1. For each polynomial $p(x)$ in the given polynomial rings $F[x]$, either find a nontrivial factorization or explain why it is irreducible:
   
   (a) $p(x) = x^2 + 2$ in $F_2[x], F_3[x], F_5[x], Q[x], R[x]$, and $C[x]$.
   (b) $p(x) = x^3 + x^2 + 2$ in $F_3[x], F_5[x]$, and $F_7[x]$.
   (c) $p(x) = x^4 + 1$ in $F_2[x], F_3[x], F_5[x]$, and $R[x]$. [Hint: This polynomial factors in each case.]

2. For each $p$ and $F[x]$ (note that these are the same as in problem 1), determine whether or not $F[x]$ modulo $p$ is a field.
   
   (a) $p(x) = x^2 + 2$ in $F_2[x], F_3[x], F_5[x], Q[x], R[x]$, and $C[x]$.
   (b) $p(x) = x^3 + x^2 + 2$ in $F_3[x], F_5[x]$, and $F_7[x]$.
   (c) $p(x) = x^4 + 1$ in $F_2[x], F_3[x], F_5[x]$, and $R[x]$. 

3. Solve the following problems:
   
   (a) Find the number of monic irreducible polynomials in $F_3$ of degrees 5, 6, 7, 8, 9, and 10.
   (b) Give an explicit construction for a field having exactly 49 elements.
   (c) Find a primitive root modulo $17^{2020}$ and the total number of primitive roots modulo $17^{2020}$, or explain why there are none.
   (d) Find a primitive root modulo $32^{2020}$ and the total number of primitive roots modulo $32^{2020}$, or explain why there are none.
   (e) Find a primitive root modulo $33^{2020}$ and the total number of primitive roots modulo $33^{2020}$, or explain why there are none.
   (f) Find a primitive root modulo $2 \cdot 5^{2020}$ and the total number of primitive roots modulo $2 \cdot 5^{2020}$, or explain why there are none.

4. Let $p(x) = x^{2020}$.
   
   (a) Find the remainder when $p(x)$ is divided by $x - 4$ in $R[x]$.
   (b) Find the remainder when $p(x)$ is divided by $x + 1$ in $R[x]$.
   (c) Find the remainder when $p(x)$ is divided by $x^2 - 3x + 4$ in $R[x]$. [Hint: Use the Chinese Remainder Theorem.]
Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. If $R/rR$ has finitely many units, then we can use the same order-calculation algorithm we used in $\mathbb{Z}/m\mathbb{Z}$ to find the order of an arbitrary unit residue class $\bar{s}$. Explicitly, $\bar{s}$ has order $n$ if and only if $\bar{s}^n = \bar{1}$ and $\bar{s}^{n/p} \neq \bar{1}$ for any prime $p$ dividing $n$.

(a) Show that $R = \mathbb{F}_5[x]$ modulo $r = x^2 + 2$ is a field with 25 elements, and deduce that the order of any nonzero residue class in $R/rR$ divides 24.

(b) Find the orders of $\bar{2}$, $\bar{x}$, and $\bar{x} + \bar{1}$ in $\mathbb{F}_5[x]$ modulo $x^2 + 2$. Are any of them primitive roots?

6. The goal of this problem is to give some examples of $R/rR$ for $R = \mathbb{F}_5[x]$ where there do and do not exist primitive roots.

(a) In $\mathbb{F}_5[x]$ modulo the irreducible polynomial $x^2 + x + 1$, show that the element $\bar{x} + 2$ is a primitive root.

(b) In $\mathbb{F}_5[x]$ modulo the reducible polynomial $x^2$, show that there are 20 units and that $\bar{x} + 3$ has order 20 (and thus is a primitive root).

(c) In $\mathbb{F}_5[x]$ modulo the reducible polynomial $x^2 + x$, show that there are 16 units but that all of them have order dividing 4. Deduce that there are no primitive roots in this case. [Hint: Show that $u^4 \equiv 1 \text{ mod } x$ and $u^4 \equiv 1 \text{ mod } x + 1$ for each possible unit, and then use the Chinese Remainder Theorem.]

(d) Based on the results of parts (a)-(c), and in analogy with the case for $\mathbb{Z}/m\mathbb{Z}$, can you conjecture when there will be a primitive root in $R/rR$ when $R = \mathbb{F}_5[x]$?