Midterm 2 Topics:

- Related Rates
- Minimum and maximum values, critical points + classification
- Increasing and decreasing behavior, concavity
- Rolle’s theorem + mean value theorem
- L’Hôpital’s rule
- Antiderivatives
- Riemann sums + properties of definite integrals
- Fundamental theorem of calculus
- Evaluating definite and indefinite integrals

(Differentiating integrals and substitution are not on the midterm, though they are fair game for the final.)
Problem 1

Compute \( \lim_{x \to 0} \frac{e^x - 1}{\sin(3x)} \).
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Answer:
Problem 1

Compute \( \lim_{x \to 0} \frac{e^x - 1}{\sin(3x)} \).

Answer: Use L’Hôpital’s rule to get \( \lim_{x \to 0} \frac{e^x}{3 \cos(3x)} = \frac{1}{3} \).
Problem 2

Find all intervals where $f(x) = x^5 + 5x^4 + 7$ is increasing, decreasing, concave up, concave down.
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Answer:
Problem 2

Find all intervals where \( f(x) = x^5 + 5x^4 + 7 \) is increasing, decreasing, concave up, concave down.

Answer: Note \( f' = 5x^3(x + 4) \) so \( f \) has critical \( \#s \) \( x = 0, -4 \). Using test points gives a sign diagram \( f' : \oplus | \ominus | \oplus \), so \( f \) is incr on \((-\infty, -4)\) and \((0, \infty)\) and decr on \((0, 4)\).

Likewise, \( f'' = 20x^2(x + 3) \) so \( f \) has inflection \( \#s \) \( x = 0, -3 \). Using test points gives a sign diagram \( f'' : \ominus | \oplus | \ominus \), so \( f \) is conc up on \((-3, 0)\) and \((0, \infty)\) and conc down on \((-\infty, -3)\).
Problem 3

The sum of three positive numbers is 12 and two of them are equal. Find the largest possible product.
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Answer:
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Answer: Suppose the numbers are $x, x, y$. Then $2x + y = 12$ meaning that $y = 12 - 2x$, and so the product is $p(x) = x^2(12 - 2x) = 12x^2 - 2x^3$. Since we must have $0 < x < 6$ we look for critical #s in this range.

Since $p'(x) = 24x - 6x^2 = 6x(4 - x)$, the only critical # of $p$ in that range is $x = 4$, which is local max via the sign diagram for $p'$. This means the maximum product is $p(4) = \boxed{64}$.
Problem 4

Evaluate \( \int (x^2 + 1)^2 \, dx. \)
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Answer:
Problem 4

Evaluate $\int (x^2 + 1)^2 \, dx$.

Answer: Expand the integrand:

$$\int (x^2 + 1)^2 \, dx = \int (x^4 + 2x^2 + 1) \, dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C.$$
Interlude!
Problem 5

Find the absolute minimum and maximum of $f(x) = x + \frac{16}{x}$ on the interval $[1, 8]$, and all places where they occur.
Problem 5

Find the absolute minimum and maximum of \( f(x) = x + \frac{16}{x} \) on the interval \([1, 8]\), and all places where they occur.

Answer:
Problem 5

Find the absolute minimum and maximum of \( f(x) = x + 16/x \) on the interval \([1, 8]\), and all places where they occur.

Answer: Since \( f'(x) = 1 - \frac{16}{x^2} \), the critical \#s occur when \( x = -4, 4 \). So the only critical \# in the interval is at \( x = 4 \).

So, including the endpoints, our point list is \( x = 1, 4, 8 \).

Since \( f(1) = 17 \), \( f(4) = 8 \), and \( f(8) = 10 \), the minimum is 8 at \( x = 4 \) and the maximum is 17 at \( x = 1 \).
Problem 6

The volume of a cylindrical block of ice is $V = \pi r^2 h$. If the radius $r$ is currently 10 cm and decreasing at $1 \frac{\text{cm}}{\text{min}}$ and the height $h$ is currently 20 cm and decreasing at $3 \frac{\text{cm}}{\text{min}}$, how fast is the volume decreasing?
Problem 6

The volume of a cylindrical block of ice is $V = \pi r^2 h$. If the radius $r$ is currently 10cm and decreasing at $1 \frac{\text{cm}}{\text{min}}$ and the height $h$ is currently 20cm and decreasing at $3 \frac{\text{cm}}{\text{min}}$, how fast is the volume decreasing?

Answer:
Problem 6

The volume of a cylindrical block of ice is $V = \pi r^2 h$. If the radius $r$ is currently 10cm and decreasing at $1 \text{ cm/min}$ and the height $h$ is currently 20cm and decreasing at $3 \text{ cm/min}$, how fast is the volume decreasing?

Answer: Taking the derivative of the volume formula gives $V'(t) = 2\pi rh \cdot r'(t) + \pi r^2 \cdot h'(t)$. The given information says $r = 10\text{cm}, \ r' = -1 \text{ cm/min}, \ h = 20\text{cm}, \ h' = -3 \text{ cm/min}$.

Plugging in gives $V' = -700\pi \text{ cm}^3/\text{min}$.
Evaluate \( \int_2^2 \sqrt{3e^{2x} + \sin(x)} \, dx \).
Problem 7

Evaluate \( \int_{2}^{2} \sqrt{3e^{2x} + \sin(x)} \, dx \).

Answer:
Problem 7

Evaluate \( \int_{2}^{2} \sqrt{3e^{2x} + \sin(x)} \, dx \).

Answer: The top and bottom limits are the same so the integral is \( 0 \), regardless of the function.
Interlude!

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Problem 8

Evaluate \[ \int_{1}^{e} \frac{x^2 - x + 1}{x} \, dx. \]
Problem 8

Evaluate \( \int_{1}^{e} \frac{x^2 - x + 1}{x} \, dx \).

Answer:
Problem 8

Evaluate \[ \int_1^e \frac{x^2 - x + 1}{x} \, dx. \]

Answer: Distributing the fraction gives
\[
\int_1^e \frac{x^2 - x + 1}{x} \, dx = \int_1^e \left( x - 1 + \frac{1}{x} \right) \, dx = \left[ \frac{1}{2}x^2 - x + \ln(x) \right]_1^e = \frac{1}{2} \left( e^2 - 2e + 3 \right) - \left( \frac{1}{2} - 1 + \ln(1) \right) = \frac{1}{2} \left( e^2 - 2e + 3 \right).
\]
Problem 9

Find all critical numbers of \( f(x) = x^3 + 9x^2 - 21x + 2 \) and classify them as local minima, local maxima, or neither.
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Answer:
Problem 9

Find all critical numbers of $f(x) = x^3 + 9x^2 - 21x + 2$ and classify them as local minima, local maxima, or neither.

Answer: Note that $f'(x) = 3x^2 + 18x - 21 = 3(x - 1)(x + 7)$ so the critical numbers are $x = -7, 1$.

Using test points gives a sign diagram $f'$: $\oplus | \ominus | \oplus$, so there is a local maximum at $x = -7$ and a local minimum at $x = 1$. 
Interlude!
Problem 10

Compute \( \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos(x)} \).
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Compute \( \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos(x)} \).

Answer:
Problem 10

Compute \( \lim_{x \to 0} \frac{\sin(x^2)}{1 - \cos(x)} \).

Answer: Use L’Hôpital’s rule: 

\[
L = \lim_{x \to 0} \frac{2x \cos(x^2)}{\sin(x)} \]

This is still indeterminate so use it again: 

\[
L = \lim_{x \to 0} \frac{2 \cos(x^2) - 4x^2 \sin(x^2)}{\cos(x)}
\]

which then evaluates to 2 at \( x = 0 \).
Problem 11

Find \( f(x) \) if \( f''(x) = 12x^2 - e^x + \sin(x) \), where \( f'(0) = 1 \) and \( f(0) = 2 \).
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Find \( f(x) \) if \( f''(x) = 12x^2 - e^x + \sin(x) \), where \( f'(0) = 1 \) and \( f(0) = 2 \).

Answer:
Problem 11

Find \( f(x) \) if \( f''(x) = 12x^2 - e^x + \sin(x) \), where \( f'(0) = 1 \) and \( f(0) = 2 \).

Answer: Take the antiderivative to get
\[
f'(x) = 4x^3 - e^x - \cos(x) + C.
\]
Then \( f'(0) = 1 \) says \( f'(0) = -1 - 1 + C \) so \( C = 3 \), and \( f'(x) = 4x^3 - e^x - \cos(x) + 3 \).

Take the antiderivative again to see
\[
f(x) = x^4 - e^x - \sin(x) + 3x + D.
\]
Then \( f(0) = 2 \) says \( f(0) = -1 + D \) so \( D = 3 \). So \( f(x) = x^4 - e^x - \sin(x) + 3x + 3 \).
Interlude!
Problem 12

Evaluate \[ \int_{\pi/6}^{\pi/3} \frac{\sin(2x)}{\cos^2(x)} \, dx. \]
Problem 12

Evaluate \( \int_{\pi/6}^{\pi/3} \frac{\sin(2x)}{\cos^2(x)} \, dx \).

Answer:
Evaluate \( \int_{\pi/6}^{\pi/3} \frac{\sin(2x)}{\cos^2(x)} \, dx \).

Answer: Here we need to use the trig identity \( \sin(2x) = 2 \sin(x) \cos(x) \). This gives

\[
I = \int_{\pi/6}^{\pi/3} \frac{2 \sin(x) \cos(x)}{\cos^2(x)} \, dx = \int_{\pi/6}^{\pi/3} \frac{2 \sin(x)}{\cos(x)} \, dx = \int_{\pi/6}^{\pi/3} 2 \tan(x) \, dx.
\]

Then \( \int_{\pi/6}^{\pi/3} 2 \tan(x) \, dx = -2 \ln(\cos(x)) \bigg|_{x=\pi/6}^{\pi/3} = \boxed{\ln 3} \).
Problem 13

Find the left-endpoint Riemann sum for $f(x) = x^2$ on $[0, 1]$ with 5 equal subintervals.
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Answer:
Problem 13

Find the left-endpoint Riemann sum for \( f(x) = x^2 \) on \([0, 1]\) with 5 equal subintervals.

Answer: The interval width is \( \frac{1 - 0}{5} = \frac{1}{5} \) so the intervals are \([0, \frac{1}{5}], [\frac{1}{5}, \frac{2}{5}], [\frac{2}{5}, \frac{3}{5}], [\frac{3}{5}, \frac{4}{5}], [\frac{4}{5}, 1]\).
So the Riemann sum is 
\[ [f(0) + f(1/5) + f(2/5) + f(3/5) + f(4/5)] \cdot \frac{1}{5}, \text{ which evaluates to} \]
\[ [0^2 + 1/25 + 4/25 + 9/25 + 16/25] \cdot \frac{1}{5} = \frac{30}{125} = \boxed{0.24}. \]
Find the absolute minimum and maximum of $f(x) = x^2 e^x$ on the interval $[-3, 1]$ and all places where they occur.
Problem 14

Find the absolute minimum and maximum of $f(x) = x^2 e^x$ on the interval $[-3, 1]$ and all places where they occur.

Answer:
Problem 14

Find the absolute minimum and maximum of \( f(x) = x^2 e^x \) on the interval \([-3, 1]\) and all places where they occur.

Answer: We have \( f'(x) = 2xe^x + x^2e^x = x(x + 2)e^x \) so the critical numbers are where \( f' \) is zero, which occurs for \( x = -2, 0 \). Including the endpoints, our point list is \( x = -3, -2, 0, 1 \).

We have \( f(-3) = \frac{9}{e^3} \), \( f(-2) = \frac{4}{e^2} \), \( f(0) = 0 \), \( f(1) = e \). The minimum is 0 at \( x = 0 \) and the maximum is \( e \) at \( x = 1 \).
Use the Intermediate Value Theorem + Rolle’s Theorem to show $f(x) = x^3 + 3x + 1$ has exactly 1 real root.
Problem 15

Use the Intermediate Value Theorem + Rolle’s Theorem to show $f(x) = x^3 + 3x + 1$ has exactly 1 real root.

Answer:
Problem 15

Use the Intermediate Value Theorem + Rolle’s Theorem to show $f(x) = x^3 + 3x + 1$ has exactly 1 real root.

Answer: Since $f$ is continuous with $f(-1) = -1$ and $f(0) = 1$, by the Intermediate Value Theorem, $f$ has a real root in $(-1, 0)$.

Also, since $f' = 3x^2 + 3$ is never zero, $f$ cannot have 2 roots since by Rolle’s theorem, $f'$ would be zero somewhere between them.

So $f$ has exactly 1 real root.
Enjoy WeBWorK #10, and I will see you on Monday for more review!