$\boxed{\text{Differential Equations (part 1s) : Additional First-Order Topics}}$ (by Evan Dummit, 2016, v. 1.01)

Contents

1 First-Order Differential Equations

1.6 Autonomous Equations, Equilibria, and Stability

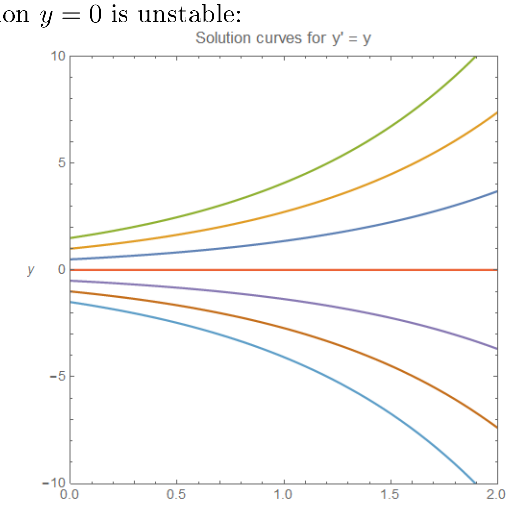
- Another class of equations that often arise are equations that do not explicitly include the independent variable:
- Definition: An autonomous equation is a first-order equation of the form $\frac{dy}{dt} = f(y)$ for some function f.
  - An equation of this form is separable, and thus solvable in theory.
  - However, it can certainly happen that the function $f(y)$ is sufficiently complicated that we cannot actually perform the integration: for example, there is no easy way to solve $\frac{dy}{dt} = y + \sin(y)$ explicitly for y because the integral $\int \frac{dy}{y + \sin(y)}$ is essentially intractable.
  - In such cases we would still like to be able to say something about what the solutions look like: fortunately, this is still possible.
- Definition: An equilibrium solution, also called a steady state solution or a critical point, is a solution of the form $y(t) = c$, for some constant c. (In other words, it is just a constant-valued solution.)
  - Clearly, if $y(t)$ is constant, then $y'(t)$ is zero everywhere.
  - So in order to find the equilibrium solutions to an autonomous equation $y' = f(y)$, we just need to solve $f(y) = 0$.
- For equilibrium solutions, we have notions of "stability":
  - An equilibrium solution $y = c$ is stable from above if, when we solve $y' = f(y)$ with the initial condition $y(0) = c + \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves toward c as t increases. This statement is equivalent to $f(c + \epsilon) < 0$.
  - A solution $y = c$ is stable from below if when we solve $y' = f(y)$ with the initial condition $y(0) = c - \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves toward c as t increases. This statement is equivalent to $f(c - \epsilon) > 0$.

- A solution $y = c$ is <u>unstable from above</u> if when we solve $y' = f(y)$ with the initial condition $y(0) = c + \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves away from c as t increases. This statement is equivalent to $f(c + \epsilon) > 0$.
- A solution $y = c$ is <u>unstable from below</u> if when we solve $y' = f(y)$ with the initial condition $y(0) = c - \epsilon$ for some small but positive ϵ, the solution $y(t)$ moves away from c as t increases. This statement is equivalent to $f(c - \epsilon) < 0$.

- <u>Definition</u>: We say an equilibrium solution is <u>stable</u> if it is stable from above and from below. We say it is <u>unstable</u> if it unstable from above and from below. Otherwise (if it is stable from one side and unstable from the other) we say it is <u>semistable</u>.

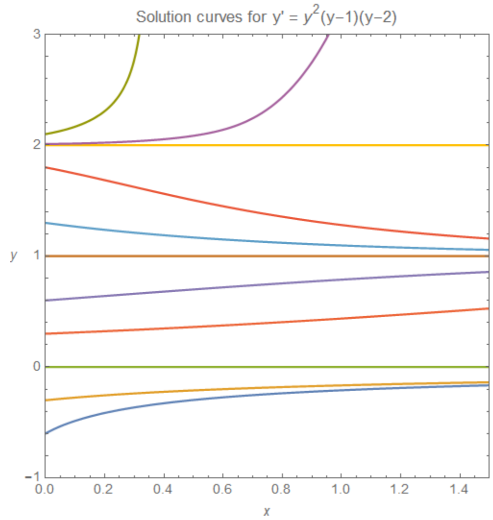
- From the equivalent conditions about the sign of f, here are the steps to follow to find and classify the equilibrium states of $y' = f(y)$:
  - <u>Step 1</u>: Find all values of c for which $f(c) = 0$, to find the equilibrium states.
  - <u>Step 2</u>: Mark all the equilibrium values on a number line, and then in each interval between two critical points, plug in a test value to f to determine whether f is positive or negative on that interval.
  - <u>Step 3</u>: On each interval where f is positive, draw right-arrows, and on each interval where f is negative, draw left-arrows.
  - <u>Step 4</u>: Using the arrows, classify each critical point: if the arrows point toward it from both sides, it is stable. If the arrows point away, it is unstable. If the arrows both point left or both point right, it is semistable.
  - <u>Step 5</u> (optional): Draw some solution curves, either by solving the equation or by using the stability information.

- <u>Example</u>: Find the equilibrium states of $y' = y$ and determine their stability.
  - Step 1: We have $f(y) = y$, which obviously is zero only when $y = 0$.
  - Step 2: We draw the line and plug in 2 test points (or just think for a second) to see that the sign diagram looks like $\ominus \underset{0}{|} \oplus$.
  - Step 3: Changing the diagram to arrows gives $\leftarrow \underset{0}{|} \rightarrow$.
  - Step 4: So we can see from the diagram that the only equilibrium point $\boxed{\text{0 is unstable}}$.
  - Step 5: We can of course solve the equation to see that the solutions are of the form $y(t) = C\,e^t$, and indeed, the equilibrium solution $y = 0$ is unstable:



- <u>Example</u>: Find the equilibrium states of $y' = y^2(y-1)(y-2)$ and determine their stability.
  - Step 1: We have $f(y) = y^2(y-1)(y-2)$, which conveniently is factored. We see it is zero when $y = 0$, $y = 1$, and $y = 2$.

- Step 2: We draw the line and plug in 4 test points (or just think for a second) to see that the sign diagram looks like $\oplus \underset{0}{|} \oplus \underset{1}{|} \ominus \underset{2}{|} \oplus$.
- Step 3: Changing the diagram to arrows gives $\to \underset{0}{|} \to \underset{1}{|} \leftarrow \underset{2}{|} \to$.
- Step 4: So we can see from the diagram that $\boxed{0 \text{ is semistable}}$, $\boxed{1 \text{ is stable}}$, and $\boxed{2 \text{ is unstable}}$.
- Step 5: In this case, it is possible to obtain an implicit solution by integration (via partial fraction decomposition); however, an explicit solution does not exist. However, we can graph some solution curves to see, indeed, our classification is accurate:



1.7 Substitution Methods

- Just like with integration, sometimes we come across differential equations which we cannot obviously solve, but which, if we change variables, will turn into a form we know how to solve.
- Determining what substitutions to try is a matter of practice, in much the same way as in integral calculus. In general, there are two kinds of substitutions: "natural" ones that arise from the form of the differential equation, and "formulaic" ones which are standard substitutions to use if a differential equation has a particular form.
- The general procedure is the following:
  - Step 1: Express the new variable v in terms of y and x.
  - Step 2: Find $\frac{dv}{dx}$ in terms of y', y, and x using implicit differentiation.
  - Step 3: Rewrite the original differential equation in y as a differential equation in v.
  - Step 4: Solve the new equation in v. (The hope is, after making the substitution, the new equation is in a form that can be solved with one of the other methods.)
  - Step 5: Substitute back for y.
- Example: Solve the equation $y' = (x+y)^2$.
  - This equation is not linear, nor is it separable as written. The obstruction is that the term $x+y$ involves both x and y.
  - Step 1: Let us try substituting $v = x + y$.
  - Step 2: Differentiating yields $\frac{dv}{dx} = 1 + \frac{dy}{dx}$, so $y' = v' - 1$.
  - Step 3: The new equation is $v' - 1 = v^2$, or $v' = v^2 + 1$.

- Step 4: The equation in v is separable. Separating it gives $\int \frac{dv}{v^2+1} = \int 1\,dx$, so that $\tan^{-1}(v) = x + C$, or $v = \tan(x+C)$.
- Step 5: Substituting back yields $y = \boxed{\tan(x+C) - x}$.

1.7.1 Bernoulli Equations

- Definition: A first-order differential equation of the form $y' + P(x)y = Q(x) \cdot y^n$ for some integer $n \neq 0, 1$ is called a Bernoulli equation.
  - The restriction that n not be 0 or 1 is not really a restriction, because if $n = 0$ then the equation is first-order linear, and if $n = 1$ then the equation is the same as $y' = (Q(x) - P(x))y$, which is separable.
  - As with first-order linear equations, sometimes Bernoulli equations can be "hidden" in a slightly different form.
  - The trick for solving a Bernoulli equation is to make the substitution $v = y^{1-n}$. The algebra is simplified if we first multiply both sides of the original equation by $(1-n) \cdot y^{-n}$, and then make the substitution.
  - We begin with $(1-n)y' \cdot y^{-n} + (1-n)P(x) \cdot y^{1-n} = (1-n)Q(x)$.
  - With $v = y^{1-n}$, we have $v' = (1-n)y^{-n} \cdot y'$.
  - Thus, we can rewrite the original equation as $v' + (1-n)P(x) \cdot v = (1-n)\,Q(x)$, which is now a first-order linear equation in v, which we can solve using the standard technique.
- Example: Solve the equation $y' + 2xy = xy^3$.
  - This equation is of Bernoulli type, with $P(x) = 2x$, $Q(x) = x$, and $n = 3$. Making the substitution $v = y^{-2}$ thus results in the equation $v' - 4xv = -2x$.
  - Next, we compute the integrating factor $I(x) = e^{\int -4x\,dx} = e^{-2x^2}$.
  - Scaling by the integrating factor gives $e^{-2x^2}v' - 4xe^{-2x^2}v = -2xe^{-2x^2}$.
  - Taking the antiderivative on both sides then yields $e^{-2x^2}v = \frac{1}{2}e^{-2x^2} + C$, so that $v = \frac{1}{2} + Ce^{-2x^2}$.
  - Finally, solving for y gives $\boxed{y = \left(\frac{1}{2} + Ce^{-2x^2}\right)^{-1/2}}$.
- Example: Solve the equation $yy' = e^x y^{-1} - y^2$.
  - The equation as written is not of Bernoulli type. However, if to both sides we add y^2 and then divide by y, we obtain the equation $y' + y = e^x y^{-2}$, which is now a Bernoulli equation with $P(x) = 1$, $Q(x) = e^x$, and $n = -2$.
  - Making the substitution $v = y^3$ results in the equation $v' + 3v = 3e^x$.
  - The integrating factor is $I(x) = e^{\int 3\,dx} = e^{3x}$, so multiplying by it yields $e^{3x}v' + 3e^{3x}v = 3e^{4x}$.
  - Taking the antiderivative then gives $e^{3x}v = \frac{3}{4}e^{4x} + C$, so $v = \frac{3}{4}e^x + Ce^{-3x}$ and then $\boxed{y = \left(\frac{3}{4}e^x + Ce^{-3x}\right)^{1/3}}$.

1.7.2 Homogeneous Equations

- Definition: An equation of the form $y' = f\left(\frac{y}{x}\right)$ for some function f is called a homogeneous first-order equation. (Note that this phrase has nothing to do with the other use of the word homogeneous when referring to differential equations.)
  - The trick to solving an equation of this form is to make the substitution $v = \frac{y}{x}$, or equivalently to set $y = vx$.

- Then differentiating $y = vx$ shows $y' = v + xv'$, hence the equation becomes $v + xv' = f(v)$, which is separable once written in the form $\dfrac{v'}{f(v) - v} = \dfrac{1}{x}$.

- Example: Solve the differential equation $2x^2 y' = x^2 + y^2$.
  - This equation is not separable nor linear, and it is not a Bernoulli equation. If we divide both sides by $2x^2$ then we obtain $y' = \dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{y}{x}\right)^2$, which is homogeneous.
  - Setting $v = y/x$ yields the equation $xv' = \dfrac{1}{2}v^2 - v + \dfrac{1}{2}$, and rearranging gives $\dfrac{2v'}{(v-1)^2} = \dfrac{1}{x}$.
  - Then integrating yields $\int \dfrac{2dv}{(v-1)^2} = \int \dfrac{1}{x}\,dx$, so $\dfrac{-2}{v-1} = \ln(x) + C$. Solving for v gives $v = 1 - \dfrac{2}{\ln(x) + C}$, so $\boxed{y = x - \dfrac{2x}{\ln(x) + C}}$.

- Example: Solve the differential equation $y' = \dfrac{x^2 + y^2}{xy}$.
  - If we divide the numerator and denominator of the fraction by x^2, we obtain $y' = \dfrac{1 + (y/x)^2}{(y/x)}$, which is homogeneous.
  - Setting $v = y/x$ yields $xv' = \dfrac{1 + v^2}{v} - v = \dfrac{1}{v}$.
  - Separating and integrating yields $\int v\,dv = \int \dfrac{1}{x}\,dx$, so that $\dfrac{1}{2}v^2 = \ln(x) + C$, so $v = \sqrt{2\ln(x) + C}$ and then $\boxed{y = x\sqrt{2\ln(x) + C}}$.

1.8 Exact First Order Equations and Integrating Factors

- Theorem (Exact Equations): For functions $M(x, y)$ and $N(x, y)$ with $M_y = N_x$ (on some rectangle), there exists a function $F(x, y)$ with $F_x = M$ and $F_y = N$ (on that rectangle). Then the solutions to the differential equation $M(x, y) + N(x, y)\, y' = 0$ are given (implicitly) by $F(x, y) = C$ where C is an arbitrary constant.
  - Note that M_y denotes the partial derivative of M with respect to y, namely $\dfrac{\partial M}{\partial y}$, and similarly for the other functions.
  - The equation $M(x, y) + N(x, y)\, y' = 0$ is also sometimes written $M(x, y)\, dx + N(x, y)\, dy = 0$. In this form, it is more symmetric between the variables x and y. I will generally do this.
  - Remark (for those who know vector calculus): The part of the theorem stating that $M_y = N_x$ implies the existence of a function F such that $F_x = M$ and $G_y = N$ is a theorem from vector calculus: the criterion $M_y = N_x$ is equivalent to the vector field $\langle M, N\rangle$ being conservative. The function F is the corresponding "potential function", with $\nabla F = \langle M, N\rangle$. The rest of the theorem is really just an application of this result.
  - Remark: Note that if $M = f(x)$ is a function only of x and $N = -\dfrac{1}{g(y)}$ is a function only of y, then our equation looks like $f(x) - \dfrac{1}{g(y)}\, y' = 0$. Rearranging it gives the general form $y' = f(x)\, g(y)$ of a separable equation. Since $M_y = 0 = N_x$ in this case, we see that separable equations are a special case of exact equations.

- We can use the theorem to solve "exact equations", where $M_y = N_x$. If the partial derivatives are not equal, we are not necessarily out of luck – like with first-order linear equations, there may exist an integrating factor $I(x, y)$ which we can multiply the equation by, in order to make the equation exact.

- Unfortunately, we don't really get much for free: trying to solve for the integrating factor is often as hard as solving the original equation. Finding $I(x, y)$, in general, requires solving the PDE $\frac{\partial I}{\partial y} \cdot M - \frac{\partial I}{\partial x} \cdot N + I \cdot (M_y - N_x) = 0$, which is just as tricky (if not trickier) to solve as the original equation. Only in a few special cases are there methods for computing the integrating factor $I(x, y)$.
- Case 1: Suppose we want to see if there exists an integrating factor that depends only on x (and not on y). Then $\frac{\partial I}{\partial y}$ would be zero, since I does not depend on y, and so $I(x)$ would need to satisfy $\frac{I'}{I} = \frac{M_y - N_x}{N}$. This can only happen if the ratio $\frac{M_y - N_x}{N}$ is a function $P(x)$ only of x (and not y); then $I(x) = e^{\int P(x)\,dx}$.
  - The form of this integrating factor should look familiar – it is the same as the one from a first-order linear equation. There is a very good reason for this; namely, a first-order linear equation is a special case of this form of equation.
- Case 2: We could also look to see if there is an integrating factor that depends only on y and not on x. We can do the same calculation, this time using $\frac{\partial I}{\partial x} = 0$, to see that such an integrating factor exists if the ratio $\frac{N_x - M_y}{M}$ is a function $Q(y)$ only of y (and not x); then $I(y) = e^{\int Q(y)\,dy}$.
- Remark: There is no really good reason only to consider these cases, aside from the fact that they're the easiest. We could just as well try to look for integrating factors that are a function of the variable $t = xy$. Or of $v = x/y$. Or of $w = y + \ln(x)$. In each case we'd end up with some other kind of condition. But we won't think about those things – we really just care about the two kinds of integrating factors above.

- Example: Solve for $y(x)$, if $(4y^2 + 2x) + (8xy)y' = 0$.
  - There is no obvious substitution to make, and it is not separable, linear, homogeneous, or Bernoulli. So we must check for exactness.
  - In differential form the equation is $(4y^2 + 2x)\,dx + 8xy\,dy = 0$. Therefore, $M = 4y^2 + 2x$ and $N = 8xy$.
  - Therefore we have $M_y = 8y$ and $N_x = 8y$. Since these are equal, the equation is exact.
  - So we want to find F with $F_x = M$ and $F_y = N$. Taking the "anti-partial-derivative" of M with respect to x yields $F(x, y) = 4xy^2 + x^2 + g(y)$ for some function $g(y)$. Checking then shows $F_y = 8xy + g'(y)$ so $g'(y) = 0$.
  - Therefore, our solutions are given implicitly by $\boxed{4xy^2 + x^2 = C}$.

- Example: Solve for $y(x)$, if $(2xy^2 - 4y) + (3x^2y - 8x)y' = 0$.
  - There is no obvious substitution to make, and it is not separable, linear, homogeneous, or Bernoulli. So we must check for exactness.
  - In differential form the equation is $(2xy^2 - 4y)\,dx + (3x^2y - 8x)\,dy = 0$. Therefore, $M = 2xy^2 - 4y$ and $N = 3x^2y - 8x$.
  - Therefore we have $M_y = 4xy - 4$ and $N_x = 6xy - 8$. These are not equal, so the equation isn't exact.
  - We look for integrating factors using the two criteria we know.
    - First, we have $\frac{M_y - N_x}{N} = \frac{-2xy + 4}{3x^2y - 8x}$ is not a function of x only.
    - Second, we have $\frac{N_x - M_y}{M} = \frac{2xy - 4}{2xy^2 - 4y} = \frac{1}{y}$ is a function of y only. Therefore we need to multiply by the integrating factor $I(y) = e^{\int (1/y)\,dy} = y$.
  - Our new equation is therefore $(2xy^3 - 4y^2) + (3x^2y^2 - 8xy)y' = 0$.
  - Now we want to find F with $F_x = 2xy^3 - 4y^2$ and $F_y = 3x^2y^2 - 8xy$. Taking the "anti-partial-derivative" of the first equation gives $F(x, y) = x^2y^3 - 4xy^2 + f(y)$ and checking in the second equation shows $f'(y) = 0$.
  - Therefore, our solutions are given implicitly by $\boxed{x^2y^3 - 4xy^2 = C}$.

1.9 First Order: General Procedure

- We can combine all of the techniques for solving first-order differential equations into a handy list of steps. For the purposes of this course, if the equation cannot be simplified via a substitution (an obvious substitution, or if it is homogeneous or Bernoulli) then it is either exact, or can be made exact by multiplying by an integrating factor. (If it's not one of those, then in this course we have no idea how to solve it.)
  - Note: It is not really necessary to check ahead of time whether the equation is separable or a first-order linear equation. Separable equations are exact, and first-order linear equations can be made exact after multiplying by an integrating factor, which will be detected using the $\frac{M_y - N_x}{N}$ test. I check for these two special types at the beginning only because it's faster to solve it using the usual methods.
- Here is the general procedure to follow to solve first-order equations:
  - Step 1: Write the equation in the two standard forms $y' = f(x, y)$ and $M(x, y) + N(x, y) \cdot y' = 0$ and check to see if it is first-order linear or separable.
    * Step 1a: If the equation is first-order linear – namely, of the form $y' + P(x)y = Q(x)$ – then multiply by the integrating factor $I(x) = e^{\int P(x)\,dx}$ and then take the antiderivative of both sides.
    * Step 1b: If the equation is separable – namely, of the form $y' = f(x) \cdot g(y)$ – then separate the y-terms and x-terms on opposite sides of the equation and then take the antiderivative of both sides.
  - Step 2: Look for possible substitutions (generally, using the $y' = f(x, y)$ form).
    * Step 2a: Check to see if there is any 'obvious' substitution that would simplify the equation.
    * Step 2b: Check to see if the equation is of Bernoulli type – namely, of the form $y' + P(x)y = Q(x) \cdot y^n$. If so, multiply both sides by $(1 - n) \cdot y^{-n}$ and then make the substitution $v = y^{1-n}$ to obtain a first-order linear equation $\frac{dv}{dx} + (1 - n)P(x) \cdot v = (1 - n)\,Q(x)$.
    * Step 2c: Check to see if the equation is homogeneous – namely, of the form $y' = F\left(\frac{y}{x}\right)$ for some function F. If so, make the substitution $v = \frac{y}{x}$ to obtain a separable equation $x \cdot \frac{dv}{dx} = F(v) - v$.
  - Step 3: If the equation is not of a special type, use the $M(x, y) + N(x, y) \cdot y' = 0$ form to find the partial derivatives M_y and N_x.
  - Step 4: If $M_y = N_x$, no integrating factor is needed. Otherwise, if $M_y \neq N_x$, look for an integrating factor I to multiply both sides of the equation by.
    * Step 3a: Compute $\frac{M_y - N_x}{N}$. If it is a function $P(x)$ only of x, then the integrating factor is $I(x) = e^{\int P(x)\,dx}$.
    * Step 3b: Compute $\frac{N_x - M_y}{M}$. If it is a function $Q(y)$ only of y, then the integrating factor is $I(y) = e^{\int Q(y)\,dy}$.
    * If neither of these methods works, you're out of luck unless you can find an integrating factor some other way.
  - Step 5: Take antiderivatives to find the function $F(x, y)$ with $F_x = M$ and $F_y = N$, and write the solutions as $F(x, y) = C$.

1.10 First Order: General Problems and Solutions

- Part of the difficulty of seeing first-order differential equations outside of a homework set (e.g., on exams) is that it is not always immediately obvious which method or methods will solve the problem. Thus, it is good to practice problems without being told which method to use.

1.10.1 Problems

- Solve the equation $xy' = y + \sqrt{xy}$.
- Solve the equation $y' = \dfrac{y - 2xy^2}{3x^2y - 2x}$.
- Solve the equation $xy' = y + \sqrt{x}$.
- Solve the equation $y' - 1 = y^2 + x^3 + x^3y^2$.
- Solve the equation $y' = -\dfrac{4x^3y^2 + y}{2x^4y + x}$.
- Solve the equation $y' = xy^3 - 6xy$.
- Solve the equation $y' = -\dfrac{2xy + 2x}{x^2 + 1}$.

1.10.2 Solutions

- Solve the equation $xy' = y + \sqrt{xy}$.
  - Step 1: The two standard forms are $y' = \dfrac{y}{x} + \sqrt{\dfrac{y}{x}}$ and $(-y - \sqrt{xy}) + xy' = 0$. The equation is not separable or first-order linear.
  - Step 2: We go down the list and recognize that $y' = \dfrac{y}{x} + \sqrt{\dfrac{y}{x}}$ is a homogeneous equation.
  - Setting $v = y/x$ (with $y = vx$ and $y' = xv' + v$) yields $xv' = \sqrt{v}$.
  - This equation is separable: we have $\int \dfrac{dv}{\sqrt{v}} = \int \dfrac{1}{x}\,dx$ hence $2v^{1/2} = \ln(x) + C$, so $v = \left(\dfrac{\ln(x)}{2} + C\right)^2$.
  - Solving for y gives $\boxed{y = x\left(\dfrac{\ln(x)}{2} + C\right)^2}$.
  - Note: The equation is also of Bernoulli type, and could be solved that way too. Of course, it will give the same answer.
- Solve the equation $y' = \dfrac{y - 2xy^2}{3x^2y - 2x}$.
  - Step 1+2: The other standard form is $(2xy^2 - y) + (3x^2y - 2x)\,y' = 0$. The equation is not separable or linear, nor is it homogeneous or Bernoulli.
  - Step 3: We have $M = 2xy^2 - y$ and $N = 3x^2y - 2x$ so $M_y = 4xy - 1$ and $N_x = 6xy - 2$.
  - Step 4: We need to look for an integrating factor, because $M_y \neq N_x$. We have $\dfrac{M_y - N_x}{N} = \dfrac{-2xy + 1}{3x^2y - 2x}$, which is not a function of x alone. Next we try $\dfrac{N_x - M_y}{M} = \dfrac{2xy - 1}{2xy^2 - y} = \dfrac{1}{y}$, so the integrating factor is $I(y) = e^{\int \frac{1}{y}\,dy} = y$.
  - Step 5: The new equation is $(2xy^3 - y^2) + (3x^2y^2 - 2xy)\,y' = 0$. Taking the "anti-partial" of the new M with respect to x gives $F(x, y) = x^2y^3 - xy^2 + f(y)$, and checking shows that $f'(y) = 0$. Hence the solutions are $\boxed{x^2y^3 - xy^2 = C}$.
- Solve the equation $xy' = y + \sqrt{x}$.
  - Step 1: The two standard forms are $y' = \dfrac{y}{x} + \dfrac{1}{\sqrt{x}}$ and $(-y - \sqrt{x}) + xy' = 0$. The equation is first-order linear.

- Rewrite in the usual first-order linear form $y' - (x^{-1})y = x^{-1/2}$.
- We have the integrating factor $I(x) = e^{\int -x^{-1}\,dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = x^{-1}$.
- Thus the new equation is $x^{-1}y' - x^{-2}y = x^{-3/2}$.
- Taking the antiderivative on both sides yields $x^{-1}y = -\frac{1}{2}x^{-1/2} + C$, so $\boxed{y = -\frac{1}{2}x^{1/2} + Cx}$.

- Solve the equation $y' - 1 = y^2 + x^3 + x^3y^2$.
  - Step 1: Adding 1 and then factoring the right-hand side gives $y' = (y^2+1)(x^3+1)$. This equation is separable.
  - Separating it gives $\dfrac{y'}{y^2+1} = x^3 + 1$.
  - Integrating yields $\int \dfrac{dy}{y^2+1} = \int (x^3+1)\,dx$, so $\tan^{-1}(y) = \dfrac{x^4}{4} + \dfrac{x^2}{2} + C$. Then $\boxed{y = \tan\left(\dfrac{x^4}{4} + \dfrac{x^2}{2} + C\right)}$.

- Solve the equation $y' = -\dfrac{(4x^3y^2 + y)}{(2x^4y + x)}$.
  - Step 1+2: The other standard form is $(4x^3y^2 + y) + (2x^4y + x)\,y' = 0$. The equation is not separable or linear, nor is it homogeneous or Bernoulli.
  - Step 3: We have $M = 4x^3y^2 + y$ and $N = 2x^4y + x$ so $M_y = 8x^3y + 1$ and $N_x = 8x^3y + 1$.
  - Step 4: No integrating factor is needed since $M_y = N_x$.
  - Step 5: Taking the "anti-partial" of M with respect to x gives $F(x,y) = x^4y + xy + f(y)$, and checking shows that $f'(y) = 0$. Hence the solutions are $\boxed{x^4y^2 + xy = C}$.

- Solve the equation $y' = xy^3 - 6xy$.
  - Step 1: The equation is of Bernoulli type when written as $y' + 6xy = xy^3$.
  - Multiply both sides by $-2y^{-3}$ to get $-2y^{-3}y' - \dfrac{12}{x}y^{-2} = -2x$.
  - Making the substitution $v = y^{-2}$ with $v' = -2y^{-3}y'$ then yields the linear equation $v' - \dfrac{12}{x}v = -2x$.
  - The integrating factor is $e^{\int -(12/x)\,dx} = e^{-12\ln(x)} = x^{-12}$.
  - The new equation is $x^{-12}v' - 12x^{-13}v = -2x^{-11}$.
  - Taking the antiderivative on both sides yields $x^{-12}v = \dfrac{1}{5}x^{-10} + C$, so $v = \dfrac{1}{5}x^2 + Cx^{12}$ and $\boxed{y = \left(\dfrac{1}{5}x^2 + Cx^{12}\right)^{-1/2}}$.

- Solve the equation $y' = -\dfrac{2xy + 2x}{x^2+1}$.

  (method #1)
  - Step 1: The equation is separable, since after factoring we see that $y' = -\dfrac{2x}{x^2+1}(y+1)$.
  - Separating and integrating gives $\int \dfrac{dy}{y+1} = -\int \dfrac{2x}{x^2+1}\,dx$, so that $\ln(y+1) = -\ln(x^2+1) + C$.
  - Exponentiating yields $y + 1 = e^{-\ln(x^2+1)+C} = \dfrac{C}{x^2+1}$, so $\boxed{y = \dfrac{C}{x^2+1} - 1}$.

  (method #2)
  - Step 1: The other standard form is $(2xy + 2x) + (x^2+1)y' = 0$.
  - Step 3: We have $M = 2xy + 2x$ and $N = x^2 + 1$ so $M_y = 2x$ and $N_x = 2x$.

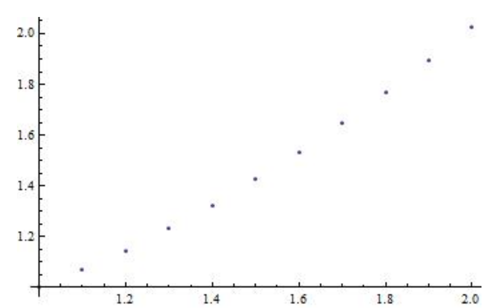
- Step 4: We have $M_y = N_x$, so the equation is exact.
- Step 5: Taking the "anti-partial" of M with respect to x gives $F(x, y) = x^2y + x^2 + f(y)$, and checking shows that $f'(y) = 1$ so $f(y) = y$. Hence the solutions are $\boxed{x^2y + x^2 + y = C}$.
- Note of course that we can solve for y explicitly, and we obtain exactly the same expression as in the other solution.

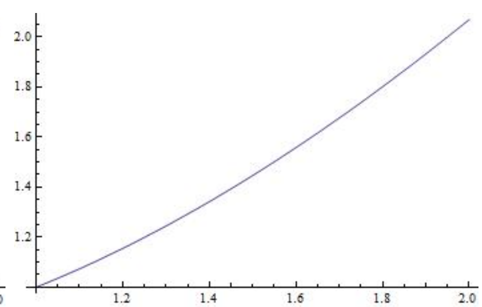
1.11 Euler's Method

- There are many first-order initial value problems $y' = f(x, y)$, $y(a) = y_0$ which we cannot solve explicitly. However, we would often like to be able to find an approximate solution on some interval $[a, b]$.
- One method we can use to find an approximate solution is Euler's Method, named after the Swiss mathematician Leonhard Euler (pronounced "oiler").
  - The general idea behind Euler's Method, which should bring back memories of basic calculus, is to break up the interval $[a, b]$ into many small pieces, and then to use a linear approximation to the function $y(x)$ on each interval to trace a rough solution to the equation.
- Here is the method, more formally:
  - Step 1: Choose the number of subintervals n, and let $h = \dfrac{b-a}{n}$ be the width of the subintervals.
  - Step 2: Define the x-values $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ... , $x_n = x_{n-1} + h = b$.
  - Step 3: Take y_0 to be the given initial value. Then compute, iteratively, the values $y_1 = y_0 + h \cdot f(x_0, y_0)$, $y_2 = y_1 + h \cdot f(x_1, y_1)$, ... , $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})$. It is easiest to organize this information in a table.
  - Step 4 (optional): Plot the points (x_0, y_0), (x_1, y_1), ... , (x_n, y_n) and connect them with a smooth curve.
- Example: Use Euler's Method to find an approximate solution on the interval $[1, 2]$ to the differential equation $y' = \ln(x + y)$ with $y(1) = 1$.
  - Step 1: Let's take 10 subintervals. Then $h = 0.1$.
  - Steps 2+3: We organize our information in the table below. We fill out the first row with the x-values. Then we fill in the empty columns one at a time: to start the next column, we add the y-value and the $h \cdot f(x, y)$ value to get the next y-value.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	1	1.0693	1.1467	1.2320	1.3249	1.4251	1.5324	1.6466	1.7674	1.8946	2.0280
$f(x, y)$	0.693	0.774	0.853	0.929	1.002	1.073	1.1418	1.208	1.272	1.334	-
$h \cdot f(x, y)$	0.0693	0.0774	0.0853	0.0929	0.1002	0.1073	0.1142	0.1208	0.1272	0.1334	-

  - Step 4: Finally, we can plot the points, and (for comparison) the actual solution curve obtained using a computer. As can be seen from the graph, the approximation is very good:





Well, you're at the end of my handout. Hope it was helpful.