^\alpha \) as claimed.

(d) Which of \( S, T, \) and \( U \) are onto? One-to-one? Isomorphisms?

- The map \( U \) is onto (since its image is all of \( P_1(\mathbb{R}) \)), but \( S \) and \( T \) are not onto.
- Also, \( S \) is one-to-one, since its kernel is trivial, but \( U \) and \( T \) both have nonzero elements in their kernels, so they are not one-to-one.
- Since none of the maps is both one-to-one and onto, none of them are isomorphisms.
4. Suppose $V = P_3(\mathbb{R})$, with standard basis $\beta = \{1, x, x^2, x^3\}$, and let $T : V \to V$ be the linear transformation with $T(1) = 1 - x + x^2 - x^3$, $T(x) = 2x - x^3$, and $T(x^2) = 3 + x - x^3$, and $T(x^3) = 1 - x^2$.

(a) Find $[T]_\beta^\beta$.

- We simply read off the coefficients from the description above, yielding $[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 3 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$

Now let $\gamma$ be the ordered basis $\gamma = \{x^3, x^2, x, 1\}$.

(b) Find the change-of-basis matrix $Q = [I]_\beta^\gamma$ and its inverse.

- We must evaluate the identity transformations on $\beta$ and write the results in terms of $\gamma$.
- We have $I(1) = 1 = (x + 1) - x$, $I(x) = x = x$, $I(x^2) = x^2$, and $I(x^3) = x^3$.
- Thus, we obtain $[I]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$. The inverse is $[I]_\gamma^\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, which we can compute either by inverting the matrix just computed, or by writing $\gamma$ in terms of $\beta$.

(c) For $v = 2 - x - 2x^2 + x^3$, compute $[v]_\beta$, $[v]_\gamma$, and verify that $[v]_\gamma = Q[v]_\beta$.

- We have $[v]_\beta = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix}$ and since $v = x^3 - 2x^2 + 2(x + 1) - 3x$, $[v]_\gamma = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -3 \end{pmatrix}$.

- Indeed, $Q[v]_\beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -3 \end{pmatrix} = [v]_\gamma$ as claimed.

(d) Find $[T]_\beta^\gamma$, $[T]_\gamma^\beta$, and $[T]_\gamma^\gamma$.

- With $Q = [I]_\beta^\gamma$ we have $[T]_\gamma^\gamma = [I]_\gamma^\beta [T]_\beta^\gamma = Q[T]_\beta^\gamma$, $[T]_\gamma^\beta = [T]_\beta^\gamma [I]_\gamma^\beta = [T]_\beta^\beta Q^{-1}$, and $[T]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\beta [I]_\gamma^\gamma = Q[T]_\beta^\beta Q^{-1}$.

- So $[T]_\beta^\gamma = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 3 & 1 \\ -2 & 2 & -2 & -1 \end{pmatrix}$, $[T]_\gamma^\beta = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 \end{pmatrix}$, $[T]_\gamma^\gamma = \begin{pmatrix} 0 & -1 & -2 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ -1 & -2 & 0 & 2 \end{pmatrix}$

5. Suppose $V$ and $W$ are finite-dimensional vector spaces and $T : V \to W$ is linear.

(a) If $\beta$ and $\gamma$ are ordered bases of $V$ and $W$ respectively such that $[T]_\beta^\gamma$ is the identity matrix, show that $T$ is an isomorphism.

- If $[T]_\beta^\gamma$ is the identity matrix, then since $[T]_\gamma^\beta$ is invertible, and so $T$ is an isomorphism (since a linear transformation is an isomorphism if its corresponding matrix is invertible).

- More explicitly, if $S : W \to V$ is the linear transformation such that $[S]_\gamma^\beta$ is the identity matrix, then $[ST]_\beta^\gamma = [S]_\gamma^\beta [T]_\beta^\gamma = I \cdot I = I$, and so $ST$ is the identity transformation on $V$. Likewise, $TS$ is the identity transformation on $W$, so $S$ is an explicit inverse for $T$.

(b) If $T$ is an isomorphism, show that there exist ordered bases $\beta$ and $\gamma$ of $V$ and $W$ respectively such that $[T]_\beta^\gamma$ is the identity matrix.

- Let $\beta = \{\beta_1, \ldots , \beta_n\}$ be any basis of $V$, and then set $\gamma = T(\beta)$ (i.e., so that $\gamma_i = T(\beta_i)$ for each $1 \leq i \leq n$). Then since $T$ is an isomorphism, $\gamma$ is indeed a basis of $W$.

- By construction, since $\gamma_i = T(\beta_i)$ for each $i$, we can see immediately that $[T]_\beta^\gamma$ is the identity matrix.
6. Suppose that \( T : V \to V \) is a linear transformation on a finite-dimensional vector space.

(a) If \( \beta \) and \( \gamma \) are two ordered bases of \( V \), show that \( \det([T]_\beta^\beta) = \det([T]_\gamma^\gamma) \).

- If \( Q = [I]_\beta^\gamma \) is the change-of-basis matrix, then \( [T]_\gamma^\gamma = Q[T]_\beta^\beta Q^{-1} \).
- Taking determinants yields \( \det([T]_\gamma^\gamma) = \det(Q[T]_\beta^\beta Q^{-1}) = \det(Q) \det([T]_\beta^\beta) \det(Q^{-1}) = \det([T]_\beta^\beta) \) since \( \det(Q^{-1}) = \frac{1}{\det(Q)} \). Hence the determinants are equal as claimed.

Per part (a), we define \( \det(T) \) to be \( \det([T]_\beta^\beta) \) for any choice of ordered basis \( \beta \).

(b) Show that \( T \) is an isomorphism if and only if \( \det(T) \) is nonzero.

- Observe \( T \) is an isomorphism \( \iff [T]_\beta^\beta \) is an invertible matrix \( \iff \det([T]_\beta^\beta) = \det(T) \) is nonzero.

7. Let \( F \) be a field and \( n \geq 2 \) be an integer. Recall that we say two matrices \( A \) and \( B \) are similar if there exists an invertible matrix \( Q \) with \( B = Q^{-1}AQ \).

(a) Show that if \( A \) and \( B \) are similar matrices in \( M_{n \times n}(F) \), then \( \det(A) = \det(B) \) and \( \text{tr}(A) = \text{tr}(B) \). [Hint: You may use the fact that \( \text{tr}(CD) = \text{tr}(DC) \).]

- If \( B = Q^{-1}AQ \), then \( \det(B) = \det(Q^{-1}AQ) = \det(Q^{-1}) \det(A) \det(Q) = \frac{1}{\det(Q)} \det(A) \det(Q) = \det(A) \).
- Likewise, using the property in the hint with \( C = Q^{-1} \) and \( D = AQ \), we see \( \text{tr}(B) = \text{tr}(Q^{-1}AQ) = \text{tr}(AQQ^{-1}) = \text{tr}(A) \).

(b) Show that “being similar” is an equivalence relation on \( M_{n \times n}(F) \).

- Reflexive: Every matrix is similar to itself, since \( A = I_n^{-1}AI_n \).
- Symmetric: If \( A \) is similar to \( B \), so that \( B = Q^{-1}AQ \), then \( A = QBQ^{-1} = (Q^{-1})^{-1}BQ^{-1} \), so \( B \) is similar to \( A \).
- Transitive: If \( A \) is similar to \( B \) and \( B \) is similar to \( C \), say with \( A = Q^{-1}BQ \) and \( B = R^{-1}CR \), then \( A = R^{-1}Q^{-1}CQR = (QR)^{-1}C(QR) \), so \( A \) is similar to \( C \).