General List of Exam Topics

- Limits (finite limits, infinite limits, limits at infinity)
- Continuity
- Limit definition of derivative, differentiability
- Computing derivatives (product, quotient, chain rule)
- Derivatives of inverse functions
- Logarithmic differentiation
- Implicit differentiation
- Parametric differentiation (velocity, speed, acceleration)
- Tangent lines and rates of change
- Linearization and linear approximation
Problem 1

Calculate the average rate of change of \( f(x) = 2x^2 + 2 \) on the interval \([1, 3]\).
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Answer:
Problem 1

Calculate the average rate of change of $f(x) = 2x^2 + 2$ on the interval $[1, 3]$.

Answer: The average rate of change is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{20 - 4}{3 - 1} = 8.$$
Problem 2

Use the limit definition of the derivative to find $s'(1)$ for $s(t) = \sqrt{3t + 1}$.
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Answer:
Problem 2

Use the limit definition of the derivative to find \( s'(1) \) for \( s(t) = \sqrt{3t} + 1 \).

Answer: By definition, this is

\[
s'(1) = \lim_{h \to 0} \frac{s(1 + h) - s(1)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{3(1 + h) + 1} - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{3h + 4} - 2}{h} \cdot \frac{\sqrt{3h + 4} + 2}{\sqrt{3h + 4} + 2}
\]

\[
= \lim_{h \to 0} \frac{3h}{h \cdot (\sqrt{3h + 4} + 2)} = \frac{3}{4}.
\]
Problem 3

Calculate \( \lim_{x \to 1} \frac{1}{(x - 1)^6} \).
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Answer:
Problem 3

Calculate \( \lim_{x \to 1} \frac{1}{(x - 1)^6} \).

Answer: As \( x \to 1^- \), the denominator approaches zero and is positive, while the numerator is positive, so the limit as \( x \to 1^- \) is \(+\infty\). As \( x \to 1^+ \), the denominator approaches zero and is positive, while the numerator is positive, so the limit as \( x \to 1^+ \) is \(+\infty\). These values are equal so the overall limit is \(+\infty\).
Problem 4

Find $f''(x)$ if $f(x) = \tan^{-1}(x)$. 
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Answer:
Problem 4

Find \( f''(x) \) if \( f(x) = \tan^{-1}(x) \).

Answer: First, \( f'(x) = \frac{1}{x^2 + 1} \). Then by the quotient rule, we see

\[
f''(x) = -\frac{2x}{(x^2 + 1)^2}.
\]
Interlude!
Problem 5

Use logarithmic differentiation to find the derivative of 
\( f(x) = \sqrt{(\sin x)^{\cos x}} \).
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Answer:
Problem 5

Use logarithmic differentiation to find the derivative of
\( f(x) = \sqrt{(\sin x)^{\cos x}}. \)

Answer: First take the natural logarithm and simplify, yielding
\[
\ln(f) = \frac{1}{2} \cos(x) \cdot \ln(\sin(x)).
\]
Now take the derivative of both sides, which gives
\[
\frac{f'}{f} = \left[ -\frac{1}{2} \sin(x) \ln(\sin(x)) + \frac{1}{2} \cos(x) \cdot \frac{\cos(x)}{\sin(x)} \right].
\]
Finally, solving for \( f' \) yields
\[
f'(x) = \sqrt{(\sin x)^{\cos x}} \left[ -\frac{1}{2} \sin(x) \ln(\sin(x)) + \frac{1}{2} \cos(x) \cdot \frac{\cos(x)}{\sin(x)} \right].
\]
Problem 6

Consider the implicit curve $x^2y + x^5y^6 = 2$, which defines $y$ implicitly as a function of $x$. Find an equation for the line tangent to the curve at the point $(x, y) = (1, 1)$.
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Answer:
Problem 6

Consider the implicit curve \( x^2y + x^5y^6 = 2 \), which defines \( y \) implicitly as a function of \( x \). Find an equation for the line tangent to the curve at the point \((x, y) = (1, 1)\).

Answer: First use implicit differentiation to find the slope of the tangent line \( \frac{dy}{dx} = y' \): we get
\[
2xy + x^2y' + 5x^4y^6 + x^5 \cdot 6y^5y' = 0,
\]
so solving yields
\[
y' = -\frac{2xy + 5x^4y^6}{x^2 + 6x^5y^5}.
\]
Then the slope is \( \frac{dy}{dx} \) at \((x, y) = (1, 1)\), which is \(-1\). Thus the equation is \( y - 1 = -(x - 1) \), or equivalently, \( y = -x + 2 \).
Problem 7

Calculate \[ \frac{d}{dt} \left[ \sqrt{\ln(\sin(t))} \right]. \]
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Answer:
Problem 7

Calculate \( \frac{d}{dt} \left[ \sqrt{\ln(\sin(t))} \right] \).

Answer: By the chain rule (repeatedly), we obtain the derivative
\[
\frac{1}{2} \left[ \ln(\sin(t)) \right]^{-1/2} \cdot \cos(t) \cdot \frac{1}{\sin(t)}.
\]
Problem 8

Find the linearization of \( f(x) = 7x^4 + 2x + 1 \) at \( x = 1 \).
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Answer:
Find the linearization of $f(x) = 7x^4 + 2x + 1$ at $x = 1$.

Answer: By definition, the linearization of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a) \cdot (x - a)$. Since $f(1) = 10$ and $f'(1) = 30$, the linearization is $L(x) = 10 + 30(x - 1)$. 

Interlude!
Problem 9

Find $f'(2)$ if $f(x) = x^3 2^x$. 

Answer: The first derivative is $f'(x) = 3x^2 \cdot 2x + x^3 \cdot 2x \ln(2)$ by the product rule. Then $f'(2) = 48 + 32 \ln(2)$. 
Problem 9

Find $f'(2)$ if $f(x) = x^32^x$.

Answer:
Problem 9

Find $f'(2)$ if $f(x) = x^32^x$.

Answer: The first derivative is $f'(x) = 3x^2 \cdot 2^x + x^3 \cdot 2^x \ln(2)$ by the product rule. Then $f'(2) = [48 + 32 \ln(2)]$. 
Problem 10

The function $f(x) = 4x + \sin(3x)$ is one-to-one, so it has an inverse function $g(x)$. Find $g'(4\pi)$. 

Answer:

This is a “differentiating an inverse function” problem. The formula to use here is $\frac{d}{dx} \left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$. Since $f(\pi) = 4\pi$ we see $g(4\pi) = f^{-1}(4\pi) = \pi$. Note $f'(x) = 4 + 3 \cos(3x)$. Then by the formula this gives $g'(4\pi) = \frac{1}{f'(\pi)} = \frac{1}{4 + 3 \cos(3\pi)} = 1$. 

The function \( f(x) = 4x + \sin(3x) \) is one-to-one, so it has an inverse function \( g(x) \). Find \( g'(4\pi) \).

Answer:
The function \( f(x) = 4x + \sin(3x) \) is one-to-one, so it has an inverse function \( g(x) \). Find \( g'(4\pi) \).

Answer: This is a “differentiating an inverse function” problem. The formula to use here is \( \frac{d}{dx} \left[ f^{-1}(x) \right] = \frac{1}{f'(f^{-1}(x))} \). Since \( f(\pi) = 4\pi \) we see \( g(4\pi) = f^{-1}(4\pi) = \pi \). Note \( f'(x) = 4 + 3 \cos(3x) \). Then by the formula this gives

\[
g'(4\pi) = \frac{1}{f'(f^{-1}(4\pi))} = \frac{1}{f'(\pi)} = \frac{1}{4 + 3 \cos(3\pi)} = 1.
\]
Problem 11a

A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find the particle’s velocity, speed, and acceleration at time $t$. 

Answer:

The velocity is $(x', y') = (6e^{2t}, 6e^{6t})$. The speed is $\sqrt{(6e^{2t})^2 + (6e^{6t})^2}$. And the acceleration is $(x'', y'') = (12e^{2t}, 36e^{6t})$. 
Problem 11a

A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find the particle’s velocity, speed, and acceleration at time $t$.

Answer:
Problem 11a

A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find the particle’s velocity, speed, and acceleration at time $t$.

Answer: The velocity is $(x', y') = (6e^{2t}, 6e^{6t})$. The speed is 
$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{(6e^{2t})^2 + (6e^{6t})^2}.$$ And the acceleration is $(x'', y'') = (12e^{2t}, 36e^{6t})$. 
A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find an equation for the tangent line to the particle’s path at time $t = 0$. 

Answer:

First, at time $t = 0$, the particle’s position is $(x(0), y(0)) = (3, 1)$. We also need the slope of the tangent line, which at time $t$ is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6e^{6t}}{6e^{2t}} = e^{4t}$. So at time $t = 0$ the slope is 1, and therefore the equation of the tangent line is $y - 1 = 1(x - 3)$. 

Problem 11b

A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find an equation for the tangent line to the particle's path at time $t = 0$.

Answer:
A particle moves through the plane so that at time $t$ seconds, its position is $(x, y) = (3e^{2t}, e^{6t})$ meters. Find an equation for the tangent line to the particle's path at time $t = 0$.

Answer: First, at time $t = 0$, the particle's position is $(x(0), y(0)) = (3, 1)$. We also need the slope of the tangent line, which at time $t$ is \[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6e^{6t}}{6e^{2t}} = e^{4t}.
\] So at time $t = 0$ the slope is 1, and therefore the equation of the tangent line is $y - 1 = 1(x - 3)$. 
Interlude!
Problem 12

Suppose that $f(1) = 5$, $f'(1) = 6$, $f(5) = 5$, $f'(5) = 2$, $g(1) = 5$, and $g'(1) = 8$. Find the derivative of $f(x)/g(x)$ at $x = 1$. 

By the quotient rule, this is

$$f'(1)g(1) - f(1)g'(1) \left[ g(1) \right]^2 = -\frac{2}{5}.$$
Problem 12

Suppose that \( f(1) = 5, \ f'(1) = 6, \ f(5) = 5, \ f'(5) = 2, \ g(1) = 5, \) and \( g'(1) = 8. \) Find the derivative of \( f(x)/g(x) \) at \( x = 1. \)

Answer:
Problem 12

Suppose that $f(1) = 5$, $f'(1) = 6$, $f(5) = 5$, $f'(5) = 2$, $g(1) = 5$, and $g'(1) = 8$. Find the derivative of $f(x)/g(x)$ at $x = 1$.

Answer: By the quotient rule, this is
\[
\frac{f'(1)g(1) - f(1)g'(1)}{[g(1)]^2} = \frac{2}{5}.
\]
Problem 13

Using an appropriate linear approximation, estimate the value of $\sqrt[5]{32.08}$. 
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Answer:
Problem 13

Using an appropriate linear approximation, estimate the value of $\sqrt[5]{32.08}$.

Answer: We use the linearization of $f(x) = \sqrt[5]{x} = x^{1/5}$ at $a = 32$, since we can easily evaluate $f(32)$. The linearization is

$L(x) = f(a) + f'(a) \cdot (x - a)$, so since $f'(x) = \frac{1}{5}x^{-4/5}$ we compute

$f(32) = 2$ and $f'(32) = \frac{1}{80}$. Then the desired estimate is

$L(32.08) = 2 + \frac{1}{80}(32.08 - 32) = 2.001$. 
Interlude!
End of Review

Good luck with your studying, and I’ll see you tomorrow for the exam!