Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $R = \mathbb{F}_3[x]$ and $p = x^2 + x$.
   
   (a) List the 9 residue classes in $R/pR$.
   
   (b) Construct the addition and multiplication tables for $R/pR$. (You may omit the bars in the residue class notation for efficiency.)
   
   (c) Identify all of the units and zero divisors in $R/pR$.
   
   (d) Verify Euler's theorem for each unit in $R/pR$.

2. Let $R = \mathbb{F}_2[x]$ and $p = x^3 + x + 1$.
   
   (a) List the 8 residue classes in $R/pR$.
   
   (b) Construct the addition and multiplication tables for $R/pR$. (You may omit the bars in the residue class notation for efficiency.)
   
   (c) Show that $R/pR$ is a field by explicitly identifying the inverse of every nonzero element. [Hint: Use the multiplication table from (b).]
   
   (d) Verify Fermat's little theorem for the elements $\pi$ and $\pi + 1$ in $R/pR$.

3. For each of the following quotient rings $R/pR$, show that the given element $u$ is a unit and find its multiplicative inverse in $R/pR$:
   
   (a) $R = \mathbb{Q}[x]$, $p(x) = x^2 + 1$, $u = x + 3$.
   
   (b) $R = \mathbb{Z}[i]$, $p = 8 + 7i$, $u = 1 - 2i$.
   
   (c) $R = \mathbb{F}_3[x]$, $p(x) = x^4 + 2x + 1$, $u = x^2 + 1$.
   
   (d) $R = \mathbb{Z}[i]$, $p = 11 - 14i$, $u = 4 + 8i$.

4. Solve the following systems of congruences:
   
   (a) $p \equiv 1 \pmod{x + 2}$ and $p \equiv 7 \pmod{x - 1}$ in $\mathbb{Q}[x]$.
   
   (b) $z \equiv 1 \pmod{2 + 2i}$ and $z \equiv -i \pmod{4 + 5i}$ in $\mathbb{Z}[i]$. 
Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Show the following things:
   (a) Show that the element $4 + 5i$ is irreducible and prime in $\mathbb{Z}[i]$.
   (b) Show that the element $x^2 + 1$ is irreducible and prime in $\mathbb{R}[x]$.
   (c) Show that the element $3 + 5i$ is neither irreducible nor prime in $\mathbb{Z}[i]$ by finding a factorization.
   (d) Show that the element $2 + \sqrt{-10}$ is irreducible but not prime in $\mathbb{Z}[\sqrt{-10}]$. [Hint: Show it divides 14 and that there are no elements of norm 2 or 7.]

6. The goal of this problem is to prove that for any integer $D > 3$, the ring $\mathbb{Z}[\sqrt{-D}]$ is not a unique factorization domain, generalizing the technique used for $D = 5$.
   (a) Show that $\sqrt{-D}$, $1 + \sqrt{-D}$, $1 - \sqrt{-D}$, and 2 are irreducible elements in $\mathbb{Z}[\sqrt{-D}]$. [Hint: For the first three, show that the only elements of norm less than $D$ are integers.]
   (b) Show that either $D$ (if $D$ is even) or $D + 1$ (if $D$ is odd) has two different factorizations into irreducibles in $\mathbb{Z}[\sqrt{-D}]$, and deduce that $\mathbb{Z}[\sqrt{-D}]$ is not a unique factorization domain.
   (c) What goes wrong if you try to use the proof to show that $\mathbb{Z}[\sqrt{D}]$ is not a UFD for positive $D > 3$?

• Remark: When combined with our previous results (showing $\mathbb{Z}[i]$ is Euclidean in class, and that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean and $\mathbb{Z}[\sqrt{-3}]$ is not on homework 7), this problem completes the characterization of the rings $\mathbb{Z}[\sqrt{-D}]$ that are Euclidean when $D$ is a positive integer.