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A CONJECTURE OF EVANS ON SUMS OF KLOOSTERMAN SUMS

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ABSTRACT. In a recent paper, Evans relates twisted Kloosterman sheaf sums to Gaussian hypergeometric functions, and he formulates a number of conjectures relating certain twisted Kloosterman sheaf sums to the coefficients of modular forms. Here we prove one of his conjectures for a fourth order twisted Kloosterman sheaf sum T_n of the quadratic character on $\mathbf{F}_p^\times$. In the course of the proof we develop reductions for twisted moments of Kloosterman sums and apply these in the end to derive a congruence relation for T_n with generalized Apéry numbers.

1. Introduction and statement of results

Fix an odd prime p, and let $\mathbf{F}_p$ be the finite field with p elements. For any multiplicative character χ on $\mathbf{F}_p^\times$, we extend χ to a function on $\mathbf{F}_p$ by setting $\chi(0) = 0$. Let ε and ϕ be the trivial and quadratic characters on $\mathbf{F}_p^\times$. Also let $\psi : \mathbf{F}_p \to \mathbf{C}$ be the additive character $x \mapsto e^{2\pi i x/p}$.

We shall write $\overline{x}$ for the multiplicative inverse of $x \in \mathbf{F}_p^\times$. For $a \in \mathbf{F}_p^\times$, the classical *Kloosterman sum* is

$$K(a) := \sum_{x \in \mathbf{F}_p^\times} \psi(x + a\overline{x}). \tag{1.1}$$

The *nth twisted Kloosterman sheaf sum* of ϕ is

(1.2)
$$T_n := \sum_{a \in \mathbf{F}_p^\times} \phi(a)(g(a)^n + g(a)^{n-1}h(a) + g(a)^{n-2}h(a)^2 + \cdots + g(a)h(a)^{n-1} + h(a)^n),$$

where $g(a)$ and $h(a)$ are the roots of the polynomial

$$X^2 + K(a)X + p. \tag{1.3}$$

We study a conjecture of Evans (see below) that relates T_4 to the coefficient $a(p)$ of the unique normalized newform in $S_4(\Gamma_0(8))$, which is given by

$$\sum_{n=1}^{\infty} a(n)q^n := q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - \cdots. \tag{1.4}$$

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To do so, we employ the Gaussian hypergeometric series. For multiplicative characters A and B on $\mathbf{F}_p$, define the normalized Jacobi sum

$$\binom{A}{B} = \frac{B(-1)}{p} J(A, \overline{B}) = \frac{B(-1)}{p} \sum_{x \in \mathbf{F}_p} A(x) \overline{B}(1-x).$$

For multiplicative characters $A_0, \ldots, A_n$ and $B_1, \ldots, B_n$ on $\mathbf{F}_p$, Greene [7] defined the *Gaussian hypergeometric series over* $\mathbf{F}_p$ by

$${}_{n+1}F_n \left(\begin{matrix} A_0, & A_1, & \ldots A_n \\ & B_1, & \ldots B_n \end{matrix} \mid x \right) = \frac{p}{p-1} \sum_{\chi} \binom{A_0 \chi}{\chi} \binom{A_1 \chi}{B_1 \chi} \cdots \binom{A_n \chi}{B_n \chi} \chi(x),$$

where the sum is over all multiplicative characters χ on $\mathbf{F}_p$ (see also [11]). We will be interested in the case where the top parameters are the quadratic character ϕ and the bottom ones the trivial character ε. For brevity we write this function as

$$(1.5) \qquad {}_{n+1}F_n(x) = {}_{n+1}F_n \left(\begin{matrix} \phi, & \phi, & \ldots \phi \\ & \varepsilon, & \ldots \varepsilon \end{matrix} \mid x \right) = \frac{p}{p-1} \sum_{\chi} \binom{\phi \chi}{\chi}^{n+1} \chi(x).$$

In [2] and [3], Ahlgren and Ono prove $p^3 {}_4F_3(1) = -a(p) - p$ by showing that the coefficients $a(p)$ and the hypergeometric values $p^3 {}_4F_3(1)$ count points on the "Calabi-Yau" variety

$$\{(x, y, z, w) \in (\mathbf{F}_p^\times)^4 \mid x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0\}.$$

With this result, Evans' Conjecture can be stated as follows.

Conjecture (Evans, [5])**.** *If p is an odd prime, then*

$$\frac{T_4}{p} = p^3 {}_4F_3(1) + p.$$

We prove the following theorem:

Theorem 1.1. *Evans' Conjecture is true.*

Ahlgren and Ono's work in [3] was motivated by Beukers' Conjecture relating the coefficient $a(p)$ to the Apéry number[1] $A\left(\frac{p-1}{2}\right)$, where

$$(1.6) \qquad A(n) := \sum_{j=0}^{n} \binom{n+j}{j}^2 \binom{n}{j}^2.$$

Beukers' conjecture says that

$$(1.7) \qquad A\left(\frac{p-1}{2}\right) \equiv a(p) \bmod p^2.$$

Ahlgren and Ono proved this conjecture using combinatorial properties of ${}_4F_3(1)$ and the relation $p^3 {}_4F_3(1) = -a(p) - p$ [3]. Motivated by these results, we derive a hypergeometric expression for the sum T_n and apply work of Ahlgren and Ono on hypergeometric congruences to prove the following.

[1]Apéry used the numbers $A(n)$ in his famous proof that $\zeta(3)$ is irrational [4].

Theorem 1.2. *For $\lambda \in \mathbf{F}_p$, define the generalized Apéry number by*

$$A(p,\lambda) := \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{j}^2 \binom{\frac{p-1}{2}+j}{j} \lambda^{pj} \tag{1.8}$$

for $\lambda \in \mathbf{F}_p^\times$ and $A(p,0)=1$. Then, if n is a positive integer,

$$T_{n+3} \equiv (-1)^{n+1} p \sum_{x_1,\ldots,x_n \in \mathbf{F}_p^\times} A\left(p, \frac{(x_1+\cdots+x_n+1)(\overline{x_1}+\cdots+\overline{x_n}+1)}{4}\right) \bmod p^2. \tag{1.9}$$

We briefly outline the structure of the paper. In the first section we provide background information on Gaussian hypergeometric functions. Next we prove reductions for T_n, including a purely hypergeometric expression as a sum of ${}_3F_2$'s. We then apply a finite field analogue of Clausen's theorem to the specific $n=4$ case. To complete the proof of Theorem 1.1, we express both T_4 and $p^3{}_4F_3(1)$ as a "trace" of squares of ${}_2F_1$'s. In the last section we prove Theorem 1.2.

2. Preliminaries on Gaussian hypergeometric functions

Here we state some basic results on Gaussian hypergeometric functions. Any Gaussian hypergeometric function can be expressed as a "Diophantine" sum over $\mathbf{F}_p$ [7]. We recall that for $x \neq 0$,

$${}_2F_1(x) = \frac{\phi(-1)}{p} \sum_{y \in \mathbf{F}_p^\times} \phi(y)\phi(1-y)\phi(1-xy), \tag{2.1}$$

$${}_3F_2(x) = \frac{1}{p^2} \sum_{y,z \in \mathbf{F}_p^\times} \phi(y)\phi(1-y)\phi(z)\phi(1-z)\phi(1-xyz). \tag{2.2}$$

Note. We note that ${}_3F_2(0) = {}_2F_1(0) = 0$ according to Greene's conventions given in the introduction. However, it is also natural to define ${}_3F_2(0)$ and ${}_2F_1(0)$ by the right hand sides of the equalities above. We will take this convention for the rest of the paper because it simplifies the statements of Theorems 3.1 and 1.2.

Like their classical counterparts, Gaussian hypergeometric functions satisfy many transformation laws. In this paper we need the following two transformations, which are applications of Theorems 4.2 and 4.4 of [7].

Lemma 2.1. *For $x \neq 0,1$, we have*

$${}_2F_1(x) = \phi(-1){}_2F_1(1-x), \tag{2.3}$$

$${}_2F_1(x) = \phi(x){}_2F_1(1/x). \tag{2.4}$$

In what follows, we will need to express a ${}_3F_2$ as a square of a ${}_2F_1$. To do so, we use a special case of a formula provided by Evans and Greene in Theorem 1.7 of [6].

Lemma 2.2. *If $t \neq 0, \pm 1$, then*

$${}_3F_2\left(\frac{1}{1-t^2}\right) = \phi(t^2-1)\left(-\frac{1}{p} + {}_2F_1\left(\frac{1-t}{2}\right)^2\right).$$

The following expression for $p^3{}_4F_3(1)$ as a trace of squares of ${}_2F_1$'s is proven in Lemma 2.2 of [3].

Lemma 2.3. *We have that*

$$p^3{}_4F_3(1) = p^2 \sum_{x \in \mathbf{F}_p^\times} \phi(x){}_2F_1(x)^2.$$

Finally, we will need two special evaluations of hypergeometric functions from Corollary 11.12 and Theorem 11.14 of [11].

Lemma 2.4. *We have that*

$$(2.5) \qquad {}_2F_1(-1) = \begin{cases} 0 & \text{if } p \equiv 3 \bmod 4, \\ \frac{2x(-1)^{\frac{x+y+1}{2}}}{p} & \text{if } p \equiv 1 \bmod 4,\ x^2+y^2=p, \text{ and } x \text{ odd}, \end{cases}$$

$$(2.6) \qquad {}_3F_2(1) = \begin{cases} 0 & \text{if } p \equiv 3 \bmod 4, \\ \frac{4x^2-2p}{p^2} & \text{if } p \equiv 1 \bmod 4,\ x^2+y^2=p, \text{ and } x \text{ odd}. \end{cases}$$

3. Reductions of twisted Kloosterman sums

Using the relations $g(a)+h(a) = -K(a)$ and $g(a)h(a) = p$, which follow from (1.3), the polynomial $g(a)^n + g(a)^{n-1}h(a) + \cdots + g(a)h(a)^{n-1} + h(a)^n$ can be written as a polynomial in $K(a)$ with coefficients in $\mathbf{Z}$. Therefore T_n can be written as a linear combination of the twisted Kloosterman moments

$$(3.1) \qquad S(m) = \sum_{a \in \mathbf{F}_p^\times} \phi(a)K(a)^m.$$

In this section we give general reductions for such exponential sums, relate them to hypergeometric functions, and specialize to the case T_4.

3.1. Reductions of T_n.

The next proposition reduces the exponential sum $S(m) = \sum_{a \in \mathbf{F}_p^\times} \phi(a)K(a)^m$ to a symmetric sum of quadratic characters.

Proposition 3.1. *For an integer $m \geq 1$, we have that*

$$S(m+1) = p\phi(-1) \sum_{x_1,\ldots,x_m \in \mathbf{F}_p^\times} \phi(x_1 + \cdots + x_m + 1)\phi(\overline{x_1} + \cdots + \overline{x_m} + 1).$$

Proof. By definition, we have that

$$\begin{aligned} &\sum_{a \in \mathbf{F}_p^\times} \phi(a)K(a)^{m+1} \\ &\quad = \sum_{a \in \mathbf{F}_p^\times} \phi(a) \sum_{x_1,\ldots,x_m,y \in \mathbf{F}_p^\times} \psi(x_1 + \cdots + x_m + y + a(\overline{x_1} + \cdots + \overline{x_m} + \overline{y})) \\ &\quad = \sum_{x_1,\ldots,x_m,y \in \mathbf{F}_p^\times} \psi(x_1 + \cdots + x_m + y) \sum_{a \in \mathbf{F}_p^\times} \phi(a)\psi(a(\overline{x_1} + \cdots + \overline{x_m} + \overline{y})). \end{aligned}$$

If $\overline{x_1} + \cdots + \overline{x_m} + \overline{y} = 0$, then the sum on a vanishes, so we may multiply by

$$\phi(\overline{x_1} + \cdots + \overline{x_m} + \overline{y})^2 = \begin{cases} 0 & \text{if } \overline{x_1} + \cdots + \overline{x_m} + \overline{y} = 0, \\ 1 & \text{otherwise} \end{cases}$$

inside the sum on $x_1, \dots, x_m, y$. Thus we have

$$\begin{aligned}\sum_{a \in \mathbf{F}_p^\times} \phi(a) K(a)^{m+1} &= \sum_{x_1,\dots,x_m,y \in \mathbf{F}_p^\times} \phi(\textstyle\sum_i \overline{x_i} + \overline{y}) \psi(\sum_i x_i + y) \\ &\qquad \times \sum_{a \in \mathbf{F}_p^\times} \phi(a(\textstyle\sum_i \overline{x_i} + \overline{y})) \psi(a(\sum_i \overline{x_i} + \overline{y})) \\ &= G(\phi) \sum_{x_1,\dots,x_m,y \in \mathbf{F}_p^\times} \phi(\textstyle\sum_i \overline{x_i} + \overline{y}) \psi(\sum_i x_i + y),\end{aligned}$$

where $G(\phi) = \sum_{a \in \mathbf{F}_p^\times} \phi(a)\psi(a)$ is the usual Gauss sum. Making the change of variables $(x_1, \dots, x_m, y) \mapsto (x_1 y, x_2 y, \dots, x_m y, y)$, we have

$$\sum_{a \in \mathbf{F}_p^\times} \phi(a) K(a)^{m+1} = G(\phi) \sum_{x_1,\dots,x_m \in \mathbf{F}_p^\times} \phi(\textstyle\sum_i \overline{x_i} + 1) \displaystyle\sum_{y \in \mathbf{F}_p^\times} \phi(y) \psi(y(\textstyle\sum_i x_i + 1)).$$

As before, we can multiply by $\phi(\sum_i x_i + 1)^2$ to get

$$\begin{aligned}\sum_{a \in \mathbf{F}_p^\times} \phi(a) K(a)^{m+1} &= G(\phi) \sum_{x_1,\dots,x_m \in \mathbf{F}_p^\times} \phi(\textstyle\sum_i \overline{x_i} + 1) \phi(\sum_i x_i + 1) \\ &\qquad \times \sum_{y \in \mathbf{F}_p^\times} \phi(y(\textstyle\sum_i x_i + 1)) \psi(y(\sum_i x_i + 1)) \\ &= G(\phi)^2 \sum_{x_1,\dots,x_m \in \mathbf{F}_p^\times} \phi(\textstyle\sum_i \overline{x_i} + 1) \phi(\sum_i x_i + 1).\end{aligned}$$

Since $G(\phi)^2 = p\phi(-1)$ by Proposition 6.3.2 of [8], this finishes the proof. $\square$

Remark. Note in particular that $S(2) = -p$. Also, the sum $S(1) = \sum_{a \in \mathbf{F}_p^\times} \phi(a) K(a)$ is not covered by the proposition, but it is easy to evaluate $S(1) = G(\phi)^2 = p\phi(-1)$ directly.

The form of the character sum in the proposition motivates the study of the auxiliary function

$$Q(a,b) = \sum_{x,y \in \mathbf{F}_p^\times} \phi(x + y + a) \phi(\overline{x} + \overline{y} + b) \tag{3.2}$$

for $a, b \in \mathbf{F}_p$. The function $Q(a,b)$ serves as the connection between twisted Kloosterman sheaf sums and hypergeometric functions.

Proposition 3.2. *If $a, b \in \mathbf{F}_p$ are not both zero, then $Q(a,b) = p^2 \phi(-1) {}_3F_2\left(\frac{ab}{4}\right) + p\phi(ab)$. Additionally, $Q(0,0) = -(p-1)\phi(-1)$.*

Proof. First, we may assume that $a \neq 0$ by interchanging a and b (if necessary) because the sum is clearly symmetric in a and b. Now, for $c \in \mathbf{F}_p^\times, d \in \mathbf{F}_p$, we note the well-known formula

$$\sum_{x \in \mathbf{F}_p} \phi(cx^2 + dx + e) = \begin{cases} (p-1)\phi(c) & \text{if } d^2 - 4ce = 0, \\ -\phi(c) & \text{if } d^2 - 4ce \neq 0. \end{cases}$$

Applying the change of variables $(x+y, xy) \mapsto (s, r)$ yields

$$\begin{aligned} Q(a,b) &= \sum_{s\in\mathbf{F}_p, r\in\mathbf{F}_p^\times} \phi(s+a)\phi(s/r+b)[1+\phi(s^2-4r)] \\ &= \sum_{s\in\mathbf{F}_p, r\in\mathbf{F}_p^\times} \phi(s+a)\phi(s/r+b) + \sum_{r\in\mathbf{F}_p} \phi(a)\phi(b)\phi(-4r) \\ &\quad + \sum_{r,s\in\mathbf{F}_p^\times} \phi(s+a)\phi(s/r+b)\phi(s^2-4r). \end{aligned}$$

The first sum equals $p\phi(ab)$ by the note above, and the second sum clearly vanishes. Substituting $(r,s) \mapsto (ra^2s^2/4, -as)$ in the third sum yields

$$\begin{aligned} &\sum_{r,s\in\mathbf{F}_p^\times} \phi(r)\phi(s+a)\phi(s+br)\phi(s^2-4r) \\ &\quad = \sum_{r,s\in\mathbf{F}_p^\times} \phi(r)\phi(-as+a)\phi(-as+bra^2s^2/4)\phi(1-r) \\ &\quad = \phi(-1) \sum_{r,s\in\mathbf{F}_p^\times} \phi(s)\phi(r)\phi(1-s)\phi(1-absr/4)\phi(1-r) \\ &\quad = \phi(-1)p^2{}_3F_2\left(\frac{ab}{4}\right). \end{aligned}$$

For the last statement of the proposition, we have

$$Q(0,0) = \sum_{x,y\in\mathbf{F}_p^\times} \phi(x+y)^2\phi(xy) = \sum_{x\in\mathbf{F}_p^\times} -\phi(-x^2) = -(p-1)\phi(-1).$$

□

Remark. The specific evaluation of $Q(a,b)$ in the case $a=b=1$ is the subject of two notable early papers. The Lehmers gave the first evaluation of the sum, directly rewriting $Q(1,1)$ as a Jacobsthal sum via a complicated change of variables [9]. A later paper by Mordell revisits their method and finds a simpler path that also eventually leads to Jacobsthal sums [10]. The more general method here also allows for direct evaluation of $Q(1,1) = \phi(-1)p^2{}_3F_2(1/4) + p$, as ${}_3F_2(1/4)$ has an explicit evaluation as given in Theorem 11.17 of [11].

If we now define for positive integers m the sum

$$G(m) := \sum_{x_1,\ldots,x_m\in\mathbf{F}_p^\times} {}_3F_2\left(\frac{(x_1+x_2+\cdots+x_m+1)(\overline{x_1}+\cdots+\overline{x_m}+1)}{4}\right), \tag{3.3}$$

we have the following result.

Theorem 3.1. *For $m \geq 1$, there is a recurrence*

$$S(m+3) = p^3G(m) + pS(m+1) - p^2C(m), \tag{3.4}$$

where $C(m)$ is the number of simultaneous solutions to the congruences $x_1+\cdots+x_m = -1$ and $\overline{x_1}+\cdots+\overline{x_m} = -1$.

Proof. Applying Propositions 3.1 and 3.2, we have

$$\begin{aligned} S(m+3) &= p\phi(-1) \sum_{x_1,\ldots,x_{m+2}\in\mathbf{F}_p^\times} \phi(x_1+\cdots+x_{m+2}+1)\phi(\overline{x_1}+\cdots+\overline{x_{m+2}}+1) \\ &= p\phi(-1) \sum_{x_1,\ldots,x_m\in\mathbf{F}_p^\times} Q(x_1+\cdots+x_m+1, \overline{x_1}+\cdots+\overline{x_m}+1) \\ &= p^3G(m)+p^2\phi(-1) \sum_{x_1,\ldots,x_m\in\mathbf{F}_p^\times} \phi\left((x_1+\cdots+x_m+1)(\overline{x_1}+\cdots+\overline{x_m}+1)\right) \\ &\quad + p\phi(-1)(Q(0,0)-p^2\phi(-1){}_3F_2(0))C(m) \\ &= p^3G(m)+pS(m+1)-p^2C(m), \end{aligned}$$

where the last term is a correction stemming from the fact that

$$Q(0,0)-p^2\phi(-1){}_3F_2(0) = -p\phi(-1),$$

and this correction must be made exactly $C(m)$ times. □

Remark. This result allows us to express T_n as a linear combination of ${}_3F_2$'s, $S(1)$, $S(2)$, $S(3)$ and the constants $C(m)$.

3.2. **Simplifications of T_4.** A calculation using the relations $g(a)+h(a) = -K(a)$ and $g(a)h(a) = p$ shows that

$$g(a)^4+g(a)^3h(a)+g(a)^2h(a)^2+g(a)h(a)^3+h(a)^4 = K(a)^4-3pK(a)^2+p^2.$$

Therefore we have that

$$T_4 = \sum_{a\in\mathbf{F}_p^\times} \phi(a)(K(a)^4-3pK(a)^2+p^2) = \sum_{a\in\mathbf{F}_p^\times} \phi(a)K(a)^4 - 3p\sum_{a\in\mathbf{F}_p^\times} \phi(a)K(a)^2. \tag{3.5}$$

We can now apply the results from the previous section to reduce T_4. To ease notation, we define a function $F : \mathbf{F}_p^\times \to \mathbf{C}$ by

$$F(a) := Q(a+1, \overline{a}+1). \tag{3.6}$$

Lemma 3.1. *We have*

$$T_4 = p\phi(-1)\sum_{x\in\mathbf{F}_p^\times} F(x)+3p^2.$$

Proof. Write $T_4 = S(4)-3pS(2)$ as in (3.5) and apply Proposition 3.1. □

Lemma 3.2. *For $a \neq 0, \pm 1$, we have that*

$$F(a) = p^2\phi(a){}_2F_1(-a)^2.$$

Proof. By Proposition 3.2, for $a \neq 0, -1$, we have that

$$F(a) = p\phi(a)+p^2\phi(-1){}_3F_2\left(\frac{(a+1)^2}{4a}\right). \tag{3.7}$$

Taking $t = \frac{a-1}{a+1}$ in Lemma 2.2, we get

$$p^2\phi(-1){}_3F_2\left(\frac{(a+1)^2}{4a}\right) = \phi(a)\left(-p+p^2{}_2F_1\left(\frac{1}{1+a}\right)^2\right),$$

and so $F(a) = p^2\phi(a){}_2F_1(1/(1+a))^2$. Applying the two transformations in Lemma 2.1 completes the proof. □

We will need two special evaluations of F at ± 1, the points excluded from the previous lemma.

Lemma 3.3. *We have that*

$$\phi(-1)F(1) = p^2\phi(-1){}_2F_1(-1)^2 - p, \tag{3.8}$$

$$\phi(-1)F(-1) = p^2{}_2F_1(1)^2 - p. \tag{3.9}$$

Proof. By Proposition 3.2 we have $F(1) = p + p^2\phi(-1){}_3F_2(1)$, and so the first identity follows directly from Lemma 2.4. Note that

$$\phi(-1)F(-1) = \phi(-1)\sum_{x,y\in\mathbf{F}_p^\times}\phi(x+y)\phi(\overline{x}+\overline{y}) = \phi(-1)\sum_{x,y\in\mathbf{F}_p^\times}\phi(xy)\phi(x+y)^2$$
$$= -\phi(-1)\sum_{x+y=0}\phi(xy) = -(p-1),$$

and the Diophantine representation in (2.1) for ${}_2F_1$ implies that $p^2{}_2F_1(1)^2 = 1$. This proves (3.9). □

4. Proof of Theorem 1.1

By successively applying Lemmas 3.1, 3.2, and 3.3, we have

$$\begin{aligned}\frac{T_4}{p} &= 3p + \phi(-1)\sum_{x\in\mathbf{F}_p^\times}F(x)\\ &= 3p + \phi(-1)F(1) + \phi(-1)F(-1) + p^2\sum_{x\in\mathbf{F}_p^\times\setminus\{\pm 1\}}\phi(-x){}_2F_1(-x)^2\\ &= 3p + (\phi(-1)F(1) - p^2\phi(-1){}_2F_1(-1)^2) + (\phi(-1)F(-1) - p^2{}_2F_1(1)^2)\\ &\quad + p^2\sum_{x\in\mathbf{F}_p^\times}\phi(x){}_2F_1(x)^2\\ &= p + p^2\sum_{x\in\mathbf{F}_p^\times}\phi(x){}_2F_1(x)^2.\end{aligned}$$

Applying Lemma 2.3 yields

$$\frac{T_4}{p} = p^3{}_4F_3(1) + p.$$

5. Apéry-type congruences

For the proof of Theorem 1.2, we recall a result of Ahlgren [1].

Theorem 5.1 (Ahlgren)**.** *If p is an odd prime and $1 \le \lambda \le p-1$, then define*

$$A(p,\lambda) := \sum_{j=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{j}^2\binom{\frac{p-1}{2}+j}{j}\lambda^{pj},$$

$$B(p,\lambda) := \sum_{j=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{j}^2\binom{\frac{p-1}{2}+j}{j}\lambda^{j}\cdot\left\{1 + \frac{1}{2}\sum_{i=\frac{p+1}{2}}^{\frac{p-1}{2}+j}\frac{1}{i} + 3j\sum_{i=1+j}^{\frac{p-1}{2}+j}\frac{1}{i}\right\}.$$

Then we have

$$p^2{}_3F_2(\lambda) \equiv A(p,\lambda) + pB(p,\lambda) \bmod p^2. \tag{5.1}$$

Remark. If we define $A(p,0) = 1$ and $B(p,0) = 0$, then the results self-evidently hold for $\lambda = 0$ as well. We will use this definition in our proof below.

Proof of Theorem 1.2. By Proposition 3.1, $pS(m) \equiv 0 \bmod p^2$. Therefore, Theorems 3.1 and 5.1 together show that

$$\begin{aligned} S(m+3) &\equiv \sum_{x_1,\ldots,x_m \in \mathbf{F}_p^\times} p^3 {}_3F_2\left(\frac{(x_1+x_2+\cdots+x_m+1)(\overline{x_1}+\cdots+\overline{x_m}+1)}{4}\right) \\ &\equiv p \sum_{x_1,\ldots,x_m \in \mathbf{F}_p^\times} A\left(p, \frac{(x_1+\cdots+x_m+1)(\overline{x_1}+\cdots+\overline{x_m}+1)}{4}\right) \bmod p^2. \end{aligned}$$

Proposition 3.1 shows that the moments $S(k)$ vanish modulo p, hence writing T_n as a linear combination of moments gives $T_n \equiv (-1)^n S(n) \bmod p^2$. □

Remark. The number of simultaneous solutions to the congruences $x_1 + x_2 + \cdots + x_p = a$ and $\overline{x_1} + \overline{x_2} + \cdots + \overline{x_p} = b$ is equal to 0 mod p for all (a,b) except for $a = b = 0$, where the number of solutions is -1 mod p. Applying this fact yields that $T_{m+p} \equiv -T_m \bmod p^2$.

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