# Contents

# 0   Complex Numbers

In this supplementary chapter, we will outline basic properties along with some applications of complex numbers.

## 0.1   Arithmetic with Complex Numbers

- Complex numbers may seem daunting, arbitrary, and strange when first introduced, but they are (in fact) very useful in mathematics and elsewhere. Plus, they're just neat.

- <u>Some History</u>: Complex numbers were first encountered by mathematicians in the 1500s who were trying to write down general formulas for solving cubic equations (i.e., equations like $x^3 + x + 1 = 0$), in analogy with the well-known formula for the solutions of a quadratic equation. It turned out that their formulas required manipulation of complex numbers, even when the cubics they were solving had three real roots.

  - It took over 100 years before complex numbers were accepted as something mathematically legitimate: even negative numbers were sometimes suspect, so (as the reader may imagine) their square roots were even more questionable.

  - The stigma is still evident even today in the terminology ("imaginary numbers"), and the fact that complex numbers are often glossed over or ignored in mathematics courses.

  - Nonetheless, they are very real objects (no pun intended), and have a wide range of uses in mathematics, physics, and engineering.

  - Among neat applications of complex numbers are deriving trigonometric identities with much less work (see later) and evaluating certain kinds of definite and indefinite integrals. For example, using the theory of functions of a complex variable, one can derive many rather unusual results, such as $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\, dx = \frac{\pi}{e}$.

- <u>Definitions</u>: A complex number is a number of the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is the "imaginary unit", defined so that $i^2 = -1$.

  - <u>Notation</u>: Sometimes, $i$ is written as $\sqrt{-1}$. In certain areas (especially electrical engineering), the letter $j$ can be used to denote $\sqrt{-1}$, rather than $i$ (which is used to denote electrical current).

  - The <u>real part</u> of $z = a + bi$, denoted $\mathrm{Re}(z)$, is the real number $a$.

  - The <u>imaginary part</u> of $z = a + bi$, denoted $\mathrm{Im}(z)$, is the real number $b$.

  - The <u>complex conjugate</u> of $z = a + bi$, denoted $\overline{z}$, is the complex number $a - bi$.

    * The notation for conjugate varies among disciplines. The notation $z^*$ is often used in physics and computer programming to denote the complex conjugate, in place of $\overline{z}$.

  - The <u>modulus</u> (also called the <u>absolute value</u>, <u>magnitude</u>, or <u>length</u>) of $z = a + bi$, denoted $|z|$, is the real number $\sqrt{a^2 + b^2}$.

  - <u>Example</u>: $\mathrm{Re}(4 - 3i) = 4$, $\mathrm{Im}(4 - 3i) = -3$, $\overline{4 - 3i} = 4 + 3i$, $|4 - 3i| = 5$.

- Two complex numbers are added (or subtracted) simply by adding (or subtracting) their real and imaginary parts: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

○ <u>Example</u>: The sum of $1 + 2i$ and $3 - 4i$ is $\boxed{4 - 2i}$. The difference is $(1 + 2i) - (3 - 4i) = \boxed{-2 + 6i}$.

- Two complex numbers are multiplied using the distributive law and the fact that $i^2 = -1$: $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$.

  ○ <u>Example</u>: The product of $1 + 2i$ and $3 - 4i$ is $(1 + 2i)(3 - 4i) = 3 + 6i - 4i - 8i^2 = \boxed{11 + 2i}$.

- For division, we rationalize the denominator: $\dfrac{a + bi}{c + di} = \dfrac{(a + bi)(c - di)}{(c + di)(c - di)} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2}\, i$.

  ○ <u>Example</u>: The quotient of $2i$ by $1 - i$ is $\dfrac{2i}{1 - i} = \dfrac{2i(1 + i)}{(1 - i)(1 + i)} = \dfrac{-2 + 2i}{2} = \boxed{-1 + i}$.

- A key property of the conjugate is that it is multiplicative: if $z = a + bi$ and $w = c + di$, then $\overline{zw} = \overline{z} \cdot \overline{w}$. (This is easy to see just by multiplying out the relevant quantities.) From this we see that the modulus is also multiplicative: $|zw| = |z| \cdot |w|$.

  ○ <u>Example</u>: If $z = 1 + 2i$ and $w = 3 - i$, then $\overline{z} = 1 - 2i$ and $\overline{w} = 3 + i$. We compute $zw = 5 + 5i$ and $\overline{z} \cdot \overline{w} = 5 - 5i$, so indeed $\overline{zw} = \overline{z} \cdot \overline{w}$. Furthermore, we have $|z| = \sqrt{5}$, $|w| = \sqrt{10}$, and $|zw| = \sqrt{50} = |z| \cdot |w|$.

  ○ This is the underlying reason for why division works in general: we write $\dfrac{z}{w} = \dfrac{z \cdot \overline{w}}{w \cdot \overline{w}} = \dfrac{z \cdot \overline{w}}{|w|^2}$, where the denominator is now the real number $|w|^2 = c^2 + d^2$.

- Using complex numbers, we can give meaning to the solutions of a quadratic equation even when they are not real numbers.

  ○ Explicitly, by completing the square, we see that the polynomial $az^2 + bz + c = 0$ has the two solutions $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ over the complex numbers. (Technically, this is not quite true when $b^2 - 4ac = 0$: in this case, the convention is to say that this polynomial still has two roots, but they are equal.)

  ○ Notice that the expression for the roots now makes sense even if $b^2 - 4ac < 0$: the roots are simply non-real complex numbers.

  ○ We can then factor the polynomial as $az^2 + bz + c = a(z - r_1)(z - r_2)$ where $r_1 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ are the two roots.

  ○ More generally, the Fundamental Theorem of Algebra says that any polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ can be completely factored into a product of linear terms over the complex numbers. (This a foundational result in algebra and it was first correctly proven by Argand and Gauss in the early 1800s.)

## 0.2 Complex Exponentials, Polar Form, and Euler's Theorem

- We often think of the real numbers geometrically, as a line. The natural way to think of the complex numbers is as a plane, with the $x$-coordinate denoting the real part and the $y$-coordinate denoting the imaginary part.

  ○ Once we do this, there is a natural connection to polar coordinates: namely, if $z = x + yi$ is a complex number which we identify with the point $(x, y)$ in the complex plane, then the modulus $|z| = \sqrt{x^2 + y^2}$ is simply the coordinate $r$ when we convert $(x, y)$ from Cartesian to polar coordinates.

  ○ Furthermore, if we are given that $|z| = r$, we can uniquely identify $z$ given the angle $\theta$ that the line connecting $z$ to the origin makes with the positive real axis. (This is the same $\theta$ from polar coordinates.)

- From our computations with polar coordinates (or simple trigonometry), we see that we can write $z$ in the form $\boxed{z = r \cdot [\cos(\theta) + i \cdot \sin(\theta)]}$.

  ○ This is called the <u>polar form</u> of $z$. The angle $\theta$ is called the <u>argument</u> of $z$ and sometimes denoted $\theta = \arg(z)$.

○ <u>Notational remark</u>: Since it comes up frequently, some people like to abbreviate $\cos(\theta) + i \cdot \sin(\theta)$ by cis($\theta$) ("<u>c</u>osine-<u>i</u>-<u>s</u>ine").

○ Conversely, if we know $z = x + iy$ then we can compute the $(r, \theta)$ form fairly easily, since $r = |z|$ and $\theta = \arg(z)$.

○ Explicitly, we have $\boxed{r = \sqrt{x^2 + y^2}}$ and $\boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right)}$ if $x > 0$, and $\boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right) + \pi}$ if $x < 0$.

  ∗ This extra $+\pi$ is needed because of the specific way we've chosen the definition of arctangent. Otherwise we'd get the wrong value for $\theta$ if $z$ lies in the second or third quadrants.

  ∗ Again, note that these are the exact same formulas for converting between rectangular and polar coordinates.

○ <u>Example</u>: If $z = 1 + i$, then the corresponding values of $r$ and $\theta$ are $r = |z| = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \dfrac{\pi}{4}$, so we can write $z$ in polar form as $z = \boxed{\sqrt{2} \cdot \left[\cos\dfrac{\pi}{4} + i \sin\dfrac{\pi}{4}\right]}$. (It is easy to multiply this out and verify that the result is simply $1 + i$.)

• We're comfortable with plugging complex numbers into polynomials, but what about other functions? We'd like to be able to say what something like $e^{a+bi}$ should mean.

  ○ We feel like $e^{a+bi}$ should obey the exponential rules, and so we want to say $e^{a+bi} = e^a \cdot e^{bi}$. So really, we only care about what $e^{bi}$ is.

• The key result is what is called <u>Euler's identity</u>: $\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$.

  ○ Notice that this is the same expression that showed up in the polar form of a complex number.

  ○ One way to derive Euler's identity is via Taylor series expansions: we know that $e^x = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ for any real number $x$. Let us blithely assume that this also holds for any complex number $x$. Setting $x = i\theta$ then produces

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right). \end{aligned}$$

  ○ But now notice that the real part is the Taylor series for $\cos(\theta)$, while the imaginary part is the Taylor series for $\sin(\theta)$.

  ○ Thus, we obtain $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, which is precisely Euler's formula.

• Euler's identity encodes <u>a lot</u> of information. Here is one application:

  ○ Exponential rules state $e^{i(\theta+\varphi)} = e^{i\theta} \cdot e^{i\varphi}$.

  ○ Expanding out both sides with Euler's identity yields

$$\cos(\theta + \varphi) + i \cdot \sin(\theta + \varphi) = [\cos(\theta) + i \cdot \sin(\theta)] \cdot [\cos(\varphi) + i \cdot \sin(\varphi)].$$

  ○ Multiplying out and simplifying yields

$$\cos(\theta + \varphi) + i \cdot \sin(\theta + \varphi) = [\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)] + i \cdot [\cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)].$$

  ○ Setting the real and imaginary parts equal yields (respectively) the equalities

$$\begin{aligned} \cos(\theta + \varphi) &= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) \\ \sin(\theta + \varphi) &= \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi) \end{aligned}$$

and notice that these are exactly the addition formulas for sine and cosine!

○ What this means is that the rather strange-looking trigonometric addition formulas, which are rather weird and arbitrary when first encountered, actually just reflect the natural structure of the multiplication of complex numbers.

- Another application is the simple relation $e^{i(n\theta)} = (e^{i\theta})^n$. Writing out both sides in terms of sines and cosines gives <u>De Moivre's identity</u> $\boxed{\cos(n\theta) + i \cdot \sin(n\theta) = [\cos(\theta) + i \cdot \sin(\theta)]^n}$.

  ○ Plugging in various values of $n$ and then expanding out the right-hand side via the Binomial Theorem allows one to obtain identities for $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

  ○ <u>Example</u>: $\cos(2\theta) + i \cdot \sin(2\theta) = [\cos(\theta) + i \cdot \sin(\theta)]^2 = \left(\cos^2\theta - \sin^2\theta\right) + i \cdot (2\sin\theta\cos\theta)$, and so we recover the double-angle formulas for sine and cosine.

- Even setting $\theta = \pi$ in Euler's identity tells us something very interesting: we obtain $e^{i\pi} = -1$, or, better, $\boxed{e^{i\pi} + 1 = 0}$.

  ○ The constants 0, 1, $i$, $e$, and $\pi$ are, without a doubt, the five most important numbers in all of mathematics.

  ○ That there exists one simple equation relating all five of them is (to the author at least) quite amazing.

- We can use Euler's identity to simply the evaluation of certain integrals, such as $\int e^{ax}\cos(bx)\,dx$.

  ○ Using Euler's identity we have $e^{ibx} = \cos(bx) + i\sin(bx)$ and $e^{-ibx} = \cos(bx) - i\sin(bx)$, so by adding the equations we see that $\cos(bx) = \dfrac{1}{2}\left[e^{ibx} + e^{-ibx}\right]$.

  ○ Then we can successively compute

  $$
  \begin{aligned}
  \int e^{ax}\cos(bx)\,dx &= \frac{1}{2}\int e^{ax}\left[e^{ibx} + e^{-ibx}\right]\,dx \\
  &= \frac{1}{2}\int \left(e^{(a+bi)x} + e^{(a-bi)x}\right)dx \\
  &= \frac{1}{2}\left[\frac{e^{(a+bi)x}}{a+bi} + \frac{e^{(a-bi)x}}{a-bi}\right] \\
  &= \frac{1}{2}\cdot\frac{(a-bi)e^{(a+bi)x} + (a+bi)e^{(a-bi)x}}{a^2 + b^2} \\
  &= \frac{1}{2}\cdot e^{ax}\frac{(a-bi)(\cos(bx) + i\sin(bx)) + (a+bi)(\cos(bx) - i\sin(bx))}{a^2 + b^2} \\
  &= \boxed{e^{ax}\cdot\frac{a\cos(bx) + b\sin(bx)}{a^2 + b^2}}
  \end{aligned}
  $$

  where we used the fact that $\int e^{(a+bi)x}\,dx = \dfrac{e^{(a+bi)x}}{a+bi}$, and then combined the fractions and simplified the resulting expression.

  ○ Using the same method we can also compute the integral $\int e^{ax}\sin(bx)\,dx = e^{ax}\cdot\dfrac{a\sin(bx) - b\cos(bx)}{a^2 + b^2}$: the only difference is that we instead write $\sin(bx) = \dfrac{1}{2i}\left[e^{ibx} - e^{-ibx}\right]$ at the beginning.

  ○ <u>Remark</u>: It is also possible to evaluate this integral using integration by parts twice.

- Using Euler's identity and the polar form of complex numbers above, we see that every complex number can be written as $\boxed{z = r \cdot e^{i\theta}}$ for some $r$ and $\theta$. We call this the <u>exponential form</u> of $z$.

  ○ <u>Example</u>: We can draw $1 + i$ in the complex plane, or use the formulas, to see that $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \dfrac{\pi}{4}$, and so we see that $\boxed{1 + i = \sqrt{2}\cdot e^{i\pi/4}}$.

○ Example: Either by geometry or trigonometry, we see that $\left|1 - i\sqrt{3}\right| = 2$ and $\arg(1 - i\sqrt{3}) = -\dfrac{\pi}{3}$, hence $\boxed{1 + i\sqrt{3} = 2 \cdot e^{-i\pi/3}}$.

- It is very easy to take powers of complex numbers when they are in exponential form: $(r \cdot e^{i\theta})^n = r^n \cdot e^{i(n\theta)}$.

  ○ Example: Compute $(1 + i)^8$.
    * From above we have $1 + i = \sqrt{2} \cdot e^{i\pi/4}$, so $(1 + i)^8 = \left(\sqrt{2} \cdot e^{i\pi/4}\right)^8 = (\sqrt{2})^8 \cdot e^{8i\pi/4} = 2^4 \cdot e^{2i\pi} = \boxed{16}$.
    * Note how much easier this is compared to multiplying $(1 + i)$ by itself eight times.
  ○ Example: Compute $(1 - i\sqrt{3})^9$.
    * From above we have $1 - i\sqrt{3} = 2 \cdot e^{-i\pi/3}$, so $(1 - i\sqrt{3})^9 = 2^9 \cdot e^{-9i\pi/3} = 512 \cdot e^{-3i\pi} = \boxed{-512}$.

- Taking roots of complex numbers is also easy using the polar form. We do need to be slightly careful, since (like having 2 possible square roots of a positive real number), there are $n$ different $n$th roots of any nonzero complex number.

  ○ The general formula says that the $n$ possible $n$th roots of $z = r \cdot e^{i\theta}$ are $\boxed{\sqrt[n]{r}e^{i\theta/n} \cdot e^{2i\pi k/n}}$, where $k$ ranges through $0, 1, \cdots, n - 1$.
  ○ One can check that the $n$th power of all of these numbers is indeed $r \cdot e^{i\theta}$, since $e^{2i\pi k} = 1$ (for $k$ an integer) by Euler's formula. And they are clearly all distinct, and so they are all of the $n$th roots.

- Example: Find all complex square roots of $2i$.

  ○ We are looking for square roots of $2i = 2 \cdot e^{i\pi/2}$. By the formula, the two square roots are $\sqrt{2} \cdot e^{i[\pi/4 + k\pi]}$ for $k = 0, 1$.
  ○ Converting from exponential to rectangular form using Euler's formula gives the two square roots as $\boxed{1 + i, -1 - i}$.
  ○ Indeed, we can easily multiply out to verify that $(1 + i)^2 = (-1 - i)^2 = 2i$, as it should be.

- Example: Find all complex numbers $z = a + bi$ with $z^3 = 1$.

  ○ We are looking for cube roots of $1 = 1 \cdot e^0$. By the formula, the three cube roots of 1 are $1 \cdot e^{2ki\pi/3}$, for $k = 0, 1, 2$.

  ○ Converting from exponential to rectangular form using Euler's formula gives the roots as $\boxed{1, \; -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i, \; -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i}$ in $x + yi$ form.

Well, you're at the end of my handout. Hope it was helpful.