Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $V$ be a vector space with scalar field $F$ and $\Phi : V \times V \to F$ be a bilinear form. Identify each of the following statements as true or false:

(a) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.
(b) Every $n \times n$ symmetric matrix over an arbitrary field $F$ is congruent to a diagonal matrix.
(c) The function $Q(x,y) = xy$ on $\mathbb{R}^2$ is a quadratic form.
(d) The function $Q(x,y) = x^2 - 4xy + xyz + z^2$ on $\mathbb{R}^3$ is a quadratic form.
(e) The function $Q(f) = \int_0^1 x f(x)^2 \, dx$ on $\mathbb{R}[x]$ is a quadratic form.
(f) The function $Q(A) = \det(A)$ on $M_{2\times2}(\mathbb{R})$ is a quadratic form.
(g) The function $Q(A) = \det(A)$ on $M_{3\times3}(\mathbb{R})$ is a quadratic form.
(h) Every quadratic form over $\mathbb{R}$ is a bilinear form.
(i) Every quadratic form over an arbitrary field is a bilinear form.
(j) The second derivatives test will classify any critical point as a local minimum, local maximum, or saddle point.
(k) If both eigenvalues of the $2 \times 2$ real symmetric matrix $S$ are positive, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be an ellipse.
(l) If one eigenvalue of the $2 \times 2$ real symmetric matrix $S$ is zero and the other is nonzero, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be a hyperbola.

2. For each symmetric matrix $S$ over each given field, find an invertible matrix $Q$ and diagonal matrix $D$ such that $Q^T S Q = D$:

(a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$ over $\mathbb{Q}$.
(b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$ over $\mathbb{Q}$.

3. Consider the bilinear form $\Phi((a,b), (c,d)) = 4ac - 2ad - 2bc + 7bd$ on $\mathbb{R}^2$ and let $Q$ be the associated quadratic form.

(a) Write down $Q$ explicitly and also find $[\Phi]_\beta$ for $\beta = \{(1,0), (0,1)\}$.
(b) Find an orthonormal basis $\gamma$ for $\mathbb{R}^2$ such that $[\Phi]_\gamma$ is diagonal, and compute the diagonalization $[\Phi]_\gamma$.
(c) Describe the shape of the quadratic variety $Q(x, y) = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.
(d) Classify the critical point of the function $Q(x, y)$ at $(0,0)$ as a local minimum, local maximum, or saddle point.
(e) Calculate the signature and index of $Q$. Is $Q$ positive definite? Positive semidefinite? Negative definite? Negative semidefinite?
4. Consider the quadratic form \( Q(x, y, z) = 11x^2 + 40xy - 16xz - 16y^2 - 16yz + 5z^2 \) on \( \mathbb{R}^3 \).

(a) Find the symmetric matrix associated to the underlying bilinear form for \( Q \) with respect to the standard basis \( \beta = \{(1,0,0), (0,1,0), (0,0,1)\} \).

(b) Give an explicit orthonormal change of basis that diagonalizes \( Q \), and find the resulting diagonalization.

(c) Describe the shape of the quadratic variety \( Q(x, y, z) = 1 \) in \( \mathbb{R}^3 \) as one of the 9 standard quadric surfaces.

(d) Classify the critical point of the function \( Q(x, y, z) \) at \((0,0,0)\) as a local minimum, local maximum, or saddle point.

(e) Calculate the signature and index of \( Q \). Is \( Q \) positive definite? Positive semidefinite? Negative definite? Negative semidefinite?

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that \( Q_1 \) and \( Q_2 \) are two quadratic forms on \( V \), and let \( \beta \in F \).

(a) Show that \( Q_1 + Q_2 \) is a quadratic form on \( V \). Note that \( Q_1 + Q_2 \) is defined pointwise, so \((Q_1 + Q_2)(v) = Q_1(v) + Q_2(v)\).

(b) Show that \( \beta Q_1 \) is a quadratic form on \( V \).

(c) Deduce that the set of all quadratic forms on \( V \) is a vector space.

6. Suppose \( T : V \to \mathbb{R} \) is a linear operator on the real inner product space \( V \) with inner product \( \langle \cdot, \cdot \rangle \). Define the map \( \Phi : V \times V \to F \) by setting \( \Phi(v, w) = \langle T(v), w \rangle \).

(a) Show that \( \Phi \) is a bilinear form on \( V \).

(b) Show that \( \Phi \) is symmetric if and only if \( T \) is Hermitian.

(c) If \( V \) is finite-dimensional, prove that \( \Phi \) is an inner product on \( V \) if and only if \( T \) is a positive-definite Hermitian operator. [Hint: Show that \( |I_{32}| \) requires all eigenvalues of \( T \) to be positive.]

7. In multivariable calculus, the following more explicit version of the second derivatives test is often taught:  

- **Theorem (Second Derivatives Test in \( \mathbb{R}^2 \)):** Suppose \( P \) is a critical point of \( f(x, y) \), and let \( D \) be the value of the discriminant \( f_{xx}f_{yy} - f_{xy}^2 \) at \( P \). If \( D > 0 \) and \( f_{xx} > 0 \), then the critical point is a minimum. If \( D > 0 \) and \( f_{xx} < 0 \), then the critical point is a maximum. If \( D < 0 \), then the critical point is a saddle point. If \( D = 0 \), then the test is inconclusive.

Using our general version of the second derivatives test, prove this variation. [Hint: Note that \( D = \det(H) = \lambda_1 \lambda_2 \); then examine what information the sign of \( D \) yields about the eigenvalues \( \lambda_1, \lambda_2 \)].

8. Let \( S \) be an \( n \times n \) real symmetric matrix.

(a) Show that \( S \) is congruent to a matrix whose diagonal entries are all in the set \( \{-1, 0, 1\} \).

(b) Prove that, up to congruence, there are exactly \( \frac{1}{2}(n + 1)(n + 2) \) different real \( n \times n \) symmetric matrices.

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\(^1\)The statement of this theorem is copied directly from my multivariable calculus course notes, in fact!