Part I: Calculation Problems

1. For each of the following relations, decide whether they are (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) irreflexive, (vi) an equivalence relation, (vii) a partial ordering, and (viii) a total ordering.

(a) \( R = \{ (1, 1), (2, 1), (2, 2) \} \) on the set \( \{ 1, 2 \} \).
(b) \( R = \{ (1, 2), (2, 1) \} \) on the set \( \{ 1, 2 \} \).
(c) \( R = \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4) \} \) on the set \( \{ 1, 2, 3, 4 \} \).
(d) The divisibility relation on the set \( \{ 2, -3, 4, -5, 6 \} \).
(e) The divisibility relation on the set \( \{ 2, -4, -12, 36 \} \).
(f) The relation \( R \) on \( \mathbb{Z} \) with \( a R b \) precisely when \( |a| \equiv |b| \) modulo 5.
(g) The relation \( R \) on \( \mathbb{R} \) with \( a R b \) precisely when \( ab > 0 \).

2. For each of the following functions \( f : A \to B \), determine whether (i) \( f \) is one-to-one, (ii) \( f \) is onto, (iii) \( f \) is a bijection.

(a) \( f(x) = 2x \) from \( A = \mathbb{R} \) to \( B = \mathbb{R} \).
(b) \( f(n) = 2n \) from \( A = \mathbb{Z} \) to \( B = \mathbb{Z} \).
(c) \( f(x) = \frac{x}{x - 1} \) from \( A = \mathbb{R} \setminus \{ 1 \} \) to \( B = \mathbb{R} \).
(d) \( f(x) = x^3 \) from \( A = \mathbb{R} \) to \( B = \mathbb{R} \).
(e) \( f = \{(1, 2), (2, 3), (3, 4), (4, 1)\} \) from \( \{1, 2, 3, 4\} \) to itself.
(f) \( f = \{(1, 3), (2, 4), (3, 1), (4, 4)\} \) from \( \{1, 2, 3, 4\} \) to itself.

3. Identify the ordered pairs in the equivalence relation that corresponds to the partition \( \{1, 2, 4\}, \{3, 5\}, \{6\} \) of \( \{1, 2, 3, 4, 5, 6\} \).

4. Show \( R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x| = |y|\} \) is an equivalence relation and list the equivalence classes of 0, 1, 2, -2, and 4.

5. Find the coefficients of \( x^{18} \) in \((2x + 1)^{24}\), in \((x^2 + x)^{13}\), and in \((x^3 - 2/x)^{14}\).

6. A U.S. social security number (SSN) is a 9-digit string of the form abc-de-fghi. Find:

(a) The number of SSNs where all of the digits are odd.
(b) The number of SSNs that have no 0s.
(c) The number of SSNs that have at least one 0.
(d) The number of SSNs that have exactly four 0s.
(e) The number of SSNs that have at least one 0 or 8.
(f) The number of SSNs with no repeated digits.
(g) The number of SSNs with no double digits (i.e., no 00, 11, etc.).
(h) The number of SSNs where abc + de = fghi.
(i) The number of SSNs where abc, de, and fghi are all multiples of 3.
(j) The number of SSNs where abc \cdot de \cdot fghi is a multiple of 3.
(k) The number of SSNs where each digit is \( \geq \) the one to its left (e.g., 997-64-3100).
Part II: Proof Problems

1. Suppose $f : A \to B$ is a function.
   
   (a) If $f$ is one-to-one, show that there is a bijection between $A$ and $\text{im}(f)$. Deduce that $\#A = \#\text{im}(f)$.
   
   (b) If $A$ and $B$ are both finite and $\#A = \#B$, show that $f$ is one-to-one implies that $f$ is onto.
   
   (c) Show that (b) is false for infinite sets by giving a function $f : \mathbb{R} \to \mathbb{R}$ where $f$ is one-to-one but not onto.

2. A real number is algebraic if it is a root of a nonzero polynomial $p(x)$ with integer coefficients, and it is transcendental if it is not the root of any such polynomial.
   
   (a) Let $S_n$ be the set of all roots of nonzero polynomials of degree at most $n$ whose coefficients are integers and at most $n$ in absolute value. Show that $S_n$ is finite and that the set of algebraic numbers is $\cup_{n \geq 1} S_n$.
   
   (b) Show that the set of algebraic numbers is countable. Deduce there are uncountably many transcendental numbers.

3. Let $A, B, C$ be sets, $R$ be a relation, $f, g$ be functions, and $m, k, n$ be nonnegative integers. Prove the following:
   
   (a) Show that the only equivalence relation $R$ on $A$ that is a function from $A$ to $A$ is the identity relation.
   
   (b) Prove that if $A$ is countable and $B$ is uncountable, then $B \setminus A$ is uncountable.
   
   (c) Prove that $\sum_{k=0}^{n} \binom{n}{k} 9^k = 10^n$.
   
   (d) Find the number of partitions of $\{1, 2, \ldots, 2n\}$ into $n$ unordered sets of 2 elements. Deduce that $2^n n!$ divides $(2n)!$.
   
   (e) Prove that the set $\mathbb{Q} \times \mathbb{Z}$ is countable and that the set $\mathbb{R} \times \mathbb{Z}$ is uncountable.
   
   (f) Prove that there exists a bijection between $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$, the set of rational numbers strictly between 0 and 1.
   
   (g) A committee of 3 people is chosen from $n$ mathematicians and $n$ computer scientists. Prove that the number of possible committees is $2 \binom{n}{3} + 2n \binom{n}{2}$ and deduce that $2 \binom{n}{3} + 2n \binom{n}{2} = \binom{2n}{3}$.
   
   (h) Prove that if $f : A \to B$ is one-to-one and $S \subseteq A$, then $f^{-1}(f(S)) = S$.
   
   (i) Prove that if $f : A \to B$ is onto and $T \subseteq B$, then $f(f^{-1}(T)) = T$.
   
   (j) Prove that there exists a bijection between $(0, 1)$ and $[0, 1]$. [Hint: Use the Cantor-Schröder-Bernstein theorem.]
   
   (k) Prove that $\sum_{k=1}^{n} k \cdot k! = (n + 1)! - 1$ for every positive integer $n$.
   
   (l) Suppose $f : A \to B$ is a bijection. Show that $\tilde{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ given by $\tilde{f}(S) = \{ f(s) : s \in S \}$ is also a bijection.
   
   (m) If $R, S : A \to B$ are relations, prove that $R^{-1} \cap S^{-1} = (R \cap S)^{-1}$.
   
   (n) Prove that $\sum_{k=0}^{n} \frac{2^k}{k!(n-k)!} = \frac{3^n}{n!}$.