Contents

4 Applications of Cauchy's Integral Formula

In this chapter, our goal is to use Cauchy's integral formula $f(z_0) = \int_\gamma \frac{f(z)}{z - z_0}\,dz$ to study the behavior of holomorphic functions in a variety of ways. We begin by giving estimates on the growth rate of holomorphic functions and then studying the various behaviors near zeroes and singularities that holomorphic functions may have. We then give a lengthy discussion of residue calculus and how to evaluate a wide range of otherwise very difficult-to-evaluate integrals on the real line.

4.1 Estimates for Holomorphic Functions

- In this section we will give various estimates resulting from Cauchy's integral formula that will help us characterize useful properties of holomorphic functions.

4.1.1 The Cauchy Estimates

- Our first goal is to use Cauchy's integral formula to provide estimates on the growth rate for the value of a holomorphic function.
  - Recall that if f is holomorphic on the interior of a simply-connected region with counterclockwise boundary γ, then for each z_0 in the region and each $n \geq 0$ we have $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz$.
- Theorem (Cauchy Estimates): Suppose R is an open region and f is holomorphic on R.

1. Suppose $z_0 \in R$ and that the closed disc $D : |z - z_0| \leq r$ is contained in R. If $|f(z)| \leq M$ on D, then $\left|f^{(n)}(z_0)\right| \leq \dfrac{M \cdot n!}{r^n}$.
   - Proof: If we take γ to be the counterclockwise boundary of the disc, then by our usual arclength estimate, we have $\left|f^{(n)}(z_0)\right| = \left|\dfrac{n!}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \dfrac{n!}{2\pi} \cdot 2\pi r \cdot \dfrac{M}{r^{n+1}} = \dfrac{M \cdot n!}{r^n}$.
2. Suppose that f is holomorphic on the closed disc $|z - z_0| \leq r$. Then the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f centered at $z = z_0$ has radius of convergence at least r.
   - Proof: Since holomorphic functions are analytic, we may write $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for constants $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, and applying (1) yields $|a_n| \leq \dfrac{M}{r^n}$.
   - Then $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1/r$, meaning the radius of convergence of the power series is at least r, as claimed.
3. Suppose that f is holomorphic on the open disc $|z - z_0| < r$ but on no larger open disc. Then the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for f centered at $z = z_0$ has radius of convergence equal to r.
   - Proof: By applying (2) to the closed discs $|z - z_0| \leq s$ for $0 < s < r$ we see that the radius of convergence is at least s for all $s < r$, so the radius of convergence must be at least r.
   - On the other hand, if the radius were greater than r, then since analytic functions are holomorphic, f would be holomorphic on a disc

- Part (3) of the theorem provides, in many cases, an easy way to identify the radius of convergence of the power series of a function whose holomorphic behavior is already known, such as a rational function:

- Example: Find the radius of convergence of the power series centered at $z = 0$ for $f(z) = \dfrac{1}{1-2z}$.
  - The function f is holomorphic for $z \neq 1/2$, so by (3) of the theorem above, the function is holomorphic for $|z| < 1/2$ but not for $|z| \leq 1/2$ and therefore not for any $|z| < r$ with any $r > 1/2$.
  - Thus, the radius of convergence must equal $\boxed{1/2}$.
  - We can confirm this fact by computing the power series itself, which is $\sum_{n=0}^{\infty}(2z)^n$ so that $a_n = 2^n$: then indeed $\limsup_{n\to\infty} |a_n|^{1/n} = 2$ so the radius is $1/2$.

- Example: Find the radius of convergence of the power series centered at $z = 0$ for $f(z) = \dfrac{1}{1+z^2}$.
  - The function f is holomorphic for $z \neq \pm i$, so by (3) of the theorem above, the function is holomorphic for $|z| < 1$ but not for $|z| \leq 1$ and therefore not for any $|z| < r$ with any $r > 1$.
  - Thus, the radius of convergence must equal $\boxed{1}$.
  - We can confirm this fact by computing the power series itself, which is $\sum_{n=0}^{\infty}(-1)^n z^{2n}$: then indeed $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ so the radius is 1.
  - Interestingly, if we consider this function's power series only on the real line, it still has radius of convergence 1, yet the function is infinitely differentiable on all of $\mathbb{R}$.
  - The reason for this otherwise mysterious lack of convergence for this power series on the entire real line, however, becomes quite obvious when we extend our view to the complex plane: as we saw above, the roots $z = \pm i$ of the denominator prevent this function from being holomorphic beyond $|z| < 1$.

4.1.2 Entire Functions, Liouville's Theorem, and the Fundamental Theorem of Algebra

- The Cauchy estimates, in particular, place very strong restrictions on the behavior of functions that are holomorphic on the entire complex plane. Such functions are given a special name:

- Definition: A function $f(z)$ holomorphic on the entire complex plane is called an entire function (or just entire).

- Examples: Any polynomial in z is entire, as are e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$.
- Non-Examples: The functions $1/z$, $\mathrm{Log}(z)$, and $\csc z$ are not entire, since none of them are holomorphic at $z = 0$. The function $\tan z$ is not entire since it is not holomorphic at $z = \pi/2$. The function $\sum_{n=0}^{\infty} z^n$ is not entire since it is not holomorphic (or even defined) for $|z| > 1$.

- A fundamental result of Liouville is that the only bounded entire functions are constants:
- Theorem (Growth Estimates): Suppose $f(z)$ is an entire function.

1. (Liouville's Theorem) A bounded entire function is constant.
   - Proof: Suppose $f(z)$ is entire and assume that $|f(z)| \le M$ on $\mathbb{C}$.
   - Applying the Cauchy estimate to f' on the circle of radius r centered at z_0 yields $|f'(z_0)| \le \dfrac{M}{r}$. Since this bound holds for all $r > 0$, taking $r \to \infty$ yields $|f'(z_0)| = 0$ hence $f'(z_0) = 0$.
   - This applies for all z_0, so f' is identically zero, and so f must be constant.
2. If there exist constants A, B, and n such that $|f(z)| \le A + B\,|z|^n$ for all z, then $f(z)$ is a polynomial of degree at most n.
   - Notice that Liouville's theorem is the special case $n = 0$ of this result.
   - Proof: Suppose $f(z)$ is entire and assume that $|f(z)| \le A + B\,|z|^n$ on $\mathbb{C}$.
   - Applying the Cauchy estimate to $f^{(n+1)}$ on the circle of radius r centered at z_0 yields $\left|f^{(n+1)}(z_0)\right| \le \dfrac{n!(A + Br^n)}{r^{n+1}} = n!(A/r^{n+1} + B/r)$. Since this bound holds for all $r > 0$, taking $r \to \infty$ yields $\left|f^{(n+1)}(z_0)\right| = 0$ hence $f^{(n+1)}(z_0) = 0$.
   - This applies for all z_0, so $f^{(n+1)}$ is identically zero. Taking the antiderivative $n+1$ times shows that f must be a polynomial of degree at most n, as claimed.

- By using Liouville's theorem, we can give a very quick proof of the fundamental theorem of algebra. We first make some preliminary remarks.
  - First, recall the remainder/factor theorem[1]: if $p(z)$ is a polynomial and $r \in \mathbb{C}$, then the remainder upon dividing $p(z)$ by $z - r$ is $p(r)$. In particular, $z - r$ divides $p(z)$ if and only if $p(r) = 0$.
  - The usual statement of the fundamental theorem of algebra over $\mathbb{C}$ is that any polynomial $p(z)$ of degree d can be factored into the form $p(z) = a(z - r_1)(z - r_2)\cdots(z - r_d)$ for some (not necessarily distinct) complex numbers r_i.
  - In order to prove this fact, it suffices to show that every nonconstant polynomial $p(z)$ has a root in $\mathbb{C}$: then an easy induction using the remainder/factor theorem to remove a linear factor $z - r$ establishes the general form above.
  - We now establish that every nonconstant complex polynomial has a root in $\mathbb{C}$.

- Theorem (Fundamental Theorem of Algebra): Every nonconstant complex polynomial has a root in $\mathbb{C}$.
  - Proof: Suppose $p(z)$ is a nonconstant complex polynomial with no root in $\mathbb{C}$. Then $1/p(z)$ is entire hence in particular continuous on $\mathbb{C}$.
  - From writing $p(z) = \sum_{n=0}^{d} a_n z^n = a_d z^d(1 + a_{d-1}/z + \cdots + a_0/z^d)$ we see $\lim_{z\to\infty} p(z)/z^d = a_d$, and since $d > 0$ this means $\lim_{|z|\to\infty} |p(z)| = \infty$.
  - Taking the reciprocal yields $\lim_{|z|\to\infty} |1/p(z)| = 0$, so in particular there exists some R such that $|1/p(z)| \le 1$ for $|z| \ge R$.
  - Additionally, for $|z| \le R$ the function $|1/p(z)|$ is continuous on a closed bounded region, hence it is bounded above, say with $|1/p(z)| \le M$.
  - Then $|1/p(z)| \le \max(1, M)$ for all $z \in \mathbb{C}$, meaning that $1/p(z)$ is a bounded entire function. It is therefore constant by Liouville's theorem. But this is a contradiction if $p(z)$ is not constant. We conclude that $p(z)$ must have a complex root, as desired.

[1] Purely for completeness, if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, then $p(z) - p(r) = \sum_{k=0}^{n} a_k(z^k - r^k)$, and since $z - r$ divides each term in the sum, it divides $p(z) - p(r)$.

4.1.3 The Maximum Modulus Principle

- Cauchy's integral formula, as we have previously mentioned, can also be interpreted as an averaging result.
  - Specifically, if $f(z)$ is holomorphic for $|z - z_0| \le r$, then the value $f(z_0)$ is equal to the average value of $f(z)$ on the circle $|z - z_0| = r$.
  - In particular, then, we see that $f(z_0)$ cannot be strictly larger in absolute value than all of the values $f(z)$ on the circle $|z - z_0| = r$ (otherwise, we would have a contradiction to the triangle inequality). Since this holds for each possible radius, we see that $f(z_0)$ cannot be strictly larger in absolute value than the value of $f(z)$ for any z in the disc $|z - z_0| \le r$.
  - By applying the argument to smaller discs centered at other points, we can see that *no* interior point of the disc can be a point where f takes its "maximum modulus".
- We may state this principle more formally:
- Theorem (Maximum Modulus Principle): Suppose $f(z)$ is holomorphic on a connected bounded region R.
  1. If $|f(z)|$ is constant on R, then $f(z)$ is constant on R.
     - Proof: We have $0 = \dfrac{\partial}{\partial z}\left[f(z)\overline{f(z)}\right] = f'(z)\overline{f(z)} + f(z)\dfrac{\partial \overline{f}}{\partial z} = f'(z)\overline{f(z)}$ since $\dfrac{\partial \overline{f}}{\partial z} = \overline{\dfrac{\partial f}{\partial \overline{z}}} = 0$ since f is holomorphic.
     - This either requires $f'(z)$ be zero or $\overline{f(z)}$ (hence $f(z)$) be zero along a sequence of points converging to a limit in R, which by the uniqueness result for power series implies $f'(z)$ or $f(z)$ must be identically zero on R.
     - In either case we see that $f(z)$ is constant on R, as claimed.
  2. If R is closed and the maximum value of $|f(z)|$ occurs at a point z_0 in the interior of R, then f is constant on R.
     - Proof: Suppose $|f(z)|$ is maximized at $z = z_0$ in the interior of R. By definition there exists some $r > 0$ such that the disc $|z - z_0| \le r$ is contained in R.
     - By Cauchy's integral formula using the parametrization of the contour γ_r winding once counterclockwise around $|z - z_0| = r$, we see $f(z_0) = \dfrac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$, meaning that $f(z_0)$ is the average value of $f(z)$ on the circle.
     - By the triangle inequality we then have $|f(z_0)| = \left|\dfrac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta\right| \le \dfrac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right| d\theta \le \dfrac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, d\theta = |f(z_0)|$, which means we have equality everywhere. This means $|f(z)|$ is constant on the circle $|z - z_0| = r$ hence by (1), $f(z)$ is constant on the circle.
     - Since $f(z)$ is constant on a curve, by uniqueness of series expansions this means f is constant on all of R, as claimed.
  3. (Maximum Modulus Principle) The maximum value of $|f(z)|$ on R occurs on the boundary of R.
     - Proof: This follows immediately by noting the result is trivial if f is constant, and then by taking the contrapositive of (2) if f is not constant.
- We can apply the maximum modulus principle to simplify certain kinds of optimization problems.
- Example: Find the maximum value of $\left|z^2 + 4z - 2\right|$ for $|z| \le 1$.
  - A natural first attempt is to use the triangle inequality to write $\left|z^2 + 4z - 2\right| \le \left|z^2\right| + |4z| + |-2| \le 7$. However, this calculation cannot be sharp, because for $|z| = 1$ we can only have $|4z - 2| = 6$ for $z = -1$, but then $\left|z^2 + 4z - 2\right| = 5$ rather than 7.
  - A direct approach would be to set $z = re^{i\theta}$ and then compute $\left|z^2 + 4z - 2\right|^2 = \left|r^2 e^{2i\theta} + 4re^{i\theta} - 2\right|^2 = \left|(r^2\cos 2\theta + 4r\cos\theta - 2) + (r^2 \sin 2\theta + 4r\sin\theta)i\right|^2 = (r^2\cos 2\theta + 4r\cos\theta - 2)^2 + (r^2\sin 2\theta + 4r\sin\theta)^2$ but this ends up being rather unpleasant to maximize for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, as one may quickly discover by attempting to take partial derivatives.

- Using the maximum modulus principle, however, we can reduce to having a single parameter: since the function $f(z) = z^2 + 4z - 2$ is holomorphic, its maximum modulus on the closed disc $|z| \leq 1$ necessarily occurs on the boundary $|z| = 1$.
- Then we have the simpler optimization $\left|f(e^{i\theta})\right|^2 = (e^{2i\theta}+4e^{i\theta}-2)(e^{-2i\theta}+4e^{-i\theta}-2) = -8\cos\theta - 4\cos 2\theta + 21$, which necessarily takes its extreme values when the derivative $8\sin\theta + 8\sin 2\theta = 8\sin\theta(1 + 2\cos\theta)$ is zero. By basic trigonometry this occurs for $\theta = 0, \pi, 2\pi/3, 4\pi/3, 2\pi$, and one may then check that $\theta = 2\pi/3$ and $4\pi/3$ yield maxima of $\left|z^2 + 4z - 2\right| = \boxed{\sqrt{27}}$ occurring for $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

* We will also remark that the maximum modulus principle does not hold on unbounded domains (at least, without modification).
  - For a specific example, consider the function $f(z) = e^{e^z}$ on the region $-\pi/2 \leq \mathrm{Im}(z) \leq \pi/2$.
  - On the boundary of this region, with $z = x \pm i\pi/2$, we have $e^z = \pm ie^x$ and so $|f(z)| = \left|e^{\pm ie^x}\right| = 1$, so f is bounded on the boundary of the region.
  - However, on the real axis, we see that f is unbounded. Thus, f does not attain its maximum modulus on the boundary of the unbounded region R.
  - The main issue is that the function $f(z)$ grows extremely quickly on the real axis. If one imposes suitable restrictions on the growth rate of f inside R, one may recover versions of the maximum modulus principle.
* One result of this nature is due to Lindelöf:
* Theorem (Lindelöf Principle): Let R be a half-strip region of the form $x_1 \leq \mathrm{Re}(z) \leq x_2$ and $\mathrm{Im}(z) \geq y_1$ for some real x_1, x_2, y_1. If f is holomorphic on R and $|f(x+iy)| \leq Ay^n$ for some constants A and n on R, then the maximum modulus of f occurs on the boundary of R.
  - This result can be substantially generalized into a result known as the Phragmén–Lindelöf principle. We will not give the details here.
  - If f is unbounded on the boundary of R then the result is trivial. So assume that $|f(z)| \leq M$ on the boundary of R. The rest of the argument then follows by making an estimation [2] relying on the maximum modulus principle applied to the function $f(z)/(z+it)^{n+1}$ on a suitable rectangle $x_1 \leq \mathrm{Re}(z) \leq x_2$, $y_1 \leq \mathrm{Im}(z) \leq y_2$.

4.2 Laurent Series and Singularities

* As we have seen in the motivation for Cauchy's integral formula, the series expansion for a holomorphic function around a given point $z = z_0$ carries a tremendous amount of information.
  - We used power series expansions to define and analyze various elementary functions such as the exponential, sine, and cosine, and we also showed during our discussion of formal power series that any rational function can be expanded as a Laurent series with finitely many negative terms.
  - More generally, as also follows from our discussion of formal series and the fact that holomorphic functions are analytic, the quotient of any two holomorphic functions can be expressed as a convergent Laurent series $\sum_{n=-k}^{\infty} a_n(z-z_0)^n$.
  - We would now like to expand our discussion to cover local expansions of all holomorphic functions at arbitrary points.

[2]For completeness: take $z_0 = x_0 + iy_0$ in R, choose any $t > -y_1$ and select $y_2 > y_0$ such that $\dfrac{Ay_2^n}{(y_2+t)^{n+1}} \leq \dfrac{M}{(y_1+t)^{n+1}}$ (possible since the left-hand side tends to zero as $y_2 \to \infty$). Then applying the maximum modulus principle to $g(z) = \dfrac{f(z)}{(z+it)^{n+1}}$ on the rectangle $x_1 \leq \mathrm{Re}(z) \leq x_2$ and $y_1 \leq \mathrm{Im}(z) \leq y_2$ shows that $|g(z_0)| \leq |g(iy_1+it)| = \dfrac{M}{(y_1+t)^{n+1}}$. In terms of f this yields $|f(z_0)| \leq M\left[\dfrac{|z_0+it|}{y_1+t}\right]^n$. Taking the limit as $t \to \infty$ yields $|f(z_0)| \leq M$, as desired.

4.2.1 Laurent Series Expansions

- There are still examples of holomorphic functions that cannot everywhere be expressed as a Laurent series of the form $\sum_{n=-k}^{\infty} a_n(z-z_0)^n$, where necessarily the center z_0 must be a point where the function is not holomorphic.
  - A simple example is $f(z) = e^{1/z}$, which is holomorphic for $z \neq 0$.
  - This function cannot have a Laurent expansion $\sum_{n=-k}^{\infty} a_n z^n$ centered at $z = 0$: if it did, multiplying by z^k and then taking the limit as $z \to 0$ would yield a finite number a_{-k}, but one can check (e.g., via L'Hôpital's rule) that $\lim_{z\to 0} z^k e^{1/z}$ does not exist for any integer k.
  - On the other hand, we can certainly write down a convergent series expansion for $f(z)$: simply plug in $1/z$ to the power series for e^z to see that $e^{1/z} = \sum_{n=0}^{\infty} \dfrac{(1/z)^n}{n!} = \sum_{n=-\infty}^{0} \dfrac{1}{|n|!} z^n$, valid for all $z \neq 0$.
  - This calculation suggests that we should broaden our focus to consider "doubly infinite" Laurent series expansions of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$.
- Definition: A Laurent series centered at $z = z_0$ is a series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for $a_n \in \mathbb{C}$. We say that the series converges at a value z if the two tails $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ both converge at z, and we say the series converges absolutely when both tails converge absolutely.
  - When the series converges, we define the value $f(z)$ of the Laurent series to be the sum of the two tail series.
  - By our previous analysis of the convergence behavior of power series, we know that if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $z = \tilde{z}$ where $|\tilde{z} - z_0| = R$, then it converges absolutely for all z with $|z - z_0| < R$.
  - By replacing $z - z_0$ with its reciprocal, we can see in the same way that if $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ converges for $z = \tilde{z}$ where $|\tilde{z} - z_0| = r$, then it converges absolutely for all z with $|z - z_0| > r$.
  - As a consequence, the region of absolute convergence of a general Laurent series is an annulus of the form $r < |z - z_0| < R$ for some r and R.
- Our main result is that a function that is holomorphic on an annulus can be expressed as a convergent Laurent series on that annulus:
- Theorem (Laurent Expansions): Suppose $f(z)$ is holomorphic on the annulus $r \leq |z - z_0| \leq R$.
  1. For any $r < s < S < R$, the function $f(z)$ has an absolutely convergent Laurent series expansion $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ that converges absolutely and uniformly to f on the region $s \leq |z - z_0| \leq S$. Moreover, the coefficients are given by $a_n = \int_{\gamma_R} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,dz$ for $n \geq 0$ and $\int_{\gamma_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,dz$ for $n < 0$, where γ_R and γ_r are the counterclockwise circles $|z - z_0| = R$ and $|z - z_0| = r$ respectively.
     - The proof of this result is essentially the same as the one we used to show that holomorphic functions are analytic: we simply expand out $1/(\zeta - z)$ as a geometric series and then integrate term by term.
     - Proof: Consider the contour γ that follows the circle of radius R once counterclockwise, traverses a line to the circle of radius r and then follows this circle once clockwise, and then returns along the same line to its starting point.
     - The integrals along the line segment cancel one another, so the integral of any function around this contour γ is the same as the integral around the counterclockwise circle γ_R of radius R minus the integral around the counterclockwise circle γ_r of radius r.
     - Since the winding number of γ around any point in the annulus is 1, Cauchy's integral formula implies that for any z in the annulus, we have $f(z) = \int_{\gamma_R} \dfrac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{\gamma_r} \dfrac{f(\zeta)}{\zeta - z}\,d\zeta$.
     - Now we expand both integrals as convergent geometric series: for $r < s \leq |z - z_0|$ we have $\dfrac{1}{\zeta - z} = \dfrac{1}{\zeta - z_0} \cdot \dfrac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \dfrac{1}{(\zeta - z_0)^{n+1}}(z - z_0)^n$, and for $|z - z_0| \leq S \leq R$ we have $\dfrac{1}{\zeta - z} = -\dfrac{1}{z - z_0} \cdot \dfrac{1}{1 - \dfrac{\zeta - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \dfrac{1}{(z - z_0)^{n+1}}(\zeta - z_0)^n = \sum_{n=-\infty}^{-1} \dfrac{1}{(\zeta - z_0)^{n+1}}(z - z_0)^n$.

- On γ_R with $|\zeta - z_0| = R$ the first series has common ratio $\left|\dfrac{z - z_0}{\zeta - z_0}\right| \le \dfrac{S}{R}$ while on γ_r with $|\zeta - z_0| = r$ the second has common ratio $\left|\dfrac{\zeta - z_0}{z - z_0}\right| \le \dfrac{r}{s}$. Thus, both geometric series converge absolutely and uniformly for $s \le |z - z_0| \le S$.
- Therefore, because $f(\zeta)$ is bounded on γ since it is continuous, the partial sums of $\sum_{n=0}^{\infty} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n$ and of $\sum_{n=-\infty}^{-1} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n$ converge absolutely and uniformly for $s \le |z - z_0| \le S$ to $\dfrac{f(\zeta)}{\zeta - z}$ on γ_R and γ_r respectively.
- Hence by our results on uniform convergence and integrals, we may switch the order of the sums and integral to see

$$\begin{aligned} f(z) = \int_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_{\gamma_r} \frac{f(\zeta)}{\zeta - r}\,d\zeta &= \int_{\gamma_R} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\,d\zeta - (-1)\int_{\gamma_r} \sum_{n=-\infty}^{-1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\,d\zeta \\ &= \sum_{n=0}^{\infty} \int_{\gamma_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\,d\zeta + \sum_{n=-\infty}^{-1} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}(z - z_0)^n\,d\zeta \end{aligned}$$

which is of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $a_n = \int_{\gamma_R} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta$ for $n \ge 0$ and $\int_{\gamma_r} \dfrac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,dz$ for $n < 0$. This is the desired Laurent series expansion for f with coefficients as claimed.

2. The coefficients a_n in a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ are unique and are also given by $a_n = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z - z_0)^{n+1}}\,dz$ where γ is any contour inside the annulus $r < |z - z_0| < R$ with winding number 1 around z_0.
   - Proof: By our observations on the convergence of Laurent expansions, the Laurent expansion converges uniformly to $f(z)$ on γ since γ is in the interior of the annulus.
   - Interchanging the sum and integral yields $\dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z - z_0)^{d+1}}\,dz = \dfrac{1}{2\pi i}\int_\gamma \left[\sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-d-1}\right] dz$ $= \dfrac{1}{2\pi i}\sum_{n=-\infty}^{\infty} a_n \left[\int_\gamma (z - z_0)^{n-d-1}\,dz\right] = a_d$ where the last step follows by our usual observation that $\int_\gamma (z - z_0)^{n-d-1}\,dz$ is $2\pi i$ for $n - d - 1 = -1$ (i.e., for $n = d$) and is 0 for all other n.
3. If f has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ for $r < |z - z_0| < R$, then the Laurent expansion for f' can be obtained by differentiating termwise: $f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}$.
   - Proof: Since the Laurent expansion converges absolutely and uniformly for $s \le |z - z_0| \le S$ for $r < s < S < R$, by our results on uniform convergence we may differentiate termwise to obtain the expansion for $f'(z)$. Taking $s \to r$ and $S \to R$ yields that on the full annulus $r < |z - z_0| < R$.
   - Finally, (2) ensures that the Laurent expansion for $f'(z)$ is unique.

- In principle, for any holomorphic function $f(z)$ on an annulus $r \le |z - z_0| \le R$ we may compute the coefficients in its Laurent expansion using the integral formulas in the theorem above.
  - However, in practice (e.g., for rational functions) it is often easier to manipulate known series, such as geometric series, to compute Laurent expansions.
- Example: Find the Laurent expansion of $f(z) = 1/(1 - z)$ on the region $|z| > 1$.
  - We have repeatedly worked out the expansion $\dfrac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$ for $|z| < 1$. We can use the same idea to obtain the expansion for $|z| > 1$, using $1/z$ in place of z.

- Explicitly, we have $f(z) = \dfrac{1}{1-z} = -\dfrac{1}{z} \cdot \dfrac{1}{1-1/z} = -z^{-1}\sum_{n=0}^{\infty} z^{-n} = \boxed{\sum_{n=-\infty}^{-1} -z^n} = \cdots - z^{-4} - z^{-3} - z^{-2} - z^{-1}$.

- Example: Find the terms from z^{-3} through z^3 in the Laurent expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ on the region $|z| < 1$, on the region $1 < |z| < 2$, and on the region $|z| > 2$.
  - Using a partial fraction decomposition, we have $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ so it suffices to expand each of these series on the given region.
  - For $|z| < 1$ we have $\dfrac{1}{z-1} = -\dfrac{1}{1-z} = \sum_{n=0}^{\infty}(-1)z^n$, whereas for $|z| > 1$ we have $\dfrac{1}{z-1} = \dfrac{1}{z} \cdot \dfrac{1}{1-1/z} = \dfrac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} z^{-1-n}$.
  - For $|z| < 2$ we have $\dfrac{1}{z-2} = -\dfrac{1}{2} \cdot \dfrac{1}{1-z/2} = -\dfrac{1}{2}\sum_{n=0}^{\infty}(z/2)^n = \sum_{n=0}^{\infty} \dfrac{-1}{2^{n+1}} z^n$, whereas for $|z| > 2$ we have $\dfrac{1}{z-2} = \dfrac{1}{z} \cdot \dfrac{1}{1-2/z} = \dfrac{1}{z}\sum_{n=0}^{\infty}(2/z)^n = \sum_{n=0}^{\infty} 2^n z^{-1-n}$.
  - So for $|z| < 1$ we get $f(z) = \sum_{n=0}^{\infty} \dfrac{-1}{2^{n+1}} z^n - \sum_{n=0}^{\infty}(-1)z^n = \sum_{n=0}^{\infty}(1 - \dfrac{1}{2^{n+1}})z^n = \boxed{\dfrac{1}{2} + \dfrac{3}{4}z + \dfrac{7}{8}z^2 + \dfrac{15}{16}z^3 + \cdots}$.
  - For $1 < |z| < 2$ we get $f(z) = \sum_{n=0}^{\infty} \dfrac{-1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} z^{-1-n} = \boxed{\cdots - z^{-3} - z^{-2} - z^{-1} - \dfrac{1}{2} - \dfrac{1}{4}z - \dfrac{1}{8}z^2 - \dfrac{1}{8}z^3 - \cdots}$.
  - Finally, for $|z| > 2$ we get $f(z) = \sum_{n=0}^{\infty} 2^n z^{-1-n} - \sum_{n=0}^{\infty} z^{-1-n} = \sum_{n=1}^{\infty}(2^n - 1)z^{-1-n} = \boxed{\cdots + 7z^{-4} + 3z^{-3} + z^{-2}}$.

- Example: Find the terms from z^{-3} through z^3 in the Laurent expansion of $f(z) = z^{-2}e^z + ze^{1/z}$ on the region $1 < |z| < 2$.
  - We have $z^{-1}e^z = z^{-2}\sum_{n=0}^{\infty} \dfrac{1}{n!} z^n = z^{-2} + z^{-1} + \dfrac{1}{2} + \dfrac{1}{6}z + \dfrac{1}{24}z^2 + \dfrac{1}{120}z^3 + \cdots$ and $ze^{1/z} = z\sum_{n=0}^{\infty} \dfrac{1}{n!} z^{-n} = \cdots + \dfrac{1}{24}z^{-3} + \dfrac{1}{6}z^{-2} + \dfrac{1}{2}z^{-1} + 1 + z$.
  - Adding yields $z^{-2}e^z + ze^{1/z} = \boxed{\cdots + \dfrac{1}{24}z^{-3} + \dfrac{7}{6}z^{-2} + \dfrac{3}{2}z^{-1} + \dfrac{3}{2} + \dfrac{7}{6}z + \dfrac{1}{24}z^2 + \dfrac{1}{120}z^3 + \cdots}$.

- Example: Find the terms up through z^3 in the Laurent expansion, along with the radius of convergence, for $f(z) = \csc(z)$ centered at $z = 0$.
  - As a formal series, $f(z) = \dfrac{1}{\sin(z)} = \dfrac{1}{z - z^3/3! + z^5/5! - z^7/7! + \cdots} = \dfrac{1}{z} \cdot \dfrac{1}{1 - z^2/3! + z^4/5! - z^6/7! + \cdots}$ $= z^{-1}(1 + \dfrac{1}{6}z^2 + \dfrac{7}{360}z^4 + \cdots) = \boxed{z^{-1} + \dfrac{1}{6}z + \dfrac{7}{360}z^3 + \cdots}$, using our previous techniques to compute the multiplicative inverse of $1 - z^2/3! + z^4/5! - z^6/7! + \cdots$.
  - For the radius of convergence, we note that $f(z)$ only fails to be holomorphic when $\sin(z) = 0$, which occurs for $z = k\pi$ for integers k.
  - Therefore, by our results on the radius of convergence of a series expansion, the radius of convergence must be $\boxed{\pi}$, the distance from $z = 0$ to the next closest point where $f(z)$ is not holomorphic.

4.2.2 Zeroes of Holomorphic Functions

- The notion of a zero (or root) of a polynomial is quite familiar, as is the notion of the multiplicity of a zero.
  - Explicitly, if $p(z)$ is a polynomial, we say that $r \in \mathbb{C}$ is a zero (or root) of p if $p(r) = 0$. By the remainder/factor theorem, if r is a root of $p(z)$, then $z - r$ divides $p(z)$, which is to say, $p(z)/(z - r)$ is equal to a polynomial.

- By repeatedly taking out factors of $z-r$ until there are none remaining (a process which must terminate eventually as long as $p(z)$ is not the zero polynomial), when r is a root of $p(z)$ we may write $p(z) = (z-r)^d q(z)$ for some polynomial $q(z)$ of which r is not a root and some unique positive integer d, which we call the <u>multiplicity</u> of that root.
- <u>Example</u>: For $p(z) = z^3 - z^2 = z^2(z-1)$, the roots of $p(z)$ are $z = 0$ and $z = 1$. The root $z = 0$ has multiplicity 2, while the root $z = 1$ has multiplicity 1.
- We may equivalently identify the multiplicity of r using the derivatives of $p(z)$: it is not hard to verify via repeated application of the product rule that r has multiplicity d if and only if the first $d-1$ derivatives of $p(z)$ vanish at r but $p^{(d)}(r) = 0$.
- <u>Example</u>: For $p(z) = z^3 - z^2$ we have $p(0) = p'(0) = 0$ but $p''(0) = -2$, so 0 has multiplicity 2 (as seen above using the factorization).

- The zeroes of holomorphic functions can be classified in a similar way.
- <u>Definition</u>: Suppose $f(z)$ is holomorphic on a region R and let $z_0 \in R$. We say that z_0 is a <u>zero</u> of f if $f(z_0) = 0$, and if f is not identically zero we say the <u>order</u> (or <u>order of vanishing</u>) of f at z_0 is the smallest positive integer d such that $p^{(i)}(z_0) = 0$ for $i = 1, 2, \dots, d-1$ but $p^{(d)}(z_0) \neq 0$.
  - For completeness, we also define the order of vanishing of the identically zero function to be ∞ everywhere.
  - It is easy to find the order of vanishing using the power series expansion of $f(z)$ around $z = z_0$: if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, then by repeatedly differentiating and evaluating at z_0, we see that the order is the smallest d for which $a_d \neq 0$.
  - This calculation also shows that the order of vanishing is always well defined when f is not the zero function (since the power series of a nonzero function is necessarily nonzero by uniqueness of series expansions) and also agrees with our earlier notion of the order of a power series.
  - Additionally, by factoring out $(z-z_0)^d$ from the power series expansion of $f(z)$ at $z = z_0$, we also obtain an analogous "factorization" result: if f is a nonzero holomorphic function with a zero of order d at $z = z_0$, then we may write $f(z) = (z-z_0)^d g(z)$ where $g(z)$ is holomorphic and does not vanish at z_0.
- <u>Example</u>: Find the zeroes of $f(z) = \sin(z^2)$ and calculate their orders.
  - Since $\sin w = 0$ if and only if $w = k\pi$ for an integer k, the zeroes of $f(z)$ are the square roots of these values, which are $\boxed{\pm\sqrt{k\pi}, \pm i\sqrt{k\pi}}$ for nonnegative integers k.
  - For the orders, we note $f'(z) = 2z\cos(z^2)$, which for $z^2 = k\pi$ is simply $2z\cos(k\pi) = \pm 2z$. This is nonzero except at $z = 0$, so all of the zeroes except $z = 0$ have $\boxed{\text{order 1}}$.
  - At $z = 0$ we have $f''(z) = 2\cos(z^2) - 4z^2\sin(z^2)$ so that $f''(z) = 2$. Thus $z = 0$ has $\boxed{\text{order 2}}$
- We will mention now one other very useful analogy between polynomials and holomorphic functions.
  - It is easy to see (e.g., by induction on the degree) that a nonzero polynomial necessarily has finitely many roots.
  - In contrast, holomorphic functions may have infinitely many zeroes: for example, $f(z) = \sin z$ has zeroes at $z = k\pi$ for all integers k.
  - Although there may be infinitely many of them, as we showed using the uniqueness of power series, the zeroes of analytic (equivalently, holomorphic) functions do retain one very convenient property: they are <u>isolated</u>, meaning that if z_0 is a zero of f and f is not identically zero, then there exists some $r > 0$ such that $f(z) \neq 0$ for $0 < |z-z_0| < r$.
  - In other words, each zero of a nonzero holomorphic function f is a positive distance away from all other zeroes of f.

4.2.3 Removable Singularities, Poles, and Essential Singularities

- We have seen various examples of functions that are holomorphic except at some isolated set of points. We now study these isolated singularities in more detail. First, we give a precise definition:

- Definition: Suppose R is an open set and $z_0 \in R$. We say that z_0 is an isolated singularity of f if f is holomorphic on all of R except at z_0.

  - Examples: The functions $\frac{1}{z}$, $\frac{e^z}{z}$, and $e^{1/z}$ all have a single isolated singularity at $z = 0$.
  - Example: The function $\csc z$ has an isolated singularity at $z = k\pi$ for each integer k.
  - Example: The function $\frac{1}{z^2+1}$ has isolated singularities at $z = i$ and $z = -i$.
  - We can also construct some more artificial examples by taking a holomorphic function and changing its value at a point, or merely omitting a point from its domain entirely.
  - Example: The function $f(z) = \begin{cases} z^2 & \text{for } z \neq 0,1 \\ 100 & \text{for } z = 0,1 \end{cases}$ has isolated singularities at $z = 0$ and $z = 1$.
  - Example: The function $g(z) = z + 5$ for $z \neq -2$ has an isolated singularity at $z = -2$ (it is not defined there).
  - Example: The function $h(z) = \frac{z^2-1}{z-1}$ for $z \neq 1$ has an isolated singularity at $z = 1$ (the expression given is not defined there).
  - Of course, for $f(z)$ above it is clear that we have taken the "wrong" value for $f(z)$ at two points, while for the function $g(z)$ we should just extend the domain to include $z = -2$, and for $h(z)$ we should simplify the expression to $h(z) = z + 1$ for $z \neq 1$ and then extend its domain in the same way.

- A less trivial but similar situation arises for the function $f(z) = \frac{\sin z}{z}$, which has an isolated singularity at $z = 0$.

  - It is natural to try to extend the domain of this function to include $z = 0$, and the most sensible way to do it is to extend the definition by setting $\tilde{f}(0) = \lim_{z\to 0} \frac{\sin z}{z} = \cos(0) = 1$ (note that the limit is simply the limit definition of the complex derivative of $\sin z$ at $z = 0$).
  - Indeed, with this choice, the function $\tilde{f}(z) = \frac{\sin z}{z}$ for $z \neq 0$ and $\tilde{f}(0) = 1$ is in fact holomorphic at $z = 0$. This can be shown directly by computing $\tilde{f}'(0)$ using the limit definition of the derivative, but we can give a much more conceptually natural approach using power series.
  - Explicitly, we have the series expansion $\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$ for all $z \neq 0$.
  - So if we just consider the function $g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$ defined by this power series, then $g(z)$ is holomorphic on all of $\mathbb{C}$ and agrees with $\tilde{f}(z)$ for all $z \neq 0$ by the series expansion for sine.
  - But $g(z)$ also agrees with $\tilde{f}$ at $z = 0$ since $g(0) = 1 = \tilde{f}(0)$ also. Therefore g and $\tilde{f}$ are exactly the same function, so since g is holomorphic at $z = 0$, so is $\tilde{f}$.

- We see that in some cases, we may "remove" the isolated singularity by defining (or redefining) the value of $f(z_0)$ so that the new function is holomorphic at z_0.

  - However, this is not always possible for all singularities. For example, for $f(z) = \frac{1}{z}$, there is no way to assign a value to $f(0)$ that makes the resulting function holomorphic, or even continuous, since $\lim_{z\to 0} \frac{1}{z}$ is not defined. In a sense we will make more precise later, it is reasonable to view this limit as being ∞, since $\lim_{z\to 0} \left|\frac{1}{z}\right| = \infty$, so the function grows uniformly large in absolute value as z approaches zero, but this still does not allow us to make the function holomorphic at $z = 0$.

- The function $f(z) = e^{1/z}$ has a similar issue at $z = 0$: there is no possible choice for $f(0)$ that makes f continuous there, let alone holomorphic: the limit $\lim_{z\to 0} e^{1/z}$ is also undefined. In fact, the behavior is actually worse than that of $1/z$, since even the absolute value limit $\lim_{z\to 0} \left|e^{1/z}\right|$ is undefined (for instance, along the positive real axis the limit is ∞ while along the imaginary axis it is 1).

- We may use the Laurent expansion of a function near an isolated singularity to classify different types of local behavior for a holomorphic function.
  - More specifically, suppose that z_0 is an isolated singularity of f. Then since z_0 is isolated, there is a positive value of R such that $f(z)$ is holomorphic on the punctured disc $0 < |z - z_0| \le R$.
  - From our discussion of Laurent expansions, for any $0 < r < R$ the function $f(z)$ has a convergent Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ on the annulus $r \le |z - z_0| \le R$.
  - By uniqueness of Laurent expansions, all of these expansions must agree with one another, so in fact, this Laurent expansion converges on the full punctured disc $0 < |z - z_0| \le R$.
  - We can then identify three different classes of behavior depending on how many of the coefficients a_n with $n < 0$ are nonzero: none of them, finitely many of them, or infinitely many of them.

- Definition: Suppose that z_0 is an isolated singularity of f and $f(z)$ has a convergent Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ on the punctured disc $0 < |z - z_0| \le R$. If all of the coefficients a_n with $n < 0$ are zero, we say f has a removable singularity at z_0. If some but only finitely many of the coefficients a_n with $n < 0$ are nonzero, we say f has a pole at z_0 (the largest k such that $a_{-k} \ne 0$ is called the order of the pole; a pole of order 1 is called simple). Finally, if infinitely many of the a_n with $n < 0$ are nonzero, we say f has an essential singularity at z_0.
  - Example: The functions $f(z) = \begin{cases} z^2 & \text{for } z \ne 0,1 \\ 100 & \text{for } z = 0,1 \end{cases}$, $g(z) = z + 5$ for $z \ne -2$, and $h(z) = \dfrac{z^2 - 1}{z - 1}$ for $z \ne 1$, all discussed above, each have removable singularities (for f at $z = 0$ and $z = 1$, for g at $z = -2$, and for h at $z = 1$).
  - Example: The function $\dfrac{\sin z}{z}$ has a removable singularity at $z = 0$, since its Laurent expansion at $z = 0$ is $1 - \dfrac{1}{6}z^2 + \dfrac{1}{120}z^4 - \cdots$ has no terms with a negative power of z.
  - Example: The functions $\dfrac{1}{z}$, $\dfrac{z-1}{z^2}$, and $\dfrac{e^z}{z}$ each have a pole at $z = 0$, since their Laurent expansions at $z = 0$ are z^{-1}, $-z^{-2} + z^{-1}$, and $z^{-1} + 1 + \dfrac{1}{2}z + \cdots$. The orders of these poles are 1, 2, and 1 respectively.
  - Example: The function $\dfrac{1}{z^2(z-1)}$ has a pole of order 2 at $z = 0$ and a pole of order 1 at $z = 1$, since the respective Laurent expansions at $z = 0$ and $z = 1$ are $-z^{-2} - z^{-1} - 1 - \cdots$ and $(z-1)^{-1} - 2 + 3(z-1) + \cdots$.
  - Example: The function $e^{1/z}$ has an essential singularity at $z = 0$, since its Laurent expansion is $\sum_{n=0}^{\infty} \dfrac{1}{n!} z^{-n}$ which has infinitely many terms with a negative power of z.
  - Example: The function $1/(e^z - 1)$ has a simple pole at $z = 2\pi i k$ for each integer k, since its Laurent expansion is $(z - 2\pi i k)^{-1} - \dfrac{1}{2} + \dfrac{1}{12}(z - 2\pi i k) + \cdots$ for each such k.

- These three types of singularities have very different properties, and we can characterize each of them based on the local behavior of $f(z)$ near the singularity. We begin with removable singularities:

- Theorem (Classification of Removable Singularities): Suppose that z_0 is an isolated singularity of f and $f(z)$ has a convergent Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ on the punctured disc $0 < |z - z_0| \le R$.
  1. If the function $f(z)$ has a removable singularity at $z = z_0$, then $f(z)$ is bounded on the punctured disc $0 < |z - z_0| \le R$. Moreover, the limit $\lim_{z\to z_0} f(z) = L$ exists, and the function $\tilde{f}(z) = \begin{cases} f(z) & \text{for } z \ne z_0 \\ L & \text{for } z = z_0 \end{cases}$ is holomorphic at z_0.

- $\circ$ Proof: By hypothesis, all of the terms a_n with $n < 0$ are zero. Therefore, if we define $\tilde{f}(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then $\tilde{f}(z) = f(z)$ on the punctured disc $0 < |z - z_0| \le R$.
- $\circ$ In particular, the radius of convergence of the series for $\tilde{f}$ is at least R. So $\tilde{f}$ is holomorphic on the entire disc $|z - z_0| \le R$, hence is continuous and thus bounded there. Since f agrees with $\tilde{f}$ on the punctured disc, that means f is bounded as well.
- $\circ$ Likewise, $L = \lim_{z \to z_0} \tilde{f}(z) = \tilde{f}(z_0)$ exists, and indeed $\tilde{f}(z)$ as described is holomorphic at z_0.

2. Conversely, suppose $f(z)$ is bounded on the punctured disc $0 < |z - z_0| \le R$. Then f has a removable singularity at $z = z_0$.
   - $\circ$ This result is often called Riemann's removable singularities theorem.
   - $\circ$ Proof: We must show that all of the coefficients a_n with $n < 0$ in the Laurent expansion of $f(z)$ are zero.
   - $\circ$ By hypothesis, there exists some M such that $|f(z)| \le M$ on the disc. Let $0 < r < R$ and take γ_r to be the counterclockwise circle of radius r centered at z_0.
   - $\circ$ By our results on Laurent coefficients, we have $a_{-n} = \dfrac{1}{2\pi i}\int_\gamma (z-z_0)^{n-1} f(z)\,dz$, so since on γ we have $\left|(z-z_0)^{n-1} f(z)\right| \le r^{n-1}M$, applying the arclength bound yields $|a_{-n}| = \dfrac{1}{2\pi}\left|\int_\gamma (z-z_0)^{n-1} f(z)\,dz\right| \le \dfrac{1}{2\pi} \cdot 2\pi r \cdot r^{n-1}M = r^n M$. Since $n \ge 1$, taking $r \to 0$ shows that $a_{-n} = 0$.
   - $\circ$ Thus, all coefficients a_n with $n < 0$ in the Laurent expansion of $f(z)$ are zero, so f has a removable singularity at z_0.

- From the results (1) and (2) together, we can see that removable singularities are characterized by having f remain bounded as we approach the singularity.
- Next, we study the behavior of poles:
- Theorem (Classification of Poles): Suppose that z_0 is an isolated singularity of f and $f(z)$ has a convergent Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ on the punctured disc $0 < |z - z_0| \le R$.

1. If f has a pole at $z = z_0$, then $f(z)$ is unbounded on the punctured disc $0 < |z - z_0| \le R$ and in fact $\lim_{z \to z_0} |f(z)| = \infty$. Moreover, the pole has order k if and only if $\lim_{z \to z_0} (z-z_0)^k f(z)$ exists and is nonzero.
   - $\circ$ The fact that $\lim_{z \to z_0} |f(z)| = \infty$ when f has a pole at $z = z_0$ is the origin of the name "pole", since if we plot the surface $z' = |f(x' + iy')|$ in a 3-dimensional coordinate system, the graph stretches up to $+\infty$ as $x' + iy' \to z_0$ (i.e., the surface looks like it has a pole holding it up).
   - $\circ$ Proof: By hypothesis, only finitely many of the terms a_n with $n < 0$ are nonzero. Therefore, assuming the pole order is k, then the Laurent expansion for f is of the form $f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$ where $a_{-k} \neq 0$.
   - $\circ$ Then for $0 < |z - z_0| \le R$, we see that $f(z) = (z-z_0)^{-k}[\sum_{n=0}^{\infty} a_{n-k}(z-z_0)^n] = (z-z_0)^{-k} g(z)$ where $g(z) = \sum_{n=0}^{\infty} a_{n-k}(z-z_0)^n$.
   - $\circ$ Note that $g(z)$ has a removable singularity at $z = z_0$ by (1), hence is holomorphic for $|z - z_0| \le R$. Note also that $g(z_0) = a_{-k}$ is nonzero by assumption.
   - $\circ$ Then $\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} |z - z_0|^{-k} |g(z)| = |g(z_0)| \lim_{z \to z_0} |z - z_0|^{-k} = \infty$ by the continuity of g at $z = z_0$, that $|g(z_0)|$ is positive, and that the quantity $|z - z_0|$ is a positive real number that tends to 0 as $z \to z_0$.
   - $\circ$ For the last statement, we have $\lim_{z \to z_0}(z-z_0)^k f(z) = \lim_{z \to z_0} g(z) = g(z_0) = a_{-k}$ which is nonzero by hypothesis. On the other hand, from this calculation we see $\lim_{z \to z_0}(z-z_0)^d f(z) = a_{-k} \lim_{z \to z_0}(z-z_0)^{d-k}$, and for $d - k < 0$ the limit does not exist while for $d - k > 0$ the limit is zero. Thus $\lim_{z \to z_0}(z-z_0)^d f(z)$ only converges to a nonzero value when $d = k$ is the pole order of f.
2. The function f has a pole of order k at $z = z_0$ if and only if there exists a holomorphic function $g(z)$ on the disc $|z - z_0| \le R$ with $g(z_0) \neq 0$ and $f(z) = \dfrac{g(z)}{(z-z_0)^k}$ for all $0 < |z - z_0| \le R$.

- Proof: If f has a pole of order k, then the Laurent expansion for f is of the form $f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$ where $a_{-k} \neq 0$.
- If as in (2) we then take $g(z) = \sum_{n=0}^{\infty} a_{n-k}(z-z_0)^n$, then g is holomorphic on the disc, $g(z_0) = a_{-k} \neq 0$, and $f(z) = \dfrac{g(z)}{(z-z_0)^k}$.
- Conversely, if $f(z) = \dfrac{g(z)}{(z-z_0)^k}$ for a holomorphic g, then g has a power series expansion $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ where by hypothesis $b_0 = g(z_0) \neq 0$.
- Then we have the Laurent expansion $f(z) = \dfrac{g(z)}{(z-z_0)^k} = \sum_{n=-k}^{\infty} b_{n+k}(z-z_0)^n = b_0(z-z_0)^{-k} + b_1(z-z_0)^{1-k} + \cdots$, which since $b_0 \neq 0$, means f has a pole of order k.

3. If f is holomorphic and has a zero of order k at $z = z_0$, then $1/f$ has a pole of order k at z_0.
   - Proof: As noted in our discussion of zeroes of holomorphic functions, if f has a zero of order k at z_0 then $f(z) = (z-z_0)^k g(z)$ for a holomorphic $h(z)$ with $h(z_0) \neq 0$.
   - From our results on analytic functions, since $h(z_0) \neq 0$, the formal power series inverse $g(z) = h(z)^{-1}$ is analytic (hence holomorphic) on a disc of positive radius centered at z_0.
   - Then $\dfrac{1}{f(z)} = \dfrac{1/h(z)}{(z-z_0)^k} = \dfrac{g(z)}{(z-z_0)^k}$, so by (3), $1/f$ has a pole of order k at z_0.
4. Conversely, if f has a pole of order k at $z = z_0$ then $1/f$ has a removable singularity at z_0, and removing the singularity yields a function that has a zero of order k at z_0.
   - Proof: Suppose f has a pole of order k at $z = z_0$. Then by (3), we have $f(z) = \dfrac{g(z)}{(z-z_0)^k}$ for a holomorphic function $g(z)$ with $g(z_0) \neq 0$. As in (4), since $g(z_0) \neq 0$ it has a holomorphic inverse $h(z)$ on a disc of positive radius centered at z_0.
   - Then $\dfrac{1}{f(z)} = (z-z_0)^k g(z)^{-1} = (z-z_0)^k h(z)$, and so since $1/f$ is the product of two holomorphic functions for $z \neq z_0$, it has a removable singularity, and by our characterization of the order of a zero, we see that $1/f$ has a zero of order k at z_0.

- These results show that zeroes and poles are two sides of the same proverbial coin, and are related simply by taking reciprocals.
  - Additionally, we can see that as we approach a pole of f, the value of $|f(z)|$ tends uniformly to ∞, in contrast to the behavior when approaching a removable singularity where f remains bounded.
  - From our characterization of removable singularities, this should not be surprising, since f must be unbounded when approaching a pole or essential singularity.
- One might imagine, then, that the behavior near an essential singularity might be similar to that of a pole (e.g., that $|f|$ will tend to ∞ while approaching the singularity).
  - That is, however, not at all the case, as can be seen by the example of $f(z) = e^{1/z}$.
  - Indeed, solving $e^{1/z} = re^{i\theta}$ produces $1/z = \ln r + i\theta + 2\pi ki$ so that $z = 1/(\ln r + i\theta + 2\pi ki)$. In particular, as $k \to \infty$ these values approach zero.
  - Thus, on any punctured disc around 0, no matter how small the radius, the values taken by $f(z)$ include every complex number of the form $re^{i\theta}$ for $r \neq 0$: in other words, every nonzero complex number!
  - It turns out that this behavior is typical of essential singularities:
- Theorem (Casorati-Weierstrass): Suppose that f is holomorphic for $0 < |z-z_0| \leq R$ and z_0 is an essential singularity of f. Then for any $0 < r < R$, the values of f taken on the punctured disc $0 < |z-z_0| < r$ are dense in $\mathbb{C}$.
  - A dense set S is one with the property that for any $z \in \mathbb{C}$ there exist $z_i \in S$ such that $\lim_{n\to\infty} z_n = z$: in other words, every point in $\mathbb{C}$ is a limit point of the set S. Equivalently, for any $z \in \mathbb{C}$ and any $\epsilon > 0$, there exists a point $z' \in S$ such that $|z-z'| < \epsilon$. Equivalently, the closure of the set S is all of $\mathbb{C}$.

- For example, the points $x+iy$ with x, y rational are a dense subset of $\mathbb{C}$.
- An equivalent way of posing the result of Casorati-Weierstrass is that for any $c \in \mathbb{C}$, there exists a sequence $\{z_n\}_{n\geq 1}$ such that $z_n \to z_0$ and $f(z_n) \to f(c)$ as $n \to \infty$.
- Proof: Suppose otherwise, so that the values of f on $0 < |z - z_0| < r$ are not dense in $\mathbb{C}$. This means there exists some $\zeta \in \mathbb{C}$ and some $\epsilon > 0$ such that $|f(z) - \zeta| \geq \epsilon$ for all z in the punctured disc $0 < |z - z_0| < r$.
- So now consider the function $g(z) = \dfrac{1}{f(z) - \zeta}$: it is holomorphic on $0 < |z - z_0| < r$ since the denominator is never zero.
- Also, since $|g(z)| = \left|\dfrac{1}{f(z) - \zeta}\right| \leq \dfrac{1}{\epsilon}$ on the punctured disc, we see g is bounded hence has a removable singularity at z_0. Therefore, $g(z)$ is actually holomorphic for $|z - z_0| \leq r$. If it is nonzero at z_0, then $\dfrac{1}{g(z)}$ is holomorphic at z_0, while if g is zero at z_0 then $\dfrac{1}{g(z)}$ has a pole at z_0 by our characterization of poles.
- But then $f(z) = \zeta + \dfrac{1}{g(z)}$ either has a removable singularity at z_0 or a pole at z_0. But this is a contradiction since z_0 is an essential singularity of f.

- In fact, the result of Casorati-Weierstrass can be very substantially strengthened:
- Theorem (Picard's Big Theorem): Suppose that f is holomorphic for $0 < |z - z_0| \leq R$ and z_0 is an essential singularity of f. Then for any $0 < r < R$, the values of f taken on the punctured disc $0 < |z - z_0| < r$ include every complex number, with at most one exception.
  - This theorem is quite difficult and we do not include a proof. We will, however, mention that this result is connected with a number of other results, some connections among which we can describe.
- To motivate the first, we observe that the image of a nonconstant entire function must be dense in $\mathbb{C}$.
  - To see this, if the image is not dense, then there exists some $\zeta \in \mathbb{C}$ and some $\epsilon > 0$ such that $|f(z) - \zeta| \geq \epsilon$ for all $z \in \mathbb{C}$.
  - Then by the same argument as in the proof above, the function $g(z) = \dfrac{1}{f(z) - \zeta}$ is entire and bounded above in absolute value by $1/\epsilon$, so by Liouville's theorem it is constant.
  - Another result of Picard generalizes this result, in the same manner as Casorati-Weierstrass:
- Theorem (Picard's Little Theorem): Suppose f is a nonconstant entire function. Then the image of f includes every complex number, with at most one exception.
  - Equivalently, if f is an entire function that omits two or more values from its image, then f is constant.
  - The allowance of one exception is certainly necessary, since $f(z) = e^z$ never takes the value 0.
  - There are several approaches to proving Picard's little theorem, but Picard's original approach was first to construct the elliptic modular function λ, which is essentially an explicit covering map of $\mathbb{C}\backslash\{0, 1\}$ by the unit disc D.
  - Then if f is entire and omits at least two values from its image, by rescaling and translating we can assume the two omitted values are 0 and 1. The next part (requiring all of the hard work!) is to show that if $f : \mathbb{C} \to \mathbb{C}\backslash\{0, 1\}$ is entire, then f is the composition of a function $g : \mathbb{C} \to D$ and the modular function $\lambda : D \to \mathbb{C}\backslash\{0, 1\}$; intuitively, the reason that this composition exists is that λ is a covering map.
  - But then g is entire and its image is bounded (since it is contained in the unit disc), so by Liouville's theorem, g is constant, and thus f is also constant.
- We can see that essential singularities tend to have rather odd analytic properties, and in particular, cannot merely be obtained as a quotient of holomorphic functions.
  - For this reason, we often restrict attention to functions that are more well-behaved near their singularities.

- Since removable singularities can be ignored by redefining the function properly, we are then mostly interested in poles.

- Definition: Let R be a region. We say a function $f(z)$ is meromorphic on R if f is holomorphic on R except for a set of isolated singularities all of which are poles (or removable).
  - If $f(z)$ and $g(z)$ are both holomorphic on R and g is not identically zero, then the quotient $\frac{f(z)}{g(z)}$ is meromorphic on R: this quotient can only fail to be holomorphic when $g(z) = 0$ and this only occurs at an isolated set of points by our results about zeroes of holomorphic functions.
  - So, for example, the functions $\frac{e^z}{z}$, $\frac{e^z + \sin z}{\sin z + 2z}$, and $\frac{\sinh(z)}{z^3 \tan z}$ are all meromorphic on $\mathbb{C}$.
  - In fact, the converse of this observation holds as well: a meromorphic function on R is necessarily the quotient of two holomorphic functions on R.
  - It is clear that this statement is true locally: if $h(z)$ is meromorphic and z_0 is a pole, then as we showed, we can write $h(z) = \frac{f(z)}{(z - z_0)^k}$ for some holomorphic function f. If $h(z)$ has only finitely many poles, then by taking an appropriate common denominator, one obtains a quotient of holomorphic functions representing $h(z)$ on its entire domain.
  - In the case $h(z)$ has infinitely many poles, more work is required to construct an appropriate "denominator function", but it can be done with a suitable infinite product and a result known as the Weierstrass factorization theorem.

4.3 Residues and Residue Calculus

- Now that we have established various useful preliminaries, we turn our attention to using these results to evaluate integrals.
  - The general idea is to combine all of the facts we have accumulated about Cauchy's integral formula, winding numbers, and meromorphic functions to give a general integration formula.

4.3.1 Residues, The Residue Theorem

- So suppose that f is meromorphic on a simply connected region R and γ is a closed contour: we would like to give as simple a formula for the value of $\int_\gamma f(z)\,dz$ as we can.
  - First, since γ is closed, it contains only finitely many singularities of f, which we may freely assume are poles.
  - Locally around each singularity z_0, we may express f as a convergent Laurent series $f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^n$. As we have seen, the integral of f on a suitable contour (inside the region of convergence) then depends only on the winding number of γ around z_0 and the coefficient a_{-1}.
  - In essence, therefore, up to needing to be careful about deforming γ so that it lies inside the region of convergence, we should be able to express the integral of f on γ in terms of the winding number of γ around each singularity along with the coefficient a_{-1} at each singularity.

- To phrase all of this more conveniently, we introduce notation for this coefficient a_{-1}:

- Definition: Suppose $f(z)$ is meromorphic on an open region R and $z_0 \in R$. If $f(z)$ has a local Laurent expansion $f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^n$ at z_0, we define the residue of f at z_0, denoted $\text{Res}_f(z_0)$, to be the coefficient a_{-1}.
  - From Cauchy's integral formula, if f is holomorphic on $S \backslash \{z_0\}$, then we have $\text{Res}_f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\,dz$ where γ is any contour inside S with winding number 1 around z_0.
  - Trivially, if f is holomorphic at z_0, then the residue is zero: the only points where f can have a nonzero residue are the poles of f.

- Example: The residues of $\frac{1}{z}$, $\frac{1}{z^2}$, and $\frac{e^z}{z}$ at $z = 0$ are 1, 0, and 1 respectively.
- Example: The residues of $\frac{1}{z-1}$, $\frac{1}{\sin(\pi z)}$, and $\frac{e^z}{z-1}$ at $z = 1$ are 1, $\frac{1}{\pi}$, and e respectively.
- In the particular case where f has a simple pole at z_0, we have $f(z) = a_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$, and thus $(z-z_0)f(z) = a_{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^{n+1}$. As $z \to z_0$ all of the terms in the sum approach zero, yielding $\text{Res}_f(z_0) = a_{-1} = \lim_{z \to z_0}(z-z_0)f(z)$.
- Indeed, since $(z-z_0)f(z)$ has a removable singularity at $z = z_0$, we can really just view this limit as being the natural value of $(z-z_0)f(z)$ evaluated at $z = z_0$.
- Equivalently, if we write $f(z) = \frac{1}{h(z)}$ then h has a simple zero at $z = z_0$, in which case $\lim_{z \to z_0}(z - z_0)f(z) = \lim_{z \to z_0} \frac{z-z_0}{h(z)} = \frac{1}{h'(z_0)}$ by L'Hôpital's rule.
- More generally, if $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$ and h has a simple zero at $z = z_0$, then the residue of f at z_0 is $\frac{g(z_0)}{h'(z_0)}$ by the same calculation.

- Example: Find the residue of $f(z) = e^z/\sin(z)$ at $z = 0$.
  - Since $\sin(z)$ has a simple zero at $z = 0$ and $e^0 = 1$, $f(z)$ has a simple pole at $z = 0$. Using the formula above with $g(z) = e^z$ and $h(z) = \sin(z)$ we see the residue is $g(0)/h'(0) = e^0/\cos 0 = \boxed{1}$.
  - Indeed, the Laurent series for $f(z)$ at $z = 0$ is $z^{-1} + 1 + \frac{2}{3}z + \frac{1}{3}z^2 + \cdots$, and the coefficient of z^{-1} is indeed 1.
- We can give a similar residue formula for poles of higher order:
- Proposition (Residue Calculations): Suppose $f(z)$ is meromorphic on an open region R and f has a pole of order k at $z_0 \in R$. Then the residue of f at z_0 is given by $\text{Res}_f(z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right]$.
  - Proof: By hypothesis f has a Laurent expansion of the form $f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^n$.
  - Then $(z-z_0)^k f(z) = a_{-k} + a_{1-k}(z-z_0) + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots$ is holomorphic at z_0 after removing the singularity.
  - Differentiating term by term $k-1$ times yields $\frac{d^{k-1}}{dz^{k-1}} \left[(z-z_0)^k f(z)\right] = (k-1)!a_{-1} + k!a_0(z-z_0) + \cdots$, and now taking the limit as $z \to z_0$ (which here amounts merely to setting $z = z_0$) yields $(k-1)!a_{-1}$.
- Example: Find the residue of $f(z) = \frac{1}{z^2(z-3)^3}$ at $z = 0$ and at $z = 3$.
  - From the expression given we see $f(z)$ has a pole of order 2 at $z = 0$.
  - Applying the residue calculation formula we have $z^2 f(z) = \frac{1}{(z-3)^3}$ so that $\frac{1}{1!}\frac{d}{dz}[z^2 f(z)] = \frac{-3}{(z-3)^4}$. Then setting $z = 0$ yields $\text{Res}_f(0) = \boxed{-\frac{1}{27}}$.
  - Likewise, f has a pole of order 3 at $z = 3$.
  - Applying the residue calculation formula yields $(z-3)^3 f(z) = \frac{1}{z^2}$, so that $\frac{1}{2!}\frac{d^2}{dz^2}[(z-3)^3 f(z)] = \frac{3}{z^4}$. Setting $z = 3$ yields $\text{Res}_f(3) = \boxed{\frac{1}{27}}$.
- Theorem (Residue Theorem): Suppose that R is a bounded simply connected region and f is a meromorphic function on R with poles $z_1, z_2, \ldots, z_n$ in R. Then for any closed contour γ not passing through any of the z_i, we have $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} W_\gamma(z_k)\text{Res}_f(z_k)$.

- The idea of the proof is simply to subtract off all of the negative-power terms from the local Laurent expansions at each pole, resulting in a holomorphic function that necessarily integrates to zero around γ. Then each of the remaining finite series can just be integrated directly.
- Proof: For each k with $1 \le k \le n$, let $s_k(z)$ be the negative-power part of the Laurent series expansion for f centered at $z = z_k$.
- Then each s_k is holomorphic for all $z \neq z_k$, and $f(z) - s_k(z)$ has a removable singularity at z_k (since we have removed all of the negative-power terms from that local Laurent series).
- This means $f(z) - \sum_{k=1}^n s_k(z)$ has removable singularities at all of the z_k, so after removing the singularities, it is holomorphic on R.
- Then by Cauchy's integral theorem, we have $\int_\gamma [f(z) - \sum_{k=1}^n s_k(z)]\, dz = 0$, so rearranging yields $\int_\gamma f(z)\, dz = \sum_{k=1}^n \int_\gamma s_k(z)\, dz$.
- Then, as we have previously shown by direct integration and the definition of the winding number, for $s_k(z) = \sum_{n=-d}^{-1} a_n (z - z_k)^n$ we have $\int_\gamma s_k(z)\, dz = 2\pi i \cdot W_\gamma(z_k) \cdot a_{-1} = 2\pi i \cdot W_\gamma(z_k) \cdot \mathrm{Res}_f(z_k)$.
- Plugging in for each $\int_\gamma s_k(z)\, dz$ and summing immediately yields the desired formula.

- In the particular special case where γ is the counterclockwise boundary of the simply connected region R (which in practice is the situation we are usually concerned with), the residue theorem has the following form:
- Corollary (Cauchy's Residue Theorem): Suppose that R is a bounded simply connected region with counterclockwise boundary γ, and f is a meromorphic function on R with poles $z_1, z_2, \dots, z_n$ in R. Then $\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^n \mathrm{Res}_f(z_k)$.
  - We emphasize here that the only residues that contribute to the sum are those in R, namely, on the interior of γ.
- The most direct application of the residue theorem is to compute contour integrals of meromorphic functions.
- Example: Evaluate $\int_\gamma \dfrac{2e^{2z} - z}{(z-2)(z-4)}\, dz$ where γ is the counterclockwise circle $|z| = 3$.
  - The integrand $f(z)$ is meromorphic with simple poles at $z = 2$ and $z = 4$.
  - However, only the first of these poles lies inside the circle $|z| = 3$, so by the residue theorem, we see $\int_\gamma f(z)\, dz = 2\pi i \cdot \mathrm{Res}_f(2)$.
  - To compute the residue at 2 we compute $(z-2)f(z) = \dfrac{2e^{2z} - z}{z - 4}$ and then evaluate at $z = 2$ to obtain $\mathrm{Res}_f(2) = \dfrac{2e^2 - 2}{-2} = 1 - e^2$. Thus, the value of the integral is $\boxed{2\pi i(1 - e^2)}$.
- Example: Evaluate $\int_\gamma \dfrac{e^z}{(z^2+1)^2}\, dz$ where γ is the counterclockwise circle $|z| = 6$ followed by the boundary of the counterclockwise upper semicircle.
  - The integrand $f(z)$ is meromorphic with poles of order 2 at $z = i$ and $z = -i$.
  - Since the contour winds around $z = i$ twice and $z = -i$ once, by the residue theorem we have $\int_\gamma f(z)\, dz = 2\pi i \cdot [2\mathrm{Res}_f(i) + \mathrm{Res}_f(-i)]$.
  - To compute the residue at i we compute $\dfrac{d}{dz}[(z-i)^2 f(z)] = \dfrac{d}{dz}[\dfrac{1}{(z+i)^2}] = \dfrac{-2}{(z+i)^3}$ and then evaluate at $z = i$ to obtain $\mathrm{Res}_f(i) = \dfrac{-2}{(2i)^3} = -\dfrac{i}{4}$.
  - Likewise, for the residue at $-i$, we have $\dfrac{d}{dz}[(z+i)^2 f(z)] = \dfrac{d}{dz}[\dfrac{1}{(z-i)^2}] = \dfrac{-2}{(z-i)^3}$ and then evaluate at $z = -i$ to obtain $\mathrm{Res}_f(i) = \dfrac{-2}{(-2i)^3} = \dfrac{i}{4}$.
  - Thus, the value of the integral is $2\pi i \cdot [2(-i/4) + i/4] = \boxed{\pi/2}$.

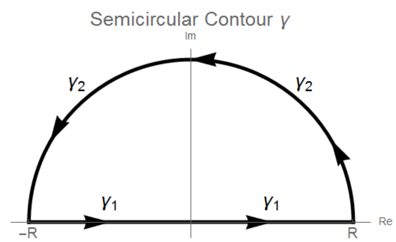
- Example: Evaluate $\int_\gamma \frac{1}{e^z - 1}\,dz$ where γ is the counterclockwise boundary of the rectangle with vertices $\pm 1 - i\pi$ and $\pm 1 + 5i\pi$.
  - The integrand $f(z)$ is meromorphic with poles at $z = 2\pi i k$ for each integer k.
  - The poles with $k = 0$, 1, and 2 lie inside the rectangle while the others lie outside, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i \cdot [\mathrm{Res}_f(0) + \mathrm{Res}_f(2\pi i) + \mathrm{Res}_f(4\pi i)]$.
  - For each residue, we compute $(z - 2\pi i k)f(z) = \frac{z - 2\pi i k}{e^z - 1}$ and then take the limit as $z \to 2\pi i k$ to obtain $\mathrm{Res}_f(2\pi i k) = \lim_{z \to 2\pi i k} \frac{z - 2\pi i k}{e^z - 1} = \lim_{z \to 2\pi i k} \frac{1}{e^z} = 1$ by L'Hôpital's rule (or equivalently, the reciprocal of the definition of the derivative).
  - Since each residue is 1, we see $\int_\gamma \frac{1}{e^z - 1}\,dz = \boxed{6\pi i}$.
- As an application of the residue theorem, we can give a procedure for evaluating trigonometric integrals[3] of the form $\int_0^{2\pi} r(\cos\theta, \sin\theta)\,d\theta$ where r is a rational function.
  - Explicitly, since $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, if we instead substitute $z = e^{i\theta}$ then the integral $\int_0^{2\pi} r(\cos\theta, \sin\theta)\,d\theta$ becomes $\int_\gamma r(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}) \cdot \frac{1}{iz}\,dz$ where γ is the unit circle.
  - Thus, setting $f(z)$ to be the rational function $f(z) = r(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}) \cdot \frac{1}{iz}$, we wish to calculate the integral of $f(z)$ around the unit circle. As an immediate application of the residue theorem, we see that this integral equals $2\pi i$ times the sum of the residues of $f(z)$ at each of its poles inside the unit circle.
- Example: Evaluate $\int_0^{2\pi} \frac{1}{2 + \cos\theta}\,d\theta$.
  - Using the method described above we calculate $f(z) = r(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}) \cdot \frac{1}{iz} = \frac{-2i}{z^2 + 4z + 1}$ which has simple poles at $z = -2 \pm \sqrt{3}$. The only one of these inside the unit circle is $z = -2 + \sqrt{3}$, and the residue of f there is $\lim_{z \to (-2+\sqrt{3})} \frac{-2i}{2z + 4|_{z=-2+\sqrt{3}}} = -\frac{i}{\sqrt{3}}$.
  - Hence by the residue theorem we see that $\int_0^{2\pi} \frac{1}{2 + \cos\theta}\,d\theta = \int_\gamma f(z)\,dz = 2\pi i \cdot (-\frac{i}{\sqrt{3}}) = \boxed{\frac{2\pi}{\sqrt{3}}}$.
- Example: Evaluate $\int_0^{2\pi} \frac{1}{1 + 4\sin^2\theta}\,d\theta$.
  - Using the method described above we calculate $f(z) = r(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}) \cdot \frac{1}{iz} = \frac{iz}{1 - 3z^2 + z^4}$ which has simple poles at $z = \frac{\pm 1 \pm \sqrt{5}}{2}$. Two of these lie inside the unit circle: $z_0 = \pm\frac{1 - \sqrt{5}}{2}$, at each of which the residue is $\frac{iz_0}{-6z_0 + 4z_0^3} = \frac{i}{-6 + 4z^2} = -\frac{i}{2\sqrt{5}}$.
  - Hence by the residue theorem we see that $\int_0^{2\pi} \frac{1}{2 + \cos\theta}\,d\theta = \int_\gamma f(z)\,dz = 2\pi i \cdot 2(-\frac{i}{2\sqrt{5}}) = \boxed{\frac{2\pi}{\sqrt{5}}}$.

[3] We also remark that there is a standard substitution $t = \tan(\theta/2)$ often called the Weierstrass substitution, which has $d\theta = \frac{2\,dt}{1 + t^2}$, $\cos\theta = \frac{1 - t^2}{1 + t^2}$, and $\sin\theta = \frac{2t}{1 + t^2}$ that allows indefinite integrals of this form to be evaluated by converting them into rational functions of t. The reader may find it enlightening to consider the similarities and differences between the Weierstrass substitution and our method using residue calculus.

4.3.2 Calculating Definite Integrals via Residue Calculus: Circular Contours

- One of the most classical and (perhaps) unexpected applications of residue calculus is its use in evaluating definite integrals on the real line that resist other techniques.
  - Rather than diving immediately into interesting examples, we will illustrate the ideas with an integral that can be easily computed using standard calculus methods.

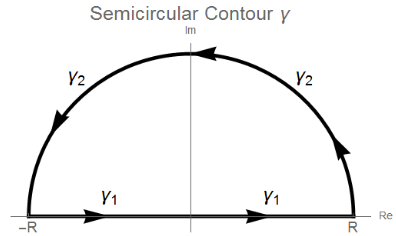
- Example: Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$.
  - By definition, this integral is the limit $\lim_{R\to\infty} \int_{\gamma_1} \frac{1}{z^2+1}\,dz$ where γ_1 is the line segment along the real axis from $-R$ to R.
  - In order to apply the residue theorem we need a closed contour, so we close this contour by adding in the contour γ_2 traversing the upper half of the circle $|z| = R$ counterclockwise from R to $-R$:



  - By the residue theorem applied to $f(z) = \frac{1}{z^2+1}$ on this closed contour $\gamma = \gamma_1 \cup \gamma_2$, since $f(z)$ has single simple poles at $z = i$ (inside the contour for large R) and $z = -i$ (outside the contour), we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \text{Res}_f(i) = 2\pi i \cdot \lim_{z\to i} \frac{z+i}{z^2+1} = 2\pi i \cdot \frac{1}{2i} = \pi$.
  - Thus, we see $\lim_{R\to\infty}[\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz] = \pi$, so to evaluate the integral we are after, we just need to find $\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz$.
  - On γ_2 we have $|f(z)| = \frac{1}{|z^2+1|} \le \frac{1}{R^2-1}$ by the triangle inequality, so applying the arclength estimate to γ_2 we have $\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{\pi R}{R^2-1} \to 0$ as $R \to \infty$.
  - Thus, $\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = 0$, and so we obtain $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_{\gamma_1} \frac{1}{z^2+1}\,dz = \boxed{\pi}$.
  - Remark: As we should expect, this calculation agrees with the result obtained directly via the fundamental theorem of calculus: $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \tan^{-1}(x)|_{x=-\infty}^{\infty} = (\pi/2) - (-\pi/2) = \pi$.

- In general, the technique consists of "complexifying" the integrand in some manner, closing the contour, making a residue calculation, and then estimating or otherwise dealing with the integral along the added pieces of the contour.

- Example: Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^5}\,dx$.
  - This integral is $\lim_{R\to\infty} \int_{\gamma_1} \frac{1}{(z^2+1)^5}\,dz$ where γ_1 is the line segment along the real axis from $-R$ to R. As above, we close this contour by adding in the contour γ_2 traversing the upper half of the circle $|z| = R$ counterclockwise from R to $-R$:



- By the residue theorem applied to $f(z) = \dfrac{1}{(z^2+1)^5}$ on this closed contour $\gamma = \gamma_1 \cup \gamma_2$, since $f(z)$ has poles of order 5 at $z = i$ (inside the contour for large R) and $z = -i$ (outside the contour), we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \mathrm{Res}_f(i)$.
- The residue is given by $\dfrac{1}{4!}\lim_{z\to i}\dfrac{d^4}{dz^4}[(z-i)^5\dfrac{1}{(z^2+1)^5}] = \dfrac{1}{4!}\lim_{z\to i}\dfrac{5\cdot 6\cdot 7\cdot 8}{(z+i)^9} = -\dfrac{35i}{256}$.
- On γ_2 we have $|f(z)| = \dfrac{1}{|z^2+1|^5} \le \dfrac{1}{(R^2-1)^5}$ by the triangle inequality, so the arclength estimate yields $\left|\int_{\gamma_2} f(z)\,dz\right| \le \dfrac{\pi R}{(R^2-1)^5} \to 0$ as $R \to \infty$.
- Thus, $\lim_{R\to\infty}\int_{\gamma_2} f(z)\,dz = 0$, and so we obtain $\int_{-\infty}^{\infty}\dfrac{1}{(x^2+1)^5}\,dx = 2\pi i\cdot(-\dfrac{35i}{256}) = \boxed{\dfrac{35\pi}{128}}$.
- Remark: One may compute this integral directly using standard real-valued calculus techniques (namely, substituting $x = \tan u$ to obtain $\int_{-\pi/2}^{\pi/2}\cos^8 u\,du$ and then reducing using double-angle identities, which does indeed yield $35\pi/128$) but it is a far messier computation.

* In both of the examples so far, the integral along a component of the contour tends to zero as the contour grows large.
  - In general, we can see that if $p(z) = a_d z^d + \cdots + a_0$ is any polynomial of degree $d \ge 2$ and γ_2 is the upper semicircle from R to $-R$, then since $\lim_{|z|\to\infty}\left|\dfrac{p(z)}{z^d}\right| = |a_d|$, for sufficiently large R with z on γ_2 we see $|p(z)| \ge cR^d$ for any constant c with $0 < c < |a_d|$.
  - Therefore, we obtain an arclength estimate $\left|\int_{\gamma_2} p(z)\,dz\right| \le 2\pi R\cdot\dfrac{1}{cR^d} = \dfrac{2\pi}{c}R^{1-d} \to 0$ as $R\to\infty$.
  - More generally, if $f(z)$ is any function such that there exist constants $c > 0$ and $\epsilon > 0$ such that $|f(z)| \le cR^{-1-\epsilon}$ for $|z| = R$ (equivalently, if $|f(z)| = O(R^{-1-\epsilon})$ in big-oh notation), we obtain a similar arclength estimate $\left|\int_{\gamma_2} f(z)\,dz\right| \to 0$ as $R\to\infty$ along any portion γ_2 of the circle $|z| = R$.

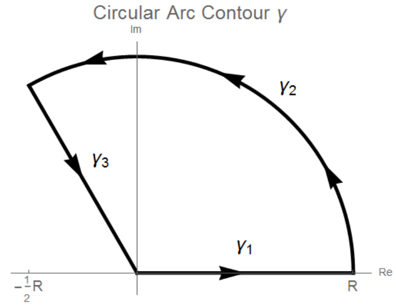
* Example: Evaluate $\int_{-\infty}^{\infty}\dfrac{1}{x^4+4}\,dx$.
  - This integral is $\lim_{R\to\infty}\int_{\gamma_1}\dfrac{1}{z^4+4}\,dz$ where γ_1 is the line segment along the real axis from $-R$ to R. As above we take γ_2 to be the upper semicircle from $-R$ to R and $\gamma = \gamma_1\cup\gamma_2$ as $R\to\infty$.
  - The integrand has simple poles at the four values of z with $z^4+4 = 0$, which are $z = \pm 1 \pm i$, and only $z = 1-i$ and $z = 1+i$ lie inside γ. Since both poles are simple, with $g(z) = 1/f(z) = 4+z^4$, the residues are $\mathrm{Res}_f(1+i) = \dfrac{1}{g'(1+i)} = \dfrac{-1-i}{16}$ and $\mathrm{Res}_f(-1+i) = \dfrac{1}{g'(-1+i)} = \dfrac{1-i}{16}$.
  - By our estimates, since $4+z^4$ has degree 4, on γ_2 we have $\left|\int_{\gamma_2} f(z)\,dz\right| \to 0$ as $R\to\infty$. Thus by the residue theorem, we see $\int_{-\infty}^{\infty}\dfrac{1}{x^4+4}\,dx = \int_\gamma\dfrac{1}{z^4+4}\,dz = 2\pi i[\mathrm{Res}_f(1+i)+\mathrm{Res}_f(1-i)] = \boxed{\dfrac{\pi}{8}}$.
  - Remark: Using partial fractions one may decompose $\dfrac{1}{x^4+4} = \dfrac{(-x+2)/8}{x^2-2x+2} + \dfrac{(x+2)/8}{x^2+2x+2}$ and then evaluate the indefinite integral: but again, this computation is quite messy.

- In some cases, we cannot obtain a suitable estimate directly, and must resort to changing the function.
- Example: Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$.
    - If we apply the method used so far, we write $\lim_{R\to\infty} \int_{\gamma_1} \frac{\cos z}{x^2+1}\,dz$ where γ_1 is the line segment along the real axis from $-R$ to R and then take γ_2 to be the upper semicircle from $-R$ to R and $\gamma = \gamma_1 \cup \gamma_2$ as $R \to \infty$.
    - However, this time, we run into a problem: the function $\frac{\cos z}{z^2+1}$ on the semicircle γ_2 no longer has a convenient estimate, since in general $\frac{\cos z}{z^2+1} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{z^2+1}$ for $z = x+iy$. We have $\left|e^{ix}e^{-y}\right| = e^{-y} \le 1$, but the other term $\left|e^{-ix}e^{y}\right| = e^{y}$ can be very large (up to e^R) on γ_2: this is problematic since it gives an arclength estimate $\left|\int_{\gamma_1} \frac{\cos z}{x^2+1}\,dz\right| \le \pi R \frac{1+e^R}{R^2-1}$ which does not go to zero as $R \to \infty$.
    - What we can do instead is observe that $\frac{\cos x}{x^2+1} = \mathrm{Re}[\frac{e^{ix}}{x^2+1}]$ when x is real, and so $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \mathrm{Re}[\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx]$.
    - Therefore, we try taking take the function $f(z) = \frac{e^{iz}}{z^2+1}$ and integrating around $\gamma = \gamma_1 \cup \gamma_2$. This function has simple poles at $z = -i$ and at $z = i$, only the latter of which lies inside γ. The residue at $z = i$ is $\lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{e^{iz}}{z+i} = -\frac{i}{2e}$.
    - Since on γ_2 we have $|f(z)| = \frac{\left|e^{iz}\right|}{|z^2+1|} \le \frac{1}{R^2-1}$, our estimates imply $\int_{\gamma_2} f(z)\,dz \to 0$ as $R \to \infty$. Thus by the residue theorem, we see $\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \int_{\gamma} \frac{e^{iz}}{z^2+1}\,dz = 2\pi i[\mathrm{Res}_f(i)] = 2\pi i \cdot (-\frac{i}{2e}) = \frac{\pi}{e}$.
    - Taking the real part then shows $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \boxed{\frac{\pi}{e}}$.
    - Remark: Unlike the previous examples, in this case the indefinite integral $\int \frac{\cos x}{x^2+1}\,dx$ is non-elementary, and thus cannot be evaluated using typical calculus techniques.
- In general, using a semicircular contour like the ones in the examples above will be effective for any meromorphic function $f(z)$ that has only finitely many poles and decreases sufficiently rapidly as $|z| \to \infty$.
    - More precisely, if there exist positive constants A and ϵ such that $|f(z)| \le A/|z|^{1+\epsilon}$ for sufficiently large $|z|$, then the integral of $f(z)$ on the semicircle γ_2 will tend to zero as $R \to \infty$.
    - Then, assuming f has only finitely many poles $z_1, \dots, z_k$ in the upper half-plane, the residue theorem yields immediately that $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_f(z_k)$.

4.3.3 Calculating Definite Integrals via Residue Calculus: Circular Contours With Detours

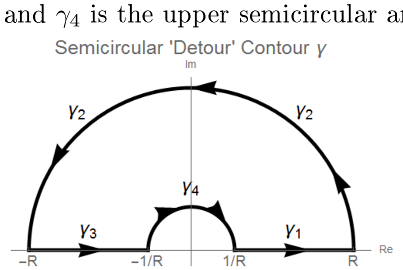
- In many other cases, we must resort to contours more complicated than a semicircle. In such situations often we will end up with several components that may all contribute to the value of the integral we are seeking.
    - The computations are often motivated by trying to evaluate the integral of a function similar to the one we seek to integrate on the real line, but finding the right choice of contour and function is as much an art form as a science.
- Example: Evaluate $\int_{0}^{\infty} \frac{1}{x^3+1}\,dx$.
    - Here, it is sensible to try $f(z) = \frac{1}{z^3+1}$, but we cannot use the upper semicircle as our contour because we only want to integrate from 0 to ∞ on the real axis.

- Instead, we can exploit the rotational symmetry of the function (namely, that $f(e^{2\pi i/3}z) = f(z)$) by using a smaller portion of the circle.
- Explicitly, take $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where γ_1 is the segment from 0 to R, γ_2 is the counterclockwise arc of $|z| = R$ from R to $Re^{2\pi i/3}$, and γ_3 is the segment from $Re^{2\pi i/3}$ back to 0:



- Letting $I_R = \int_0^R \frac{1}{x^3+1}\,dx$ and $I = \int_0^\infty \frac{1}{x^3+1}\,dx$, by parametrizing the segments we see $\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{1}{t^3+1}\,dt = I_R$ and $\int_{\gamma_3} f(z)\,dz = -\int_0^R \frac{1}{(te^{2\pi i/3})^3+1} \cdot e^{2\pi i/3}\,dt = -e^{2\pi i/3} I_R$.
- Additionally, since $f(z)$ is a polynomial of degree 3, we see that $\int_{\gamma_2} f(z)\,dz \to 0$ as $R \to \infty$.
- The function $f(z)$ has simple poles at $z = e^{i\pi/3}, e^{i\pi}, e^{5i\pi/3}$, and only the first one lies inside γ, with residue given by $\lim_{z\to e^{i\pi/3}} \frac{z - e^{i\pi/3}}{z^3+1} = \frac{1}{3e^{2\pi i/3}} = e^{-2\pi i/3}/3$ via L'Hôpital's rule.
- Then by the residue theorem, we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \text{Res}_f(e^{i\pi/3}) = \pi(-3i\sqrt{3} - 3i)$. Taking $R \to \infty$ produces $2\pi i \cdot 3e^{2i\pi/3} = \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = I + 0 + (-e^{2\pi i/3})I$.
- Solving for I yields $I = \frac{2\pi i \cdot e^{-2\pi i/3}/3}{1 - e^{2\pi i/3}} = \boxed{\frac{2\pi}{3\sqrt{3}}}$.

- <u>Example</u>: Evaluate the sinc integral $\int_0^\infty \frac{\sin x}{x}\,dx$.
  - First we remark that this integral converges near 0 since $\frac{\sin z}{z}$ has a removable singularity at 0 as we have previously seen. It also converges as $x \to \infty$ via integration by parts: $\int \frac{\sin x}{x}\,dx = -\frac{\cos x}{x^2} - \int \frac{\cos x}{x^2}\,dx$ and the latter integral is absolutely convergent as $x \to \infty$.
  - In order to use a circular contour, we must change the function, since as we saw above, $\sin z$ does not obey the necessary bound on $|z| = R$.
  - So we will again try using the function $f(z) = \frac{e^{iz}}{z}$, since $\text{Im}[\frac{e^{ix}}{x}] = \frac{\sin x}{x}$ for real x.
  - However, $f(z)$ has a pole at $z = 0$, so we cannot take a contour that passes through 0. So instead, we will take a contour that "detours" a small amount around zero: explicitly, take $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where γ_1 is the segment from $1/R$ to R, γ_2 is the upper semicircular arc of radius R from R to $-R$, γ_3 is the segment from $-R$ to $-1/R$, and γ_4 is the upper semicircular arc from $-1/R$ to $1/R$:



  - Then since f has no poles inside γ we see $\int_\gamma f(z)\,dz = 0 = \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$. Now we calculate the integral on each piece.

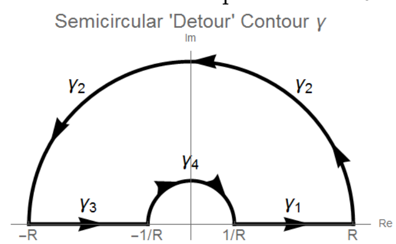
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^{R} \frac{e^{it}}{t}\,dt$.
- On γ_2 we have $\int_{\gamma_2} f(z)\,dz = \int_0^{\pi} \frac{e^{iR(\cos t + i\sin t)}}{Re^{it}} \cdot iRe^{it}\,dt = i\int_0^{\pi} e^{iR\cos t - R\sin t}\,dt$ which is bounded above in absolute value by $\int_0^{\pi} e^{-R\sin t}\,dt \le \int_0^{\pi} e^{-Rt}\,dt = (1-e^{-\pi R})/R$, which goes to zero as $R \to \infty$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = \int_{-R}^{-1/R} \frac{e^{it}}{t}\,dt = -\int_{1/R}^{R} \frac{e^{-it}}{t}\,dt$ upon negating t.
- Finally, on γ_4 (noting that the path is traversed clockwise, so we must negate the integral), we have $\int_{\gamma_4} f(z)\,dz = -\int_0^{\pi} \frac{e^{i(\cos t + i\sin t)/R}}{e^{it}/R} \cdot \frac{i}{R} e^{it}\,dt = -i\int_0^{\pi} e^{(-\sin t + i\cos t)/R}\,dt$. As $R \to \infty$ the integrand tends uniformly to $e^0 = 1$, so the integral approaches $-i\pi$.
- In particular we can compute $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = \int_{1/R}^{R} \frac{e^{it}}{t}\,dt - \int_{1/R}^{R} \frac{e^{-it}}{t}\,dt = \int_{1/R}^{R} \frac{2i\sin t}{t}\,dt$, and so as $R \to \infty$ this sum approaches $2i\int_0^{\infty} \frac{\sin t}{t}\,dt$.
- Thus, taking $R \to \infty$ in $\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz = 0$ produces $2i\int_0^{\infty} \frac{\sin t}{t}\,dt + 0 + (-i\pi) = 0$, from which we conclude $\int_0^{\infty} \frac{\sin t}{t}\,dt = \boxed{\frac{\pi}{2}}$.
- Remark: Using the entire semicircle, which at first seems like the natural approach for evaluating $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$ (which is twice our desired integral by symmetry of the integrand), in fact allows us to find a convenient "cancellation" when integrating $\frac{e^{iz}}{z}$ along the two components of γ on the real axis. Our selection of a symmetric contour is precisely what allows us to sum the contributions and thereby "cancel" the pole at $z = 0$ (which only causes divergence in the integral for the real part of $\frac{e^{ix}}{x}$ as $x \to 0$).

- In the example above, we used a contour that took a small detour around one of the poles of the function, and obtained an integral along a semicircular arc around the pole whose value conveniently came out to be $-i\pi$.
  - Roughly speaking, the contour is half of a full path enclosing the pole (the other half being the bottom semicircle), and the corresponding integral comes out to be half of $2\pi i$ times the value of the winding number around the pole (-1, since the circle was oriented clockwise).
  - In fact, whenever we have a contour that detours along a circular arc around a simple pole, we obtain a similar result:

- Lemma (Fractional Residues): Suppose f is meromorphic with a simple pole at z_0 and γ_r is the circular arc parametrized by $\gamma_r(t) = z_0 + re^{it}$ for $\theta_1 \le t \le \theta_2$. Then $\lim_{r\to 0+} \int_{\gamma_r} f(z)\,dz = i(\theta_2 - \theta_1)\mathrm{Res}_f(c)$.
  - The reason for the name of the lemma is that the circular-arc integral equals $\frac{1}{2\pi}(\theta_2 - \theta_1)$ times the value obtained by integrating around a full circle: in other words, it behaves as if we are computing the appropriate "fractional residue" based on what fraction of the circle we are including.
  - Proof: By hypothesis, the local Laurent expansion for $f(z)$ is $f(z) = a_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $0 < |z - z_0| \le R$.
  - Then $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is holomorphic hence has an antiderivative $G(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$ so by the fundamental theorem of line integrals we see $\int_{\gamma_r} g(z)\,dz = G(\gamma_r(\theta_2)) - G(\gamma_r(\theta_1))$. Then $\lim_{r\to 0+} \int_{\gamma_r} g(z)\,dz = \lim_{r\to 0}[G(\gamma_r(\theta_2)) - G(\gamma_r(\theta_1))] = G(z_0) - G(z_0) = 0$ since γ_r shrinks towards z_0 and G is continuous.
  - Now $\int_{\gamma_r} f(z)\,dz = \int_{\gamma_r} a_{-1}(z-z_0)^{-1}\,dz + \int_{\gamma_r} g(z)\,dz$, and since $\int_{\gamma_r} a_{-1}(z-z_0)^{-1}\,dz = a_{-1}\int_{\theta_1}^{\theta_2} \frac{1}{re^{it}} \cdot ire^{it} = i(\theta_2 - \theta_1)a_{-1}$, taking $r \to 0+$ yields the desired result.

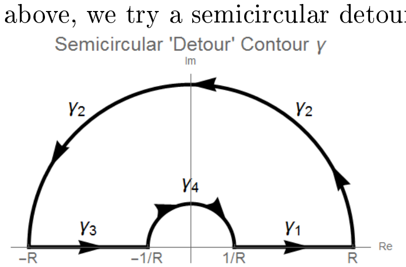
- Example: Evaluate $\int_{-\infty}^{\infty} \frac{\sin 2x}{x(x^2+1)}\,dx$.

- First we remark that this integral converges absolutely, since the integrand has a removable singularity at 0 and is bounded by $\frac{1}{|x|^3}$ for large $|x|$.
- As in the last examples, the sine function is not well behaved on the circle $|z| = R$ so we instead work with $f(z) = \frac{e^{2iz}}{z(z^2+1)}$ and observe that $\text{Im}[\frac{e^{ix}}{x}] = \frac{\sin x}{x}$ for real x.
- As before we take a contour that detours around the pole at $z = 0$:



- Then γ encloses the single simple pole at $z = i$ of $f(z) = \frac{e^{2iz}}{z(z^2+1)}$ at which the residue equals $\lim_{z\to i} \frac{z-i}{z^2+1}\frac{e^{2iz}}{z} = \frac{1}{2i}\cdot\frac{e^{-2}}{i} = -\frac{1}{2}e^{-2}$, so by the residue theorem we see $\int_\gamma f(z)\,dz = -\pi i e^{-2}$.
- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^{R} \frac{e^{2it}}{t(t^2+1)}\,dt$ which tends $\int_0^\infty \frac{e^{2it}}{t(t^2+1)}\,dt$ as $R\to\infty$.
- On γ_2 we have $|f(z)| = \left|\frac{e^{2iR(\cos t + i\sin t)}}{(Re^{it})(R^2e^{2it}+1)}\right| \le \frac{e^{-2R\sin t}}{R(R^2-1)} \le \frac{1}{R(R^2-1)}$ so since the arclength of γ_2 is πR we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-2}) \to 0$ as $R\to\infty$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R}^{R} \frac{e^{-2it}}{t(t^2+1)}\,dt$ which tends $-\int_0^\infty \frac{e^{-2it}}{t(t^2+1)}\,dt$ as $R\to\infty$.
- On γ_4, since it is a clockwise semicircle tending to zero around the simple pole at $z = 0$ of $f(z)$ at which the residue equals $\lim_{z\to 0} zf(z) = 1$, by the fractional residues lemma the integral tends to $-\pi i$ as $R\to\infty$.
- So taking $R\to\infty$ and putting all of this together yields $-\pi i e^{-2} = \int_0^\infty \frac{e^{2it}}{t(t^2+1)}\,dt + 0 - \int_0^\infty \frac{e^{-2it}}{t(t^2+1)}\,dt - \pi i$ which simplifies to $\pi i(1-e^{-2}) = \int_0^\infty \frac{2\sin 2t}{t(t^2+1)}\,dt$ so that $\int_0^\infty \frac{\sin 2t}{t(t^2+1)}\,dt = \boxed{\frac{\pi i(1-e^{-2})}{2}}$.

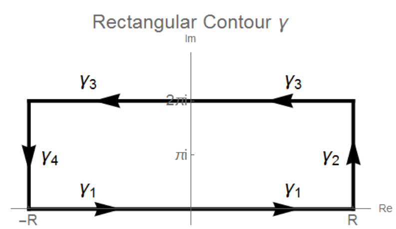
- Example: Evaluate $\int_0^\infty \frac{\sqrt[5]{x}}{x^2+1}\,dx$.
  - The most natural integrand is $f(z) = \frac{z^{1/5}}{z^2+1} = \frac{e^{\text{Log}(z)/5}}{z^2+1}$. However, this presents an obvious difficulty: namely, that $\text{Log}(z)$ is not holomorphic on the interval $[0,\infty)$, which is precisely the path we want to integrate along!
  - But we can easily rectify this issue by selecting a different branch cut of the logarithm whose branch cut is not along the real axis, but elsewhere, such as along the negative imaginary axis: namely, to take $\log(z)$ with imaginary part in $[-\pi/2, 3\pi/2)$ rather than $[0, 2\pi)$.
  - So, with $f(z) = \frac{e^{\log(z)/5}}{z^2+1}$, we see that $f(z)$ has a pole at $z = i$ and singularities along $[0, -i\infty)$: thus our contour must cut around zero. As above, we try a semicircular detour:



- Then γ encloses the single simple pole at $z = i$ of $f(z) = \dfrac{e^{\log(z)/5}}{z^2+1}$ at which the residue equals $\lim_{z\to i} \dfrac{z-i}{z^2+1} e^{\log(z)/5} = \dfrac{e^{\log(i)/5}}{2i} = \dfrac{e^{i\pi/10}}{2i}$, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i \cdot \dfrac{e^{i\pi/10}}{2i}$.
- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^{R} \dfrac{e^{\log(t)/5}}{t^2+1}\,dt = \int_{1/R}^{R} \dfrac{\sqrt[5]{t}}{t^2+1}\,dt$ which tends to our desired integral I as $R \to \infty$.
- On γ_2 we have $|f(z)| = \left|\dfrac{e^{\log(Re^{it})/5}}{(Re^{it})^2+1}\right| \le \dfrac{R^{1/5}}{R^2-1} = O(R^{-9/5})$ so since the arclength of γ_2 is πR we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-4/5}) \to 0$ as $R \to \infty$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = \int_{1/R}^{R} \dfrac{e^{\log(-t)/5}}{(-t)^2+1}\,dt = \int_{1/R}^{R} \dfrac{e^{i\pi/5}\sqrt[5]{t}}{t^2+1}\,dt$ which tends to $e^{i\pi/5}I$ as $R \to \infty$.
- On γ_4 we have $|f(z)| = \left|\dfrac{e^{\log(e^{it}/R)/5}}{(e^{it}/R)^2+1}\right| \le \dfrac{R^{-1/5}}{1-1/R^2} = O(R^{-1/5})$ so since the arclength of γ_4 is π/R we see that $\int_{\gamma_4} f(z)\,dz = O(R^{-6/5}) \to 0$ as $R \to \infty$.
- So taking $R \to \infty$ and putting all of this together yields $\pi e^{i\pi/10} = I + 0 - e^{i\pi/5}I + 0$ so that $I = \dfrac{\pi e^{i\pi/10}}{1+e^{i\pi/5}} = \dfrac{\pi}{e^{-i\pi/10}+e^{i\pi/10}} = \boxed{\dfrac{\pi}{2\cos(\pi/10)}}$.

4.3.4 Calculating Definite Integrals via Residue Calculus: Other Contours

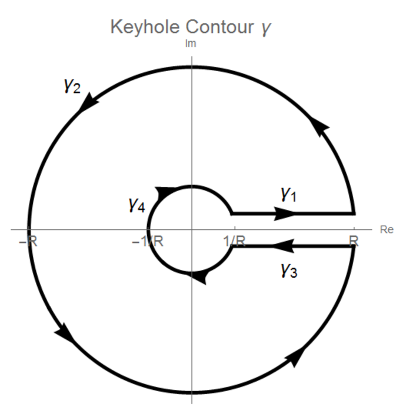
- There are many other integrals we can evaluate with a more inspired selection of contour. We illustrate with some examples.
- Example: For $0 < a < 1$, evaluate $\int_{-\infty}^{\infty} \dfrac{e^{ax}}{e^x+1}\,dx$.
  - First, note that this integral converges (absolutely): as $x \to -\infty$ the integrand is asymptotic to $e^{-a|x|}$ while as $x \to +\infty$ the integrand is asymptotic to $e^{(a-1)x}$, whose integrals both converge in the given limit.
  - A sensible choice of integrand is $f(z) = \dfrac{e^{az}}{e^z+1}$. This function has simple poles at $z = \pi i(1+2k)$ for integers k, which will cause difficulties if we integrate over a large semicircular contour since we will have to sum the residues over all of the poles. This is not necessarily a problem, but a more serious issue is that we do not get a good estimate for $|f(z)|$ on the circle $|z| = R$, since the denominator can take very small values (or even the value zero!) depending on the value of R.
  - We can avoid these issues if instead we pick a rectangular contour enclosing just the pole at $z = \pi i$ that has one side extending far along the real axis. Furthermore, since the denominator is periodic with period $2\pi i$, while the numerator changes by a factor of $e^{2\pi i a} \neq 1$ upon increasing z by $2\pi i$, if we select the height of the rectangle to be $2\pi i$, then the parallel side of the contour will be expressible in terms of the original integral.
  - So we take the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where γ_1 is the segment from $-R$ to R, γ_2 is the segment from R to $R+2\pi i$, γ_3 is the segment from $R+2\pi i$ to $-R+2\pi i$, and γ_4 is the segment from $-R+2\pi i$ to R:



- Then γ encloses the single simple pole at $z = \pi i$ of $f(z) = \dfrac{e^{az}}{e^z+1}$ at which the residue equals $\lim_{z\to\pi i} \dfrac{z-\pi i}{e^z+1} e^{az} = \dfrac{e^{a\pi i}}{\frac{d}{dz}(e^z+1)|_{z=\pi i}} = -e^{a\pi i}$, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i(-e^{a\pi i}) = \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$.
- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \dfrac{e^{at}}{e^t+1}\,dt$ which tends to our desired integral I as $R \to \infty$.
- On γ_2 we have $\int_{\gamma_2} f(z)\,dz = \int_0^{2\pi} \dfrac{e^{a(R+it)}}{e^{R+it}+1}\,idt$, and since $\left|\dfrac{e^{a(R+it)}}{e^{R+it}+1}\right| \le \dfrac{e^{aR}}{e^R-1} \to 0$ as $R \to \infty$ the integral tends to 0.
- On γ_3, noting the reversed orientation, we have $\int_{\gamma_3} f(z)\,dz = -\int_{-R}^{R} \dfrac{e^{a(t+2\pi i)}}{e^{(t+2\pi i)}+1}\,dt = -e^{2\pi i a}\int_{-R}^{R} \dfrac{e^{at}}{e^t+1}\,dt$, which tends to $e^{-2\pi ai}I$ as $R\to\infty$.
- On γ_4 we have $\int_{\gamma_2} f(z)\,dz = \int_0^{2\pi} \dfrac{e^{a(-R+it)}}{e^{-R+it}+1}\,idt$, and since $\left|\dfrac{e^{a(-R+it)}}{e^{-R+it}+1}\right| \le \dfrac{e^{-aR}}{2} \to 0$ as $R \to \infty$ the integral tends to 0.
- So taking $R\to\infty$ and putting all of this together yields $-e^{a\pi i} = I + 0 - e^{2\pi i a} I + 0$ so that $I = \dfrac{-2\pi i e^{a\pi i}}{1-e^{2a\pi i}} = \dfrac{-2\pi i}{e^{-a\pi i}-e^{a\pi i}} = \dfrac{-2\pi i}{-2i\sin(a\pi)} = \boxed{\dfrac{\pi}{\sin(a\pi)}}$.

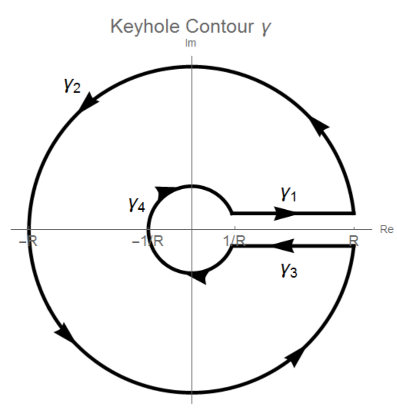
• <u>Example</u>: Evaluate $\int_0^\infty \dfrac{1}{x^2+3x+2}\,dx$.

- It seems obvious that we would want to select the function $f(z) = \dfrac{1}{z^2+3z+2}$ and then include the real interval $[0, R]$ as a portion of our contour.
- However, unlike the examples above where we exploited a convenient property of the integrand (e.g., rotational symmetry, periodicity) to convert the integral along a segment returning to the origin to something again involving the original integral I, here there is no obvious way to do that: neither a rotation $(re^{i\theta})^2 + 3(re^{i\theta}) + 2$ nor a translation $(r+it)^2 + 3(r+it) + 2$ is expressible as a nice multiple of $r^2 + 3r + 2$.
- What we will do instead is to work with the function $f(z) = \dfrac{\mathrm{Log}(z)}{z^2+3z+2}$. The main idea is that $f(z)$ has a jump discontinuity across the positive real axis that causes the values of $f(z)$ to differ by $2\pi i$ across the discontinuity, so if we integrate along a contour that passes just below the axis, its value will be $2\pi i$ greater than the corresponding integral just above the axis.
- We also want to have a component of the contour along the circle of radius R (so that the integral on that component tends to zero by the arclength estimate) and we must also avoid $z = 0$. Fitting all of these pieces together eventually leads to a contour often called the "keyhole contour" due to its visual similarity to a keyhole:



- Explicitly, we take $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where γ_1 is the segment from $1/R' + \epsilon i$ to $R'' + \epsilon i$, γ_2 is the counterclockwise arc along $|z| = R$ from $R'' + \epsilon i$ to $R'' - \epsilon i$, γ_3 is the segment from $R'' - \epsilon i$ to $1/R' - \epsilon i$, and γ_4 is the counterclockwise arc along $|z| = 1/R$ from $1/R' - \epsilon i$ to $1/R' + \epsilon i$, where $R' = \sqrt{1/R^2 - \epsilon^2}$, $R'' = \sqrt{R^2 - \epsilon^2}$, and ϵ is small relative to $1/R$ (e.g., $\epsilon = 1/R^2$).
- Then γ encloses the two simple poles at $z = -1$ and $z = -2$ of $f(z) = \dfrac{\text{Log}(z)}{z^2 + 3z + 2}$ whose residues are $\text{Res}_f(-1) = \lim_{z \to -1} \dfrac{z+1}{z^2 + 3z + 2} \text{Log}(z) = i\pi$ and $\text{Res}_f(-2) = \lim_{z \to -2} \dfrac{z+1}{z^2 + 3z + 2} \text{Log}(z) = -\ln 2 - i\pi$, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i[i\pi - \ln 2 - i\pi] = -2\pi i \ln 2$.
- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R'}^{R''} \dfrac{\text{Log}(t + \epsilon i)}{(t + \epsilon i)^2 + 3(t + \epsilon i) + 2}\,dt$, which tends to $\int_0^\infty \dfrac{\ln(t)}{t^2 + 3t + 2}\,dt$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = \left|\dfrac{\log(Re^{it})}{(Re^{it})^2 + 3(Re^{it}) + 2}\right| \le \dfrac{\ln(R)}{R^2 - 3R - 2} = O(R^{-2} \ln R)$ so since the arclength of γ_2 is $< 2\pi R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-1} \ln R) \to 0$ as $R \to \infty$.
- On γ_3, noting the reversed orientation, we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R'}^{R''} \dfrac{\text{Log}(t - \epsilon i)}{(t - \epsilon i)^2 + 3(t - \epsilon i) + 2}\,dt$, which tends to $-\int_0^\infty \dfrac{\ln(t) + 2\pi i}{t^2 + 3t + 2}\,dt$ as $R \to \infty$.
- On γ_4 we have $|f(z)| = \left|\dfrac{\log(e^{it}/R)}{(e^{it}/R)^2 + 3(e^{it}/R) + 2}\right| \le \dfrac{\ln(R)}{2} = O(\ln R)$ so since the arclength of γ_2 is $< 2\pi/R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-1} \ln R) \to 0$ as $R \to \infty$.
- So taking $R \to \infty$ and putting all of this together yields $-2\pi i \ln 2 = \int_0^\infty \dfrac{\ln(t)}{t^2 + 3t + 2}\,dt - \int_0^\infty \dfrac{\ln(t) + 2\pi i}{t^2 + 3t + 2}\,dt = -2\pi i \int_0^\infty \dfrac{1}{t^2 + 3t + 2}\,dt$ so that $\int_0^\infty \dfrac{1}{t^2 + 3t + 2}\,dt = \boxed{\ln 2}$.
- <u>Remark</u>: Of course, this integral is very easy to evaluate using partial fraction decomposition: $\dfrac{1}{t^2 + 3t + 2} = \dfrac{1}{t+1} - \dfrac{1}{t+2}$, so $\int_0^\infty \dfrac{1}{t^2 + 3t + 2}\,dt = \ln(\dfrac{t+1}{t+2})\Big|_{t=0}^\infty = \ln 2$. But this technique of using the keyhole contour is applicable to a wide range of other integrals that are not so easy to compute in the same manner.

- <u>Example</u>: Evaluate $\int_0^\infty \dfrac{\ln x}{x^2 + 3x + 2}\,dx$.
  - We take $f(z) = \dfrac{\text{Log}(z)^2}{z^2 + 3z + 2}$ and integrate around the keyhole contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ used above:



  - As before γ encloses the two simple poles at $z = -1$ and $z = -2$ of $f(z) = \dfrac{\text{Log}(z)^2}{z^2 + 3z + 2}$ whose residues are $\text{Res}_f(-1) = \lim_{z \to -1} \dfrac{z+1}{z^2 + 3z + 2} \text{Log}(z)^2 = (i\pi)^2$ and $\text{Res}_f(-2) = \lim_{z \to -2} \dfrac{z+1}{z^2 + 3z + 2} \text{Log}(z)^2 = -(\ln 2 + i\pi)^2$, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i[(i\pi)^2 - (\ln 2 + i\pi)^2] = 4\pi^2 \ln 2 - 2\pi i (\ln 2)^2$.

- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R'}^{R''} \frac{\text{Log}(t+\epsilon i)^2}{(t+\epsilon i)^2+3(t+\epsilon i)+2}\,dt$, which tends to $\int_0^\infty \frac{\ln(t)^2}{t^2+3t+2}\,dt$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = \left|\frac{\log(Re^{it})^2}{(Re^{it})^2+3(Re^{it})+2}\right| \le \frac{\ln(R)^2}{R^2-3R-2} = O(R^{-2}\ln^2 R)$ so since the arclength of γ_2 is $< 2\pi R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-1}\ln^2 R) \to 0$ as $R \to \infty$.
- On γ_3, noting the reversed orientation, we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R'}^{R''} \frac{\text{Log}(t-\epsilon i)^2}{(t-\epsilon i)^2+3(t-\epsilon i)+2}\,dt$, which tends to $-\int_0^\infty \frac{(\ln(t)+2\pi i)^2}{t^2+3t+2}\,dt$ as $R \to \infty$.
- On γ_4 we have $|f(z)| = \left|\frac{\log(e^{it}/R)^2}{(e^{it}/R)^2+3(e^{it}/R)+2}\right| \le \frac{\ln(R)^2}{2} = O(\ln^2 R)$ so since the arclength of γ_2 is $< 2\pi/R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-1}\ln^2 R) \to 0$ as $R \to \infty$.
- So taking $R \to \infty$ and putting all of this together yields $4\pi^2 \ln 2 - 2\pi i(\ln 2)^2 = \int_0^\infty \frac{\ln(t)^2}{t^2+3t+2}\,dt - \int_0^\infty \frac{(\ln(t)+2\pi i)^2}{t^2+3t+2}\,dt = \int_0^\infty \frac{4\pi^2}{t^2+3t+2}\,dt + \int_0^\infty \frac{-2\pi i \ln t}{t^2+3t+2}\,dt$. So comparing the imaginary parts yields the desired $\int_0^\infty \frac{\ln t}{t^2+3t+2}\,dt = \boxed{(\ln 2)^2}$.
- <u>Remark</u>: Notice that comparing the real parts yields $\int_0^\infty \frac{1}{t^2+3t+2}\,dt = \ln 2$, as we calculated above.

- We make some remarks about the effectiveness of using various different contours for evaluating integrals based on the shape and behavior of the function. For example, suppose we wish to evaluate an integral of the form $\int_0^\infty x^a \frac{p(x)}{q(x)}\,dx$ where $p(x)$ and $q(x)$ are polynomials and $0 < a < 1$. (Such integrals arise often as Mellin transforms.)
  - In order for this integral to converge, $q(x)$ must have no roots in $(0,\infty)$ so that there are no singularities on the real axis, and $\deg(p) - \deg(q) + a$ must be less than -1 so that the integral converges as $x \to \infty$.
  - Then the keyhole contour will be effective in evaluating this integral using integrand $f(z) = e^{a \cdot \text{Log}(z)} \frac{p(z)}{q(z)}$, since the contributions on the small and large circles will both tend to zero, while the contributions on the line segments running parallel to the real axis will differ by a multiplicative factor of $e^{2\pi i a} \neq 1$, so they will not cancel.
  - If instead we want to integrate a rational function directly, then including an extra factor of $\text{Log}(z)$ as we illustrated in examples above, will prevent the two integrals along the real axis from cancelling one another. A similar approach works for evaluating integrals of the form $\int_0^\infty (\ln x)^k \frac{p(x)}{q(x)}\,dx$. Depending on the precise nature of the cancellation that occurs, it may also be necessary to increase the power of $\ln x$ by 1.

- <u>Example</u>: Evaluate $\int_0^\infty \frac{x^{1/2}\ln x}{x^2+1}\,dx$.
  - We take $f(z) = \frac{e^{\text{Log}(z)/2}\text{Log}(z)}{z^2+1}$ and integrate around the keyhole contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.
  - There are simple poles at $z = \pm i$; the residue at $z = i$ is $\frac{e^{i\pi/4}(i\pi/2)}{2i} = \frac{\pi}{4}e^{i\pi/4}$ while the residue at $z = -i$ is $\frac{e^{3i\pi/4}(3i\pi/2)}{-2i} = -\frac{3\pi}{4}e^{3i\pi/4}$, so by the residue theorem the integral around the full contour is $\frac{\pi^2}{\sqrt{2}}(1+2i)$.

- For large $|z|$ we see the integral on $|z| = R$ is $O(R^{-1/2} \ln R) \to 0$ and for small $|z|$ the integral on $z = 1/R$ is $O(R^{-3/2} \ln R) \to 0$ also.
- On the segment above the real axis the integral tends to $\int_0^\infty \frac{e^{\ln x/2} \ln x}{x^2+1}\,dx = \int_0^\infty \frac{x^{1/2} \ln x}{x^2+1}\,dx$ as $R \to \infty$, while on the segment below the real axis the integral tends to $-\int_0^\infty \frac{e^{(\ln x + 2\pi i)/2}(\ln x + 2\pi i)}{x^2+1}\,dx = \int_0^\infty \frac{x^{1/2} \ln x}{x^2+1}\,dx + 2\pi i \int_0^\infty \frac{x^{1/2}}{x^2+1}\,dx$.
- Comparing yields $\frac{\pi^2}{\sqrt{2}}(1+2i) = 2\int_0^\infty \frac{x^{1/2}\ln x}{x^2+1}\,dx + 2\pi i \int_0^\infty \frac{x^{1/2}}{x^2+1}\,dx$, so extracting real parts yields $\int_0^\infty \frac{x^{1/2}\ln x}{x^2+1}\,dx = \boxed{\frac{\pi^2}{2\sqrt{2}}}$.

- As another class of examples, suppose we wish to evaluate a Fourier transform $\hat{f}(a) = \int_{-\infty}^\infty f(x) e^{iax}\,dx$ of some real-valued function $f(x)$. By taking real and imaginary parts, we equivalently obtain the values of the Fourier coefficients $\int_{-\infty}^\infty f(x)\cos(ax)\,dx$ and $\int_{-\infty}^\infty f(x)\sin(ax)\,dx$.
  - For integrals like these, using a rectangular contour with vertices $\pm R$ and $\pm R + iR$ will be effective as long as there exists a positive constant A such that $|f(z)| \le A/|z|$ for all sufficiently large $|z|$, since (as one may check using estimates like the ones we gave earlier) the integrals of $g(z) = f(z)e^{iaz}$ on the other three sides of the rectangle will then tend to zero.
  - Then as long as $g(z)$ (equivalently, $f(z)$) has only finitely many poles $z_1, z_2, \dots, z_n$ in the upper half-plane, we obtain a general integration formula $\int_{-\infty}^\infty f(x)e^{iax}\,dx = 2\pi i \sum_{k=1}^n \operatorname{Res}_g(z_k) = 2\pi i \sum_{k=1}^n e^{iaz_n} \operatorname{Res}_g(z_k)$.
  - This method can also be adjusted to work with a semicircular contour, and as we have seen it can also be used in the situation where the function $f(z)$ has a pole on the real axis: we simply have the contour take a semicircular detour around the singularity and use the fractional residue lemma to account for the associated contribution.

- <u>Example</u>: Evaluate $\int_{-\infty}^\infty \frac{x \sin 2x}{(x^2+1)^2}\,dx$.
  - We take $f(z) = \frac{ze^{2iz}}{(z^2+1)^2}$ and integrate around the rectangle with vertices $\pm R$ and $\pm R + iR$.
  - There are poles at $z = \pm i$, but only the pole at $z = i$ lies inside the contour. The residue at $z = i$ is $\lim_{z\to i} \frac{d}{dz}[(z-i)^2 f(z)] = \lim_{z \to i} \frac{d}{dz}\left[e^{2iz}(z+i)^{-2} + 2ize^{2iz}(z+i)^{-2} - 2ze^{2iz}(z+i)^{-3}\right] = \frac{1}{2}e^{-2}$, so by the residue theorem the integral on the contour is $i\pi e^{-2}$.
  - For $z = \pm R + it$ and $iR + t$ with $-R \le t \le R$, we see that $|f(z)| = O(R^{-3})$ so the integral on each component tends to 0 as $R \to \infty$.
  - On the real axis the integral tends to $\int_{-\infty}^\infty \frac{xe^{2ix}}{(x^2+1)^2}\,dx$ as $R \to \infty$.
  - Comparing yields $\int_{-\infty}^\infty \frac{xe^{2ix}}{(x^2+1)^2}\,dx = i\pi e^{-2}$, and extracting imaginary parts yields $\int_{-\infty}^\infty \frac{x\sin 2x}{(x^2+1)^2}\,dx = \boxed{\frac{\pi}{e^2}}$.

Well, you're at the end of my handout. Hope it was helpful.