Final Exam Topics:

- Average rate of change, limit definition of derivative
- Computing derivatives (product/quotient/chain rules)
- Logarithmic, inverse, implicit differentiation
- Parametric curves and derivatives, velocity/speed/acceleration
- Related rates
- Minimum and maximum values, crit points + classification, increasing and decreasing behavior, concavity, inflection points
- L’Hôpital’s rule
- Applied optimization
- Antiderivatives
- Riemann sums + definite integrals, Fund Thm of Calculus
- Evaluating definite and indefinite integrals, substitution
- Areas under and between curves
Problem 1

(Fa14, #10) A box with an open top is to be constructed with 600 in$^2$ of material. The length of the base is to be twice its width. Find the dimensions that maximize the volume of the box.
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Answer:
(Fa14, #10) A box with an open top is to be constructed with 600 in$^2$ of material. The length of the base is to be twice its width. Find the dimensions that maximize the volume of the box.

Answer: Width $w$, height $h$, length $l = 2w$, vol $V = lwh = 2w^2h$. Base has area $2w^2$, two sides have area $wh$, and the other two sides have area $2wh$. Hence total area is $2w^2 + 6wh$, so $2w^2 + 6wh = 600$, thus $h = \frac{300-w^2}{3w}$.

Then $V = 2w^2 \cdot \frac{300-w^2}{3w} = 200w - \frac{2}{3}w^3$ so $V' = 200 - 2w^2$ which is zero when $w = 10$ in. Sign diagram for $V'$ shows $w = 10$ in is a global max. So dimensions are $w = 10$ in, $l = 20$ in, $h = \frac{20}{3}$ in.
Problem 2

Find the area of the region lying under the curve $y = 2x - x^2$ and above the $x$-axis.
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Answer:
Find the area of the region lying under the curve \( y = 2x - x^2 \) and above the \( x \)-axis.

Answer: The curve intersects the \( x \)-axis when \( 2x - x^2 = 0 \) so that \( x = 0, 2 \). Then the desired area is

\[
\int_{0}^{2} (2x - x^2) \, dx = x^2 - \frac{1}{3}x^3 \bigg|_{x=0}^{2} = \frac{4}{3}.
\]
(Fa14, #11a) Compute \( \int (7 + 8x)^{49} \, dx \).
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Answer:
Problem 3

(Fa14, #11a) Compute \( \int (7 + 8x)^{49} \, dx \).

Answer: Substituting \( u = 7 + 8x \) with \( du = 8 \, dx \) yields

\[
I = \int u^{49} \cdot \frac{1}{8} \, du = \frac{1}{400} u^{50} + C = \frac{1}{400} (7 + 8x)^{50} + C.
\]
Interlude!
(Fa14, #13) Find the area bounded by $y = x^2 - 5x + 3$ and $y = -x^2 + x - 1$. 

Answer: The curves intersect when $x^2 - 5x + 3 = -x^2 + x - 1$ so that $2x^2 - 6x + 4 = 0$. Factoring gives $2(x - 1)(x - 2) = 0$ so intersection points are at $x = 1, 2$. Testing at $x = 3/2$, or comparing the graphs, shows that $y = -x^2 + x - 1$ is the top curve and $y = x^2 - 5x + 3$ is the bottom curve. Hence area is $\int_1^2 [\text{top} - \text{bottom}] \, dx = \int_1^2 (-2x^2 + 6x - 4) \, dx = \left[ -\frac{2}{3}x^3 + 3x^2 - 4x \right]_1^2 = \frac{1}{3}$. 
Problem 4

(Fa14, #13) Find the area bounded by \( y = x^2 - 5x + 3 \) and \( y = -x^2 + x - 1 \).

Answer:
Problem 4

(Fa14, #13) Find the area bounded by \( y = x^2 - 5x + 3 \) and \( y = -x^2 + x - 1 \).

Answer: The curves intersect when \( x^2 - 5x + 3 = -x^2 + x - 1 \) so that \( 2x^2 - 6x + 4 = 0 \). Factoring gives \( 2(x - 1)(x - 2) = 0 \) so intersection points are at \( x = 1, 2 \).

Testing at \( x = 3/2 \), or comparing the graphs, shows that \( y = -x^2 + x - 1 \) is the top curve and \( y = x^2 - 5x + 3 \) is the bottom curve.

Hence area is \( \int_1^2 [\text{top} - \text{bottom}] \, dx = \int_1^2 (-2x^2 + 6x - 4) \, dx = \left[ -\frac{2}{3}x^3 + 3x^2 - 4x \right]_1^2 = \frac{1}{3} \).
(Fa14, #11b) Compute \( \int_{\pi}^{2\pi} \frac{3 \sin x}{2 + \cos x} \, dx \).
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Answer:
Problem 5

(Fa14, #11b) Compute \( \int_{\pi}^{2\pi} \frac{3 \sin x}{2 + \cos x} \, dx \).

Answer: Substitute \( u = 2 + \cos x \) with \( du = -\sin x \, dx \). Then \( x = \pi \) corresponds to \( u = 1 \) and \( x = 2\pi \) corresponds to \( u = 3 \), so then we obtain \( I = \int_{1}^{3} \frac{-3}{u} \, du = -\ln(u) \bigg|_{1}^{3} = -3 \ln 3 \).
Interlude!

![Four cats](image-url)
Problem 6

(Fa15, #10) Suppose we need to construct a coffee cup in the shape of a circular cylinder that holds $128\pi$ cubic centimeters. The cup has no top! The cost per square centimeter of material for the sides of the cup is 1 cent, and for the bottom of the cup the cost per square centimeter is 2 cents. Find the radius $r$ and height $h$ of the cup that minimizes the cost.
Problem 6

(Fa15, #10) Suppose we need to construct a coffee cup in the shape of a circular cylinder that holds $128\pi$ cubic centimeters. The cup has no top! The cost per square centimeter of material for the sides of the cup is 1 cent, and for the bottom of the cup the cost per square centimeter is 2 cents. Find the radius $r$ and height $h$ of the cup that minimizes the cost.

Answer:
Problem 6

(Fa15, #10) Suppose we need to construct a coffee cup in the shape of a circular cylinder that holds $128\pi$ cubic centimeters. The cup has no top! The cost per square centimeter of material for the sides of the cup is 1 cent, and for the bottom of the cup the cost per square centimeter is 2 cents. Find the radius $r$ and height $h$ of the cup that minimizes the cost.

Answer: Volume is $V = \pi r^2 h$ cm$^3$, so $r^2 h = 128$ hence $h = \frac{128}{r^2}$. Area of sides is $2\pi rh$ cm$^2$, area of base is $\pi r^2$ cm$^2$, so total cost is $C = 2\pi rh + 2\pi r^2 = 2\pi (\frac{128}{r} + r^2)$ cents.

So $C'(r) = 2\pi (-\frac{128}{r^2} + 2r)$ which is zero for $r = 4$ cm, and is the global min by $C'$ sign diagram. So $\boxed{r = 4 \text{ cm and } h = 8 \text{ cm}}$. 
(Fa14, #7) Consider the function $f(x) = x^3 + \frac{1}{2}x^2$. Find $f(2)$ and $(f^{-1})'(10)$.
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Answer:
(Fa14, #7) Consider the function \( f(x) = x^3 + \frac{1}{2}x^2 \). Find \( f(2) \) and \( (f^{-1})'(10) \).

**Answer:** Note that \( f'(x) = 3x^2 + x \).

Clearly \( f(2) = 10 \), meaning that \( f^{-1}(10) = 2 \).

Then by the inverse function differentiation formula,

\[
(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))} = \frac{1}{f'(2)} = \frac{1}{14}.
\]
Interlude!
(Fa14, #10) Find the area of the region bounded by the graphs of 
\( f(x) = x^2 - x - 1 \) and \( g(x) = x + 2 \).
Problem 8

(Fa14, #10) Find the area of the region bounded by the graphs of \( f(x) = x^2 - x - 1 \) and \( g(x) = x + 2 \).

Answer:
Problem 8

(Fa14, #10) Find the area of the region bounded by the graphs of \( f(x) = x^2 - x - 1 \) and \( g(x) = x + 2 \).

Answer: The curves intersect when \( x^2 - x - 1 = x + 2 \) so that \( x^2 - 2x - 3 = 0 \). Factoring gives \( (x - 3)(x + 1) = 0 \) so intersection points are at \( x = -1, 3 \).

Testing at \( x = 0 \), or comparing the graphs, shows that \( y = x + 2 \) is the top curve and \( y = x^2 - x - 1 \) is the bottom curve.

Hence area is \( \int_{-1}^{3} \left[ \text{top} - \text{bottom} \right] \, dx = \int_{-1}^{3} (-x^2 + 2x + 3) \, dx = \left[ -\frac{1}{3}x^3 + x^2 + 3x \right]_{x=-1}^{3} = \left[ -\frac{1}{3} \cdot 3^3 + 3^2 + 3 \cdot 3 \right] - \left[ -\frac{1}{3} \cdot (-1)^3 + (-1)^2 + 3 \cdot (-1) \right] = \frac{32}{3} \).
(Fa14, #12) Compute the midpoint Riemann sum for $f(x) = x^2$ for the partition of the interval $[-\frac{1}{2}, 1]$ into 3 subintervals of equal length.
Problem 9

(Fa14, #12) Compute the midpoint Riemann sum for \( f(x) = x^2 \) for the partition of the interval \([-\frac{1}{2}, 1]\) into 3 subintervals of equal length.

Answer:
(Fa14, #12) Compute the midpoint Riemann sum for \( f(x) = x^2 \) for the partition of the interval \([-\frac{1}{2}, 1]\) into 3 subintervals of equal length.

Answer: The width of the subintervals is \( \frac{1 - (-1/2)}{3} = \frac{1}{2} \), and the subintervals are \([-\frac{1}{2}, 0], [0, \frac{1}{2}], [\frac{1}{2}, 1]\).

Then the Riemann sum is

\[
RS_{\text{mid}} = f(-\frac{1}{4}) \cdot \frac{1}{2} + f(\frac{1}{4}) \cdot \frac{1}{2} + f(\frac{3}{4}) \cdot \frac{1}{2} = \boxed{\frac{11}{32}}.
\]
Interlude!
Problem 10

(Sp17, #10c) Evaluate \( \int_{-\pi}^{\pi} \sin(x) \cos^2(x) \, dx \).

Answer:
Substitute \( u = \cos(x) \) so that \( du = -\sin(x) \, dx \). Then \( x = -\pi \) corresponds to \( u = -1 \) and \( x = \pi \) also corresponds to \( u = -1 \), so the integral is
\[
I = \int_{-1}^{-1} u^2 \, du = \left. -\frac{1}{3} u^3 \right|_{-1}^{-1} = 0.
\]
Problem 10

(Sp17, #10c) Evaluate $\int_{-\pi}^{\pi} \sin(x) \cos^2(x) \, dx$.

Answer:
(Sp17, #10c) Evaluate \( \int_{-\pi}^{\pi} \sin(x) \cos^2(x) \, dx \).

Answer: Substitute \( u = \cos(x) \) so that \( du = -\sin(x) \, dx \). Then \( x = -\pi \) corresponds to \( u = -1 \) and \( x = \pi \) also corresponds to \( u = -1 \), so the integral is \( I = \int_{-1}^{-1} u^2 \, du = -\frac{1}{3} u^3 \bigg|_{u=-1}^{-1} = 0 \).
Problem 11

Find the area of the finite region enclosed between the curves $y = 5x$ and $y = x^2 + 4$. 

Answer:

The curves intersect when $5x = x^2 + 4$ so $x = 1, 4$. Using a test point or comparing graphs shows that $y = 5x$ is the top curve and $y = x^2 + 4$ is the bottom curve for $1 \leq x \leq 4$. Then the desired area is

$$\int_1^4 (5x - (x^2 + 4)) \, dx = \left( \frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x \right) |_{x=1}^{x=4} = \frac{9}{2}.$$
Problem 11

Find the area of the finite region enclosed between the curves $y = 5x$ and $y = x^2 + 4$.

Answer:
Find the area of the finite region enclosed between the curves $y = 5x$ and $y = x^2 + 4$.

Answer: The curves intersect when $5x = x^2 + 4$ so $x = 1, 4$.

Using a test point or comparing graphs shows that $y = 5x$ is the top curve and $y = x^2 + 4$ is the bottom curve for $1 \leq x \leq 4$.

Then the desired area is

$$\int_1^4 (5x - x^2 - 4) \, dx = \left( \frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x \right) \bigg|_{x=1}^{4} = \frac{9}{2}.$$
Interlude!
Problem 12

Evaluate \( \int \tan^3 x \sec^2 x \, dx \).
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Evaluate \( \int \tan^3 x \sec^2 x \, dx \).

Answer:

Substitute \( u = \tan(x) \) so that \( du = \sec^2(x) \, dx \). Then

\[
I = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3(x) + C.
\]
Problem 12

Evaluate \( \int \tan^3 x \sec^2 x \, dx \).

Answer: Substitute \( u = \tan(x) \) so that \( du = \sec^2(x) \, dx \). Then

\[
I = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3(x) + C.
\]
Problem 13

(Fa14, #11d) Compute \( \int \frac{4t^4 - 3t + \sqrt[3]{t}}{t^2} \, dt \).
(Fa14, #11d) Compute \( \int \frac{4t^4 - 3t + \sqrt[3]{t}}{t^2} \, dt \).

Answer:
Problem 13

(Fa14, #11d) Compute $\int \frac{4t^4 - 3t + \sqrt[3]{t}}{t^2} \, dt$.

Answer: Distribute the integrand to obtain

$$I = \int \left( 4t^2 - \frac{3}{t} + t^{-5/3} \right) \, dt = \frac{4}{3} t^3 - 3 \ln t - \frac{3}{2} t^{-2/3}.$$
End

Enjoy WeBWorK #12, and happy last day of fall classes!