Differential Equations (part 1) : First-Order Differential Equations (by Evan Dummit, 2016, v. 2.10)

Contents

1 First-Order Differential Equations

In this chapter we will outline the general theory of first-order differential equations (including a general existence-uniqueness theorem) and techniques for solving some of the basic classes of first-order equations: separable equations and first-order linear equations. We will then discuss a few applications of first-order equations, to population modeling, mixing problems, Newtonian mechanics, Newton's law of cooling, and electrical circuit analysis.

1.1 Introduction and Terminology

- In general, a differential equation is merely an equation involving a derivative (or several derivatives) of a function or functions.
  - Examples: $y' + y = 0$, or $y'' + 2y' + y = 3x^2$, or $f'' \cdot f = (f')^2$, or $f' + g' = x^3$.
  - "Most" differential equations are difficult if not impossible to find exact solutions to, in the same way that "most" random integrals or infinite series are hard to evaluate exactly.
  - In every branch of science, from physics to chemistry to biology (as well as other fields such as engineering, economics, and demography), virtually any interesting kind of process is modeled by a differential equation or a system of differential equations.
  - Morally, the reason for this is that most anything interesting involves change of some kind, and the derivative measures the rate of change. Derivatives appear in the guise of a growth rate for a population, the velocity and acceleration of a physical object, the diffusion rates of molecules involved in a reaction, marginal cost and marginal profit in economics, and hundreds of other places.
- Sometimes we will be looking for the "general solution" to a differential equation (i.e., every possible function that satisfies the equation), and other times we will be looking for the one "particular solution" that satisfies some additional conditions. We can also consider systems of equations rather than single equations: in that case we will be seeking a collection of several functions which satisfy all the equations at once.
  - A very common example of "additional conditions" is what is called an initial value problem (often abbreviated "IVP"): the additional conditions are the values of the function y and its derivatives at some initial point $x = a$.
- Here are some examples of single differential equations and systems of differential equations, with and without additional conditions. (Do not expect to be able to solve any of them immediately!)

- Example: Find all functions $y(x)$ such that $y' + y = 0$.
  - Answer: The general solution is $y(x) = Ce^{-x}$ for any constant C.
- Example: Find the function $y(x)$ such that $y'' + 2y' + y = 3x^2$ and with $y(0) = y'(0) = 1$.
  - Answer: The particular function requested is $y(x) = 3x^2 - 12x + 18 - 4xe^{-x} - 17e^{-x}$.
  - This is an example of an initial value problem: the additional conditions are the values of the function y and its derivative y' at the initial point $x = 0$.
- Example: Find all functions $f(x)$ such that $f'' \cdot f = (f')^2$.
  - Answer: The general solution is $f(x) = Ae^{Bx}$ for any constants A and B.
- Example: Find the functions $f(x)$ and $g(x)$ such that $f' = 2f - g$ and $g' = f + 2g$, where $f(0) = g(0) = \sqrt{2}$.
  - Answer: The unique solution is $f(x) = 2e^{2x}\cos(x + \frac{\pi}{4})$ and $g(x) = 2e^{2x}\sin(x + \frac{\pi}{4})$.
  - This is another example of an initial value problem: the additional conditions are the values of the functions f and g at the initial point $x = 0$.
- Example: Find all functions $f(s,t)$ such that $\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} = s + t$.
  - Answer: Many solutions. Two examples are $f(s,t) = st$ and $f(s,t) = \frac{1}{2}s^2 + \frac{1}{2}t^2$.

- Most differential equations (very much unlike the carefully chosen ones above) are difficult if not impossible to find exact solutions to, in the same way that "most" random integrals or infinite series are hard to evaluate exactly.
  - A prototypical example is something like $(f'')^7 - 2e^{f'} + x\ln(f + 2f \cdot f') = \sin^{20}(x) + e^{-x}$: there is no simple function that we can write down that will solve this equation, although it is generally possible to find accurate approximate solutions using numerical techniques or by using Taylor series.

- To organize all of this, we now introduce some terminology used to classify different types of differential equations.

- Definition: If a differential equation involves functions of only a single variable (i.e., if y is a function only of x) then it is called an ordinary differential equation (or ODE).
  - We will only talk about ODEs in these notes. But for completeness, differential equations involving functions of several variables are called partial differential equations, or PDEs. (Recall that the derivatives of functions of more than one variable are called partial derivatives, hence the name.)
  - PDEs, obviously, arise when functions depend on more than one variable. They occur often in physics (with functions that depend on space and time) and economics (with functions that depend on time and other parameters).

- Definition: An nth-order differential equation is an equation in which the highest derivative is the nth derivative.
  - Example: The equations $y' + xy = 3x^2$ and $y' \cdot y = 2$ are first-order.
  - Example: The equation $y'' + y' + y = 0$ is second-order.
  - Example: The equation $e^y = y''''$ is fourth-order.

- Definition: A differential equation is linear if it a linear function of y and its derivatives y', y'', y''', $\cdots$: in other words, if there are no terms like y^2, or $(y')^3$, or $y \cdot y'$, or $\ln(y)$, or e^y. (An equation that is not linear is nonlinear.)
  - Example: The equations $y'' + y' + y = 0$, $y' + xy = 3x^2$, and $e^{\sqrt{x}}y''' = x^{100}y - 3\tan^{-1}(x^2)$ are linear.
  - Example: The equations $y' \cdot y = 3x^2$, $x^2 + (y')^2 = 1$, and $y'' = -\sin(y)$ are nonlinear.

- Using the language above we can convert a description of a problem into a differential equation, or a system of differential equations.

- Example: A population of animals (unrestricted by space or resources) will grow at a rate proportional to its size[1]. Translate this description into a differential equation and classify its type.
  - If $P(t)$ is the population at time t and k is a constant the growth rate, the description says that $\boxed{\frac{dP}{dt} = k \cdot P}$.
  - This is a $\boxed{\text{homogeneous first-order linear differential equation with constant coefficients}}$.
  - It's not hard to see that one population model that works is $P(t) = Ce^{k \cdot t}$ for any constant C, and (as we will see) these are the only solutions. So this system is likely to see exponential population growth.
- Example: A simple pendulum consists of a weight suspended on a string, with gravity the only force acting on the weight. Translate this description into a differential equation and classify its type.
  - If θ is the angle the pendulum's string makes with a vertical line, and m is the mass of the pendulum, then the horizontal force on the weight toward the vertical is proportional to $m \sin(\theta)$ by basic geometry.
  - By Newton's second law $F = ma$, and the fact that the angular acceleration is essentially the second derivative $\theta''(t)$, we obtain the equation $m\frac{d^2\theta}{dt^2} = -km\sin(\theta)$, or $\frac{d^2\theta}{dt^2} = -k\ \sin(\theta)$.
  - This is a $\boxed{\text{nonlinear second-order differential equation}}$.
  - We will remark that this equation cannot be solved exactly for the function $\theta(t)$. However, a reasonably good approximation can be found by using the rough estimate $\sin(\theta) \approx \theta$, which turns the problem into the linear second-order differential equation $\frac{d^2\theta}{dt^2} = -k \cdot \theta$ whose solutions are much easier to find.
- Example: A simple ecosystem has two species: cats and mice. The mice breed at a rate proportional to their population, and each cat eats a fixed number of mice every year. The growth rate of the number of cats will be proportional to the number of mice (since each cat has to catch mice to survive and reproduce). Translate this description into a system of differential equations.
  - If $M(t)$ and $K(t)$ are the populations of mice and cats, with k_1, k_2, k_3 are some constants, then the descriptions say that $\frac{dM}{dt} = k_1 \cdot M - k_2 \cdot K$, and $\frac{dK}{dt} = k_3 \cdot M$.
  - This is a system of two linear differential equations; we will return to study the solutions to such systems later.
  - The conditions here are fairly natural for a simple predator-prey system. But in general, one would expect there to be non-linear terms too – perhaps when two cats meet, they fight with each other and cause injury, which would change the equation to $\frac{dK}{dt} = k_3 \cdot M - k_4 \cdot K^2$.
  - This system would get even more complicated (and more difficult to solve!) if we wanted to consider additional species each of which interacts in some way with the others.

1.2 Qualitative Analysis: Existence-Uniqueness Theorem, Slope Fields

- Before we discuss how to solve various classes of first-order differential equations quantitatively, we would like to be able to say some qualitative things about solutions to general first-order equations.
- Our first goal is to decide when (in general) we can say that a first-order differential solution has an equation, and when we can say that solution is unique.
- Theorem (First-Order Existence-Uniqueness): The initial value problem $y' = f(x, y)$ with $y(a) = b$ has at least one solution (on some interval containing a) if the function f is continuous on a rectangle containing (a, b). The IVP has exactly one solution (on some interval containing a) if the partial derivative $\frac{\partial f}{\partial y}$ is continuous on a rectangle containing (a, b).

[1]Imagine each male pairing off with a female and having a fixed number of offspring each year.

- ○ The proof of the theorem is fairly difficult and quite technical. The general idea is to construct a sequence of functions (defined on some small interval around a), such that taking the limit of the sequence yields a solution to the differential equation.
  - ○ The continuity of f ensures that the sequence will converge: roughly speaking, it forces functions far out in the sequence to eventually become very close together.
  - ○ The continuity of the partial derivative $\frac{\partial f}{\partial y}$ ensures that the solution function is unique: roughly speaking, one can use the existence of the derivative to show that the integral of the absolute value of the differences of two solutions is zero on an interval containing a (meaning that the difference would have to identically zero there).

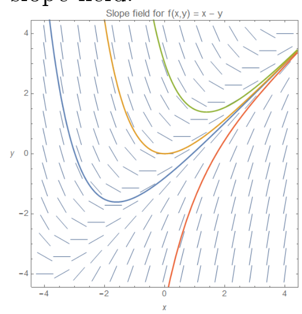
- Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = e^y + xy$ is guaranteed to have a solution, and where it is guaranteed to have a unique solution.
  - ○ All initial conditions lead to a solution, because $f(x, y) = e^y + xy$ is continuous everywhere.
  - ○ In fact, all initial conditions lead to a unique solution, because the partial derivative $f_y(x, y) = e^y + x$ is also continuous everywhere.
  - ○ It is, in fact, not possible to solve this equation explicitly using any of the techniques we will learn. Nonetheless, the theorem guarantees that it has a unique solution!

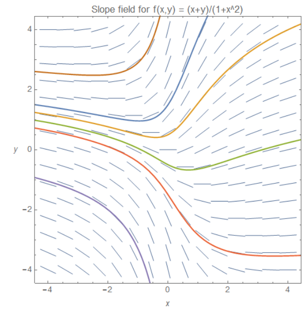
- Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = y^{2/3}$ is guaranteed to have a solution, and where it is guaranteed to have a unique solution.
  - ○ All initial conditions lead to a solution, because since $f(x, y) = y^{2/3}$ is continuous everywhere.
  - ○ However, the partial derivative $f_y = \frac{2}{3}y^{-1/3}$ is not continuous near $y = 0$, and so the solution is not guaranteed to be unique around $(a, 0)$ for any a, but unique otherwise.
  - ○ Remark: In fact, we can even write down two different solutions to the IVP $y' = y^{2/3}$ with $y(0) = 0$: namely, the constant function $y = 0$ and the function $y = \frac{1}{27}x^3$. (They both satisfy the equation and take the value zero at $x = 0$, but are clearly not the same function.)

- Example: Determine the initial conditions $y(a) = b$ for which the differential equation $y' = \sqrt{y - x}$ is guaranteed to have a solution, and where it is guaranteed to have a unique solution (where our solutions are taken to be real-valued).
  - ○ In order to have a solution, we need $f(x, y) = \sqrt{y - x}$ to be continuous in a rectangle containing (a, b). The function is not defined if $x > y$, and it is not continuous near any point with $x = y$ either, because any rectangle around a point (a, a) will capture some points with $x > y$.
  - ○ Thus, the solution is guaranteed to exist only for (a, b) with $b > a$.
  - ○ The partial derivative $f_y(x, y) = \frac{1}{2\sqrt{y - x}}$ is not defined if $x \geq y$ (since in addition to taking the square root of a negative number, we cannot divide by zero). So the solution is unique for (a, b) with $b > a$.
  - ○ Note: If we allow complex-valued solutions, then the function $f(x, y) = \sqrt{y - x}$ is defined and continuous on the entire plane, but gives non-real values when $x > y$. It is also possible to study differential equations over the complex numbers, but our theorem only applies when the functions are real-valued.

- If we know that a given initial value problem $y' = f(x, y)$ with $y(a) = b$ has a unique solution, we can use a geometric tool called a slope field to get a rough picture of the solution.
  - ○ To draw a slope field for the differential equation $y' = f(x, y)$, we choose a grid of points and, at each point (x_0, y_0) in the grid, we draw a short line segment having length $f(x_0, y_0)$.
  - ○ A solution curve to $y' = f(x, y)$ that passes through a point (x_0, y_0) in the grid will by definition have slope $f(x_0, y_0)$ at that point, so solution curves will (roughly speaking) follow the line segments as they move through the plane.

- Here are some typical slope fields with solution curves superimposed; notice how the solution curves "follow" the segments in the slope field:





  - Generally, one uses a computer when plotting slope fields, since it is very time-consuming to produce slope field plots by hand: even a 5×5 grid of points, which does not give much usable detail, will require evaluating 25 function values and then plotting 25 separate line segments.

1.3 Separable First-Order Equations

- One type of first-order equations we can solve explicitly is the class of separable equations. Before giving the formal definition, we will give an example.
- Example: Solve the initial value problem $y' = 2xy$ with $y(1) = 1$.
  - We rearrange the equation as $\dfrac{y'}{y} = 2x$, and then integrate both sides: $\int \dfrac{y'}{y}\,dx = \int x\,dx = x^2 + C_1$.
  - In the left integral we can make the substitution $u = y(x)$, with $u' = y'\,dx$, to obtain $\ln(y) + C_2 = x^2 + C_1$.
  - Moving the constants around gives $\ln(y) = x^2 + C$ for some constant C.
  - Plugging in the condition $y(1) = 1$ gives $0 = 1^2 + C$, so $C = -1$.
  - Thus, $\boxed{\ln(y) = x^2 - 1}$, so that $y = \boxed{e^{x^2-1}}$.
  - Remark: We can simplify the procedure slightly if instead we convert the statement $\dfrac{dy}{dx} = 2xy$ into the statement $\dfrac{dy}{y} = 2x\,dx$. We can then integrate both sides directly, to obtain the statement $\ln(y) = x^2 + C$.
- As one might expect, the above procedure can be generalized to a broad class of equations:
- Definition: A separable equation is of the form $y' = f(x) \cdot g(y)$ for some functions $f(x)$ and $g(y)$, or an equation equivalent to something of this form.
  - We can rearrange such an equation and then integrate both sides, in the same way as in the example above. We can simply the solving procedure slightly, as noted above: instead of making a substitution, we can use differentials.
- Here is the method for solving such equations:
  - Step 1: Replace y' with $\dfrac{dy}{dx}$, and then write the equation as $\dfrac{dy}{g(y)} = f(x)\,dx$.
  - Step 2: Integrate both sides (indefinitely), and place the $+C$ on the x side.
  - Step 3: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)
  - Step 4: Solve for y as a function of x, if required.
- Example: Solve $y' = k \cdot y$, where k is a constant.
  - Step 1: Rewrite as $\dfrac{dy}{y} = k\,dx$.

- Step 2: Integrate to get $\int \frac{dy}{y} = \int k\,dx$, which gives $\ln(y) = kx + C_1$.
- Step 4: Exponentiate to get $y = e^{kx+C_1} = \boxed{C_2 \cdot e^{kx}}$ for an arbitrary constant C_2.

- Example: Solve the differential equation $y' = e^{x-y}$.
  - Step 1: Using the identity $e^{x-y} = e^x/e^y$, we can rewrite the equation as $e^y\,dy = e^x\,dx$.
  - Step 2: Integrate to get $\int e^y\,dy = \int e^x\,dx$, which gives $e^y = e^x + C$.
  - Step 4: Take the natural logarithm to get $y = \boxed{\ln(e^x + C)}$.

- Example: Find y given that $y' = x + xy^2$ and $y(0) = 1$.
  - Step 1: Rewrite as $\frac{dy}{1+y^2} = x\,dx$.
  - Step 2: Integrate to get $\int \frac{dy}{1+y^2} = \int x\,dx$, which gives $\tan^{-1}(y) = \frac{1}{2}x^2 + C$.
  - Step 3: Plug in the initial condition to get $\tan^{-1}(1) = C$, so that $C = \pi/4$.
  - Step 4: Taking the natural logarithm gives $y = \boxed{\tan\left(\frac{1}{2}x^2 + \frac{\pi}{4}\right)}$.

1.4 Linear First-Order Equations

- Another type of first-order equations we can solve explicitly is the class of first-order linear equations, which (upon dividing by the coefficient of y') can be written in the general form $y' + P(x) \cdot y = Q(x)$, where $P(x)$ and $Q(x)$ are some functions of x.
- It would be very convenient if we could just integrate both sides to solve the equation. However, in general, we cannot: the y' term is easy to integrate, but the $P(x) \cdot y$ term is not.
- To fix this issue, we use an "integrating factor": we multiply by a function $I(x)$ which will turn the left-hand side into the derivative of a single function.
  - In other words, what we would want is for $I(x) \cdot y' + I(x)P(x) \cdot y$ to be the derivative of something nice.
  - When written this way, this sum looks sort of like the output of the product rule. If we can find $I(x)$ so that the derivative of $I(x)$ is $I(x)P(x)$, then this sum will be the derivative $\frac{d}{dx}[I(x) \cdot y]$.
  - To make this work, we need $I(x)P(x) = I'(x)$. This is a separable equation for the function $I(x)$, and we can see by inspection that one solution is $I(x) = e^{\int P(x)\,dx}$.
- Motivated by the above logic, here is the method for solving first-order linear equations:
  - Step 1: Put the equation into the form $y' + P(x) \cdot y = Q(x)$.
  - Step 2: Multiply both sides by the integrating factor $e^{\int P(x)\,dx}$ to get $e^{\int P(x)\,dx} y' + e^{\int P(x)\,dx} P(x) \cdot y = e^{\int P(x)\,dx} Q(x)$.
  - Step 3: Observe that the left-hand side is $\frac{d}{dx}\left[e^{\int P(x)\,dx} \cdot y\right]$, and take the antiderivative on both sides. (Don't forget the constant of integration C.)
  - Step 4: If given, plug in the initial condition to solve for the constant C. (Otherwise, just leave it where it is.)
  - Step 5: Solve for y as a function of x.
- Example: Find y given that $y' + 2xy = x$ and $y(0) = 1$.
  - Step 1: We have $P(x) = 2x$ and $Q(x) = x$.

- Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{x^2}$ to get $e^{x^2}y' + e^{x^2} \cdot 2x \cdot y = x \cdot e^{x^2}$.
- Step 3: Taking the antiderivative on both sides yields $e^{x^2}y = \frac{1}{2}e^{x^2} + C$.
- Step 4: Plugging in yields $e^0 \cdot 1 = \frac{1}{2}e^0 + C$ hence $C = \frac{1}{2}$.
- Step 5: Solving for y gives $y = \boxed{\frac{1}{2} + \frac{1}{2}e^{-x^2}}$.

- Example: Find all functions y for which $xy' = x^4 - 4y$.
  - Step 1: We have $y' + \frac{4}{x}y = x^3$, so $P(x) = \frac{4}{x}$ and $Q(x) = x^3$.
  - Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{4\ln(x)} = x^4$ to get $x^4y' + 4x^3y = x^7$,
  - Step 3: Taking the antiderivative on both sides yields $x^4y = \frac{1}{8}x^8 + C$.
  - Step 5: Solving for y gives $y = \boxed{\frac{1}{8}x^4 + C \cdot x^{-4}}$.

- Example: Find y given that $y' \cdot \cot(x) = y + 2\cos(x)$ and $y(0) = -\frac{1}{2}$.
  - Step 1: We have $y' - y\,\tan(x) = 2\sin(x)$, with $P(x) = -\tan(x)$ and $Q(x) = 2\sin(x)$.
  - Step 2: Multiply both sides by $e^{\int P(x)\,dx} = e^{\ln(\cos(x))} = \cos(x)$ to get $y' \cdot \cos(x) - y \cdot \sin(x) = 2\sin(x)\cos(x)$.
  - Step 3: Taking the antiderivative on both sides yields $[y \cdot \cos(x)] = -\frac{1}{2}\cos(2x) + C$.
  - Step 4: Plugging in yields $-\frac{1}{2} = -\frac{1}{2} \cdot 1 + C$ hence $C = 0$.
  - Step 5: Solving for y gives $y = \boxed{-\frac{\cos(2x)}{2\cos(x)}}$.

1.5 Applications of First-Order Equations

- In this section we discuss a few common applications of first-order differential equations: specifically, to population modeling, mixing problems, Newtonian mechanics, Newton's law of cooling, and electrical circuit analysis.

1.5.1 Population Modeling

- A basic population model is to have the growth rate of the population be proportional to the size of the population.
- Example: Solve the differential equation $P' = kP$ where k is a positive constant.
  - This is a separable equation, which we can rewrite as $\frac{dP}{P} = k\,dt$.
  - Integrating both sides yields $\ln(P) = kt + C_1$, and exponentiating gives $\boxed{P(t) = C_2e^{kt}}$.
  - If we want to satisfy the initial condition $P(0) = P_0$, we simply have $C_2 = P_0$ so the solution is $\boxed{P(t) = P_0e^{kt}}$.
- The above population model is rather unrealistic because once the population is sufficiently large, the individuals will begin competing for resources.
  - In general, the number of "competitions" would grow at a rate proportional to P^2, since that is the approximate number of pairs of individuals that will be competing with one another.

- So a better population model has the form $P' = aP - bP^2$ for some constants a and b. For various reasons, it is more useful to write this equation in the form $P' = kP(M-P)$ where $M = a/b$ and $k = b$.

- Example: Solve the differential equation $P' = kP(M-P)$, where k and M are positive constants.
  - This equation is separable, and we can rewrite it as $\dfrac{M\,dP}{P(M-P)} = kM\,dt$.
  - Integrating both sides yields $\int \dfrac{M}{P(M-P)}\,dP = \int kM\,dt$.
    - To evaluate the P-integral, we use partial fraction decomposition: $\dfrac{M}{P(M-P)} = \dfrac{1}{P} + \dfrac{1}{M-P}$.
    - The result is $\ln(P) - \ln(M-P) = kMt + C_1$.
  - We can combine the logarithms to obtain $\ln\left(\dfrac{P}{M-P}\right) = kMt + C_1$, and then exponentiate to get $\dfrac{P}{M-P} = C_2 e^{kMt}$.
  - Solving for P yields, finally, $\boxed{P(t) = \dfrac{M}{1 + C_2 e^{-kMt}}}$.
  - If we want to satisfy the initial condition $P(0) = P_0$, then plugging in shows $C = \dfrac{M}{P_0} - 1$, and then the solution can be rewritten in the form $\boxed{P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kMt}}}$.

- The differential equation $P' = kP(M-P)$ that we just analyzed is called the logistic equation, and is an example of an autonomous equation, which is a first-order equation of the form $y' = f(y)$ for some function f. (It is called "autonomous" because the independent variable does not appear anywhere.)
  - Any autonomous equation is separable and thus solvable in principle: however, it can happen that the resulting integral is too difficult to evaluate exactly.
  - We can get some idea of the solutions by looking for equilibrium solutions (also called steady state solutions or critical points): they are the solutions of the form $y(t) = c$, for some constant c.
  - The autonomous equation $y' = f(y)$ will have an equilibrium solution precisely when $f(y) = 0$.
  - For the logistic equation $P' = kP(M-P)$, we can easily see that $P = 0$ and $P = M$ are the only equilibrium solutions.
  - From the expression $P' = kP(M-P)$, we see that if $0 < P < M$ then $P' > 0$, and that $P > M$ then $P' < 0$. In other words, any positive starting population P will have $P(t)$ tending to move toward the value M as t increases.
  - This can also be seen from the explicit formula: as $t \to \infty$ we can easily compute that $P(t) \to M$ provided that $P_0 > 0$.

1.5.2 Mixing Problems

- There are a number of common examples that can all be classified as "mixing problems" whose setup is as follows:
  - We have some reservoir (pool, lake, ocean, planet, room) of liquid (water, gas) which has some substance (pollution, salt) dissolved in it.
  - The reservoir starts at an initial volume V_0 and there is an initial amount of substance y_0 in the reservoir.
  - We have some amount of liquid $\text{In}(t)$ flowing in with a given concentration $k(t)$ of the substance, and some other amount of liquid $\text{Out}(t)$ flowing out.
  - We assume that the substance is uniformly and perfectly mixed in the reservoir, and are asked to find the amount $y(t)$ of the substance that remains in the reservoir after time t.

- Of course, the amount of liquid flowing in or out may be constants (i.e., not depend on time), and similarly the concentration of the liquid flowing in could also be a constant.

- The only difficulty in solving the general mixing problem is that it requires a bit of careful bookkeeping to convert between concentrations and total amounts:
  - Let $V(t)$ be the total volume of the reservoir and $y(t)$ be the total amount of substance in the reservoir, with initial volume V_0 and initial amount y_0. Also let $\text{In}(t)$ be the volume flowing in, with concentration $k(t)$ of the substance, and $\text{Out}(t)$ be the volume flowing out.
  - The net change in the volume at time t is $\text{In}(t) - \text{Out}(t)$, so $V'(t) = \text{In}(t) - \text{Out}(t)$.
  - Since the concentration of substance moving into the reservoir is $k(t)$ and the total volume of liquid moving into the reservoir is $\text{In}(t)$, the total amount of substance moving in is $k(t) \cdot \text{In}(t)$.
  - Similarly, the concentration of substance in the reservoir is $\dfrac{y(t)}{V(t)}$, so since the volume of liquid moving out of the reservoir is $\text{Out}(t)$, the total amount of substance moving out is $\dfrac{y(t)}{V(t)} \cdot \text{Out}(t)$.
  - Thus, the net change in the amount of substance at time t is $k(t) \cdot \text{In}(t) - \dfrac{y(t)}{V(t)} \cdot \text{Out}(t)$, so $y'(t) = k(t) \cdot \text{In}(t) - \dfrac{y(t)}{V(t)} \cdot \text{Out}(t)$.
  - So we obtain two equations: $V'(t) = \text{In}(t) - \text{Out}(t)$ and $y'(t) = k(t) \cdot \text{In}(t) - \dfrac{y(t)}{V(t)} \cdot \text{Out}(t)$.
  - Solving these equations is then straightforward: we can simply integrate to find $V(t)$ explicitly, and then we can rewrite the other equation as $y' + \dfrac{\text{Out}(t)}{V(t)} \cdot y = k(t) \cdot \text{In}(t)$, which is first-order linear.

- <u>Example</u>: A small room with a volume of 1000 liters contains air containing 40% oxygen. The air is filtered at a rate of 10 liters per second and returned to the room with a concentration of 10% oxygen. Assuming the air is uniformly mixed, determine the amount of time necessary before the air in the room drops to a 20% oxygen concentration.
  - We have $\text{In}(t) = \text{Out}(t) = 10\text{L/s}$ and $V(t) = 1000\text{L}$, since it is stated that the air pumped out is immediately returned. We also have $k(t) = 0.1$, and $y_0 = (1000\text{L})(0.4) = 400\text{L}$.
  - Then from our analysis of the general case, we have $y'(t) + \dfrac{10\text{L/s}}{1000\text{L}} \cdot y = (0.1) \cdot (10\text{L/s})$, which can be rewritten as $y'(t) + \dfrac{1\,s^{-1}}{100} y(t) = 1\text{L/s}$.
  - We obtain the differential equation $y'(t) + \dfrac{1}{100} y(t) = 1$, in liters per second.
  - The integrating factor is $I = e^{\int (1/100)\,dt} = e^{t/100}$, so scaling gives the equation $e^{t/100} y'(t) + \dfrac{1}{100} e^{t/100} y(t) = e^{t/100}$.
  - Integrating both sides yields $e^{t/100} y(t) = 100 e^{t/100} + C$.
  - Plugging in the initial condition $y(0) = 400$ gives $1 \cdot 400 = 100 + C$ so $C = 300$.
  - Thus, we obtain $e^{t/100} y(t) = 100 e^{t/100} + 300$, so $y(t) = 100 + 300 e^{-t/100}$.
  - The concentration is then $\dfrac{y(t)}{V(t)} = \dfrac{1}{10} + \dfrac{3}{10} e^{-t/100}$. This quantity is equal to $20\% = 0.2$ when $\dfrac{3}{10} e^{-t/100} = \dfrac{1}{10}$, so that $e^{t/100} = 3$, whence $t = \boxed{100 \ln(3)\,\text{s} \approx 110\text{s}}$.

- <u>Example</u>: A large tank contains 600 liters of a 0.01kg/L salt solution. A 0.2kg/L salt solution is pumped into the tank at a rate of 10 liters per second and, simultaneously, the tank is drained at a rate of 5 liters per second. Determine the concentration of the salt solution in the tank as a function of time.
  - We have $\text{In}(t) = 10\text{L/s}$, $\text{Out}(t) = 5\text{L/s}$, $V_0 = 600\text{L}$, $k(t) = 0.2\text{kg/L}$, and $y_0 = 0.01\text{kg/L} \cdot 600\text{L} = 6\text{kg}$.

- From our analysis of the general case, we have $V'(t) = 5\,\text{L/s}$ and $V_0 = 600\text{L}$, so $V(t) = (600 + 5t)\text{L}$.
- Furthermore, we have $y'(t) + \dfrac{5\,\text{L/s}}{(600+5t)\text{L}} y(t) = (0.2\,\text{kg/L})(10\,\text{L/s})$, which can be rewritten as $y'(t) + \dfrac{1\,\text{s}^{-1}}{120+t} y(t) = 2\,\text{kg/s}$.
- We obtain the differential equation $y'(t) + \dfrac{1}{120+t} y(t) = 2$, in kilograms per second.
- The integrating factor is $I = e^{\int 1/(120+t)\,dt} = e^{\ln(120+t)} = 120 + t$, so we obtain $(120+t)y'(t) + y(t) = 240 + 2t$.
- Integrating both sides yields $(120+t)y(t) = 240t + t^2 + C$.
- Plugging in the initial condition $y(0) = 6$ gives $120 \cdot 6 = C$ so $C = 720$.
- Thus, $(120+t)y(t) = 240t + t^2 + 720$, so $y(t) = \dfrac{t^2 + 240t + 720}{t + 120}$, and then the concentration at time t is $\dfrac{y(t)}{V(t)} = \boxed{\dfrac{t^2 + 240t + 720}{(t+120)(600+5t)}\,\text{kg/L}}$.
- The concentration can be simplified to the expression $\boxed{0.2 - \dfrac{2736}{(t+120)^2}\,\text{kg/L}}$ after some arithmetic. In particular, we can see that as $t \to \infty$ the concentration approaches $0.2\,\text{kg/L}$. (This makes intuitive sense since as $t \to \infty$, most of the solution in the tank will have originated from the $0.2\,\text{kg/L}$ solution that was pumped in.)

1.5.3 Applications of Differential Equations in Physics

- Much of classical physics can be essentially reduced to the analysis of various differential equations.
- In kinematics, which is the branch of classical physics involving the study of motion and forces in a Newtonian (i.e., nonrelativistic) universe, the key ingredient is Newton's second law, which states that $F = ma$, where F is the total sum of forces acting on an object whose mass is m and whose acceleration is a.
  - By writing down all the forces acting on an object and relating them to the object's position and velocity, one obtains a differential equation whose solution will characterize the object's motion.
  - Typically the resulting differential equation is second-order (since acceleration is the second derivative of position) but in some cases it is possible to solve the equation using the techniques we have developed for first-order equations.
- Example: A ball of mass $m\,\text{kg}$ is dropped in a vacuum from an initial height h meters and initial velocity $v_0\,\text{m/s}$ upward, and the force exerted by gravity is mg newtons (downward). Find the position of the ball t seconds after it is dropped.
  - If $x(t)$ is the particle's position then its velocity is $v(t) = x'(t)$ and its acceleration is $a(t) = v'(t) = x''(t)$.
  - The only force acting on the ball is the force of gravity, so from Newton's second law we see that $-mg\,\text{N} = (m\,\text{kg})\,a(t)$, from which we see that $a(t) = -g\,\text{m/s}^2$.
  - We therefore have a (very easy) initial value problem $v'(t) = -g\,\text{m/s}^2$ with $v(0) = v_0\,\text{m/s}$: to solve it, we simply integrate and plug in the initial value to see that $v(t) = (-gt + v_0)\,\text{m/s}$.
  - This yields another initial value problem for the position: $x'(t) = (-gt + v_0)\,\text{m/s}$ with $x(0) = x_0$. Again, integrating and plugging in the initial value yields the solution $x(t) = \boxed{\left(-\frac{1}{2}gt^2 + v_0 t + x_0\right)\text{m}}$.
- Example: A ball of mass $m\,\text{kg}$ is dropped in atmosphere from an initial height h meters and initial velocity $v_0\,\text{m/s}$ upward, and the force exerted by gravity is again mg newtons (downward). The force of air resistance is proportional to the velocity: specifically, it is equal to $k\,v$ newtons (opposite the direction of motion) where v is the velocity. Find the position and velocity of the ball t seconds after it is dropped.

- Note that the velocity of the ball is negative, since it is moving downward. The force of air resistance is oriented upward since it is opposite the direction of motion.
- So from Newton's second law we see that $-mg\,\text{N} - kv\,\text{N} = (m\,\text{kg})\,a(t)$, from which $a(t) = \left(-g - \frac{k}{m}v\right)\,\text{m/s}^2$.
- Once again we have an initial value problem: $v'(t) = -g - \frac{k}{m}v$, with $v(0) = v_0$. This equation is first-order linear, and putting it in standard form yields $v' + \frac{k}{m}v = -g$.
- The integrating factor is $I = e^{\int (k/m)\,dt} = e^{(k/m)t}$.
- Scaling by it gives $e^{(k/m)t}v' + \frac{k}{m}e^{(k/m)t}v = -ge^{(k/m)t}$, and then integrating both sides yields $e^{(k/m)t}v = -\frac{gm}{k}e^{(k/m)t} + C$ so that $v = -\frac{gm}{k} + C\,e^{-(k/m)t}$.
- Plugging in the initial condition $v(0) = v_0$ yields $v_0 = -\frac{gm}{k} + C$, whence $C = v_0 + \frac{gm}{k}$.
- So our solution is $v = \boxed{-\frac{gm}{k} + \left(v_0 + \frac{gm}{k}\right)e^{-(k/m)t}}$.
- For the position we can now simply integrate and plug in the initial condition. After some simplification one obtains the result $x = \boxed{h - \frac{gm}{k}t - \frac{m}{k}\left(v_0 + \frac{gm}{k}\right)\left[1 - e^{(-k/m)t}\right]}$.
- It is a bit difficult to analyze the behavior of this quantity directly: for example, even with specific values for all the parameters, it is not generally possible to solve exactly for when the ball will hit the ground.
- However, we can certainly make a few qualitative observations: first, notice that as $t \to \infty$, the velocity approaches the value $-\frac{gm}{k}$: this quantity is called the "terminal velocity" of an object falling in air resistance.
  - In fact, one can compute the terminal velocity without using the solution to the differential equation: when an object is falling at terminal velocity, the force of gravity is exactly balanced by the air resistance.
  - This occurs precisely when $-mg - kv = 0$: that is, when $v = -\frac{gm}{k}$.
- Second, observe that if $g = 0$ (i.e., there is no gravity, and the only force is from air resistance), the position of the object approaches the value $h - \frac{mv_0}{k}$ as $t \to \infty$: in other words, the object will only move the finite distance $\frac{mv_0}{k}$ away from its starting position.

- Another common problem in physics is studying the properties of heat and energy transfer.
- As a simple example, Newton's law of cooling states that, under thermal conduction, the rate of heat transfer between a body and its environment is proportional to the difference in temperature between the body and the environment.
  - If T is the temperature of the object, k is the proportionality coefficient (with units of inverse time), and T_{env} is the temperature of the environment, then Newton's law of cooling can be written as $\frac{dT}{dt} = -k(T - T_{\text{env}})$.
  - If the temperature of the environment is constant, then the resulting differential equation is both separable and first-order linear. Using either method yields the solution $T(t) = \boxed{T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}}$, where T_0 is the initial temperature of the object.
  - We can see that, for any starting temperature, the difference $T - T_{\text{env}}$ will follow an exponential decay, meaning that temperature of the object will approach the temperature of the environment exponentially fast.
- Example: A boiling pot of water (100 Celsius) is placed outside in below-freezing weather (-20 Celsius) and it takes 1 hour for the temperature to drop to 40 Celsius. What will be the temperature of the water after another 1 hour outside? Another 2 hours?

- We could set up the problem and use the formula above to compute the rate constant and eventually obtain the formula $T(t) = -20 + 120 \cdot e^{-t\ln(2)}$, where t is measured in hours.
- However, it is not necessary to do this much work: we can instead observe that after 1 hour, the initial temperature difference of 120 degrees has dropped to 60 degrees, representing a 50% decrease.
- After each additional hour, the exponential decay dictates that the temperature difference will decrease by the same proportion: so after 1 hour, the difference will drop to 30 degrees, and after 2 hours, the difference will be 15 degrees.
- So after 1 hour, the temperature of the water is $\boxed{10°\text{C}}$, and after 2 hours the temperature is $\boxed{-5°\text{C}}$.

* A third problem in physics is to study the properties of electrical circuits involving various different types of components.

* In circuit analysis, the key ingredient is Kirchhoff's second law, which states that the total sum of the voltage drops around any closed circuit is zero. In order to use this law, one needs to be given the values of voltage drops across circuit components.
  - Recall that $q(t)$ denotes electrical charge measured in coulombs (C), and $i(t) = dq/dt$ denotes electrical current measured in amperes (A).
  - A resistor will resist the flow of charge through it in direct proportion to the current. Specifically, by Ohm's law, the voltage drop across a resistor is $\Delta V_R = Ri$ where R is the resistance in ohms (Ω).
  - A capacitor will store charge and resist the passage of current in direct proportion to the amount of electrical charge. The voltage drop across a capacitor is $\Delta V_C = \frac{1}{C}q$ where C is the capacitance of the capacitor in farads (F).
  - An inductor will resist a change in the electrical current in direct proportion to the rate of change of the current. The voltage drop across an inductor is $\Delta V_L = L\frac{di}{dt}$, where L is the inductance of the inductor in henrys (H).
  - A voltage source (such as a battery creating a direct current, or an alternator creating an alternating current) will produce an electromotive force. The voltage drop across the source is equal to $\Delta V_E = -E(t)$ for some function $E(t)$ in volts (V).

* In a simple RLC circuit, containing a resistor, capacitor, inductor, and a voltage source connected in series in a circle, applying Kirchhoff's law immediately yields the relation $L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$.
  - If we write everything in terms of q, we get a second-order linear differential equation for q, namely $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$.
  - At present we cannot give the general solution to a second-order linear equation, but in the special cases of an RL circuit or an RC circuit, the equation becomes a first-order linear equation in i or in q (respectively) that we can solve using the techniques we have developed.

* Example: A battery giving a constant voltage of $E(t) = 40$V is connected in series to a resistor of resistance 20Ω and an inductor of inductance 1H. If the initial current in the circuit is $i(0) = 3$A, find the current after t seconds.
  - From the analysis above, we obtain the equation $\frac{di}{dt} + 20i = 40$, with both sides measured in volts.
  - This is a first-order linear equation with integrating factor $I = e^{\int 20\,dt} = e^{20t}$, so scaling by it yields $e^{20t}\frac{di}{dt} + 20e^{20t}i = 40e^{20t}$.
  - Integrating both sides gives $e^{20t}i = 2e^{20t} + C$, so that $i = 2 + C\,e^{-20t}$. Plugging in the initial condition yields $3 = 2 + C$ from which $C = 1$, so $i(t) = \boxed{2 + e^{-20t}}$.

Well, you're at the end of my handout. Hope it was helpful.