Kakeya Sets Over non-Archimedean Local Rings

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Outline:

1. Classical Kakeya problem
2. Finite-field Kakeya problem
3. Kakeya problem over local rings
4. Open questions

This is joint work with M. Hablicek (UW-Madison).
Classical Kakeya Problem

Definition

A **Kakeya set** (or Besicovitch set) is a set of points in Euclidean space which contains a unit line segment in every direction.
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For $n \geq 2$, there exists a Kakeya set of measure 0 in $\mathbb{R}^n$. 
The Kakeya conjecture

**Definition**

The Minkowski dimension of a set $K$ is defined to be

$$\dim(K) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

where $N(\epsilon)$ is the number of squares of size $\epsilon$ needed to cover $K$. 
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**Conjecture (Kakeya conjecture)**

Any Kakeya set in $\mathbb{R}^n$ is of Minkowski dimension $n$.

The conjecture is known to hold when $n = 2$ but only lower bounds are known for $n > 2$. 
Kakeya sets over finite fields

Definitions:

- Let $\mathbb{F}_q$ be a finite field, and $n$ a fixed positive integer.
- Space of interest: $S = \mathbb{F}_q^n$.
- Lines in $S$ are of the form $\{x + sy : s \in \mathbb{F}_q, x, y \in S\}$.
- A direction in $S$ is an equivalence class of $y$ giving the same line.
- A **Kakeya set** is a set containing a line in every direction.

Theorem (Dvir; 2008) Any Kakeya set in $\mathbb{F}_q^n$ contains at least $q^n$ points. In other words, a Kakeya set over a finite field always has positive measure. (This proves the Kakeya conjecture over $\mathbb{F}_q$.)
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**Theorem (Dvir; 2008)**

Any Kakeya set in $\mathbb{F}_q^n$ contains at least $\frac{q^n}{n!}$ points.

In other words, a Kakeya set over a finite field always has positive measure. (This proves the Kakeya conjecture over $\mathbb{F}_q$.)
Between $\mathbb{R}$ and $\mathbb{F}_q$

Over $\mathbb{R}$ there exist Kakeya sets of measure zero, but over $\mathbb{F}_q$, there exists a hard lower bound on measure (independent of $q$). What's in between?

**Question (J. Ellenberg, R. Oberlin, T. Tao; 2009)**

*Are there Besicovitch phenomena in $\mathbb{F}_q[[t]]^n$ or in $\mathbb{Z}_p^n$?*

In other words, do there exist Kakeya sets of measure 0 in these spaces?
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**Theorem (E.D., Hablicek; 2011)**

*There exists a Kakeya set of measure 0 in $\mathbb{F}_q[[t]]^n$ for each $n \geq 2$.*

Proof: Explicit construction.
Preliminaries

Definitions:

- Space of interest: \( S = \mathbb{F}_q[[t]]^n \).
- Lines in \( S \) are of the form \( \{x + sy : s \in \mathbb{F}_q[[t]], x, y \in S\} \).
- A direction in \( S \) is an equivalence class of \( y \), for which at least one coordinate is a unit.

A **Kakeya set** is a set containing a line in every direction.

The Haar measure \( \mu \) on \( S \) is generated by the projections \( \pi_k : \mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]/(t^k) \), where the measure on \( (\mathbb{F}_q[[t]]/(t^k))^n \) is the probability measure.
Construction, I

Observations:

- Only need to construct a Kakeya set $K$ of measure zero in $\mathbb{F}_q[[t]]^2$, then just take $K \times \mathbb{F}_q[[t]]^{n-2}$ in general.
- In $\mathbb{F}_q[[t]]^2$, only need to construct a set $H$ containing all lines with direction vectors $(1, b)$; then $K = \{(x, y) : (x, y) \text{ or } (y, x) \in H\}$ is Kakeya.
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Construction, 1

Observations:

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- In $\mathbb{F}_q[[t]]^2$, only need to construct a set $H$ containing all lines with direction vectors $(1, b)$; then $K = \{(x, y) : (x, y) \text{ or } (y, x) \in H\}$ is Kakeya.

- For any odd map $\ast : \mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]$, the set $H_\ast = \{(x, y) : ax + y = a^\ast \text{ for some } a \in \mathbb{F}_q[[t]]\}$ will contain a line with direction vectors $(1, b)$.

- Reason: For any $b \in \mathbb{F}_q[[t]]$, the points $(x, y) = (0, -b^\ast) + s(1, b)$ are in $H_\ast$. 
Construction, II

We need to find a choice of $a^*$ that makes $H_*$ have measure zero.
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We need to find a choice of $a^*$ that makes $H_*$ have measure zero. Solution:

- For any $a \in \mathbb{F}_q[[t]]$, let $a_i$ denote the coefficient of $t^i$.
- For any $a \in \mathbb{F}_q[[t]]$, define $a^* \in \mathbb{F}_q[[t]]$ by setting
  \[
a_i^* = \begin{cases} 
0 & \text{if } i = 2^k - 2 \text{ for some natural number } k, \\
 a_{i+1} & \text{otherwise}.
\end{cases}
\]

Proposition

With notation as previous, the set

$$H := \{(x, y) \in \mathbb{F}_q[[t]]^2 : ax + y = a^* \text{ for some } a \in \mathbb{F}_q[[t]]\}$$

contains a line with direction vector $(1, b)$ for each $b \in \mathbb{F}_q[[t]]$, and has measure 0.
The point \((x, y)\) is in \(H\) if and only if there exists an \(a = a_0 + a_1 t + a_2 t^2 + \cdots\) satisfying the system

\[
\begin{align*}
    a_0 x_0 + y_0 &= 0, \quad [0] \\
    a_1 x_0 + a_0 x_1 + y_1 &= a_2, \quad [1] \\
    a_2 x_0 + a_1 x_1 + a_0 x_2 + y_2 &= 0, \quad [2] \\
    &\vdots \\
    a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n + y_n &= a^*_n, \quad [n] \\
    &\vdots
\end{align*}
\]
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    \vdots \quad & \vdots \\
    a_n x_0 + a_{n-1} x_1 + \cdots + a_0 x_n + y_n &= a_n^*, \quad & [n] \\
    \vdots \quad & \vdots
\end{align*}
\]

Now count solutions to equations \([0]-[n]\) as \(n\) grows, and show that the measure of the set of points \((x, y)\) such that there exists an \(a\) satisfying \([0]-[n]\) tends to 0 as \(n\) grows.
We can also pose the Kakeya conjecture over the local ring setting.
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**Definition**

The **Minkowski dimension** of a subset $E$ of $\mathbb{F}_q[[t]]^n$ is

$$\lim_{k \to \infty} \frac{\log |E_k|}{\log q^k}$$

where $E_k$ is the image of $E$ under $\mathbb{F}_q[[t]] \to \mathbb{F}_q[[t]]/(t^k)$.

(We have an analogous definition for subsets of $\mathbb{Z}_p^n$.)

**Conjecture (Kakeya Conjecture)**

For $n \geq 2$, the Minkowski dimension of a Kakeya set in $R^n$ where $R = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$ is $n$. 
The Minkowski dimension of a Kakeya set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$ is 2.

In dimensions $n \geq 3$ over these rings, the Kakeya conjecture remains open.
Theorem (E.D., M. Hablicek; 2011)

The Minkowski dimension of a Kakeya set in $\mathbb{F}_q[[t]]^2$ or $\mathbb{Z}_p^2$ is 2.

In dimensions $n \geq 3$ over these rings, the Kakeya conjecture remains open. Key result in proof:

Proposition

Let $E$ be a Kakeya set in $R^2$ where $R = \mathbb{F}_q[t]/t^k$ or $\mathbb{Z}/p^k\mathbb{Z}$. Then

$$|E| \geq \frac{|R|^2}{2k}.$$ 

This proposition is a sharpening of a counting result of Ellenberg-Oberlin-Tao.
Open Questions

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Does a Kakeya set of measure 0 exist in \( \mathbb{Z}_p^2 \)? (Or other rings?)
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Kakeya sets over \( \mathbb{R} \) are the central ingredient in a theorem on \( L^p \) spaces in classical analysis:

Theorem (C. Fefferman; 1971)

The truncated Fourier operator \( T \) on \( L^p(\mathbb{R}^n) \), defined by \( \hat{T}f(x) = \chi_p(x)\hat{f}(x) \), where \( \chi_p \) is the characteristic function of the unit ball, is bounded only on \( L^2 \).

If a Kakeya set of measure zero exists over \( \mathbb{Z}_p \), it may be possible to obtain a similar theorem over \( \mathbb{Z}_p \).
End of Talk

Thank you!