1. (Calculations, I) Calculate (no justification required):
   (a) Find the Jordan canonical forms of
       \[
       \begin{pmatrix}
         -5 & 9 \\
         -4 & 7
       \end{pmatrix},
       \begin{pmatrix}
         3 & 1 \\
         -2 & 5
       \end{pmatrix},
       \begin{pmatrix}
         1 & 1 & -1 \\
         -2 & 3 & -2 \\
         -1 & 0 & 1
       \end{pmatrix},
       \begin{pmatrix}
         1 & 2 & 3 \\
         2 & 3 & 1 \\
         3 & 1 & 2
       \end{pmatrix}
     \] over \( \mathbb{C} \).
   (b) Find the Jordan canonical form of the \( n \times n \) all 1s matrix
       \[
       \begin{pmatrix}
         1 & 1 & \ldots & 1 \\
         1 & 1 & \ldots & 1 \\
         \vdots & \vdots & \ddots & \vdots \\
         1 & 1 & \ldots & 1
       \end{pmatrix}
     \] over \( \mathbb{Q} \) and over \( \mathbb{F}_p \) (the field with \( p \) elements, \( p \) prime). [For the latter, the answer will depend on whether \( p \) divides \( n \) or not.]
   (c) Find the minimal and characteristic polynomials of the \( n \times n \) Jordan block with eigenvalue \( \lambda \).

2. (Calculations, II) Suppose the characteristic polynomial of the \( 5 \times 5 \) matrix \( A \) is \( p(t) = t^3(t - 1)^2 \).
   (a) Find the eigenvalues of \( A \), and list all possible dimensions for each of the corresponding eigenspaces.
   (b) List all possible Jordan canonical forms of \( A \) up to equivalence.
   (c) If \( \ker(A) \) and \( \ker(A - I) \) are both 2-dimensional, what is the Jordan canonical form of \( A \)?

3. (Minimal Polynomial, II) Suppose \( A \) is an \( n \times n \) matrix over \( \mathbb{C} \) and that \( \lambda \) is an eigenvalue of \( A \).
   (a) Prove that \( \lambda \) is a root of the minimal polynomial of \( A \). Deduce that the minimal polynomial and the characteristic polynomial have the same roots.
   (b) Show that the exponent of \( t - \lambda \) in the minimal polynomial \( m(t) \) of \( A \) is the size of the largest Jordan block of eigenvalue \( \lambda \) in the Jordan canonical form of \( T \).
   (c) Show that the minimal polynomial of a \( 2 \times 2 \) matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the \( 2 \times 2 \) matrices with minimal polynomials \( m(t) = t^2 - t, t^2 + 1 \), and \( t - \lambda \).
   (d) Show the minimal and characteristic polynomials of a \( 3 \times 3 \) matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the \( 3 \times 3 \) matrices with \( (m(t), p(t)) \) equal to \( (t, t^3), (t^2, t^3), (t^3, t^3), \) and \( (t^2 - 1, t^3 - t^2 - t + 1) \).

4. Let \( A \in M_{n \times n}(\mathbb{C}) \). Show that the following are equivalent:
   (a) The ranks of \( A \) and \( A^2 \) are equal.
   (b) The multiplicity of 0 as a root of the minimal polynomial of \( A \) is at most 1.
   (c) There is an \( n \times n \) matrix \( X \) such that \( AXA = A, XAX = X, \) and \( AX = XB \).
   (d) If there are any, every Jordan block with eigenvalue 0 in the Jordan canonical form of \( A \) has size 1.
5. (Matrix Orders, I) Let $F$ be an algebraically closed field and $G = GL_n(F)$ be the group of $n \times n$ invertible matrices over $F$.

(a) If $A$ has finite order $k$ in $G$, prove that all eigenvalues of $A$ are $k$th roots of unity in $F$ (i.e., elements $\alpha \in F$ for which $\alpha^k = 1$). [Hint: Use 3(a) and the fact that the minimal polynomial divides any polynomial annihilating $A$.]

(b) Show that if $\text{char}(F) = 0$, then the only Jordan blocks of finite order are the $1 \times 1$ blocks.

(c) Deduce that if $\text{char}(F) = 0$, then $A$ has finite order if and only if it is diagonalizable and all its eigenvalues are roots of unity.

(d) [Optional] If $\text{char}(F) > 0$, prove that $A$ has finite order if and only if all its eigenvalues are roots of unity.

• Remark: If $F$ is not algebraically closed, the results of this problem still essentially hold, provided one works with the diagonalization and eigenvalues over a field extension $K/F$ containing all the eigenvalues of $G$ (e.g., the algebraic closure of $F$), since the order of $G \in GL_n(F)$ is the same in $GL_n(K)$.

6. (Matrix Orders, II) Let $F$ be a field and $G = GL_n(F)$.

(a) If $\text{char}(F) \neq 2$, show that $G$ has precisely $n$ conjugacy classes of elements of order 2. [Hint: If $g$ has order 2, then $g^2 - I = 0$.]

(b) If $\text{char}(F) = 2$, show that $G$ has precisely $\lfloor n/2 \rfloor$ (the greatest integer $\leq n/2$) conjugacy classes of elements of order 2.

7. (Matrix Orders, III) Let $K$ be a field where $-1$ is not a square, and let $G = GL_2(K)$.

(a) If $g \in G$, show that $g$ has order 4 if and only if $\det(g) = 1$ and $\text{tr}(g) = 0$. [Hint: Consider the eigenvalues of $g$.]

(b) Find an explicit element $g \in G$ of order 4.

(c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that $G$ contains two elements $g, h$ of order 4 such that $gh$ also has order 4.