Contents

1 Complex Numbers and Derivatives

In this chapter, our goal is to discuss functions of a complex variable, and to construct various elementary functions such as polynomials, rational functions, and the exponential function. We then give various approaches to studying the notion of differentiation in the context of functions of a complex variable.

1.1 Complex Arithmetic

- In this section we review the arithmetic of complex numbers.

1.1.1 The Complex Numbers

- Before defining the arithmetic operations for complex numbers, we will give a brief review of their history in mathematics.
  - Complex numbers were first encountered by mathematicians in the 1500s who were trying to write down general formulas for solving cubic equations (i.e., equations like $x^3 + x + 1 = 0$), in analogy with the well-known formula for the solutions of a quadratic equation. It turned out that their formulas required manipulation of complex numbers, even when the cubics they were solving had three real roots.
  - It took over 100 years before complex numbers were accepted as something mathematically legitimate: even negative numbers were sometimes suspect, so (as the reader may imagine) their square roots were even more questionable.
  - The stigma is still evident even today in the terminology ("imaginary numbers"); nonetheless, complex numbers are very real objects (no pun intended), and have a wide range of uses in mathematics, physics, and engineering.
- Definition: A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the so-called "imaginary unit", defined so that $i^2 = -1$. The real part of $z = a + bi$, denoted $\mathrm{Re}(z)$, is the real number a, while the imaginary part of $z = a + bi$, denoted $\mathrm{Im}(z)$, is the real number b. The set of all complex numbers is denoted $\mathbb{C}$.

- The notation $\sqrt{-1}$ is also often used to denote the imaginary unit i. In certain disciplines (especially electrical engineering), the letter j may instead be used to denote $\sqrt{-1}$, rather than i (which is instead used to denote electrical current).
- Examples: Some complex numbers are $4+3i$, $3-\pi i$, $6i = 0+6i$, and $-5 = -5+0i$. Their real parts are 4, 3, 0, and -5 respectively, while their imaginary parts are 3, $-\pi$, 6, and 0 respectively.

- Definition: The complex conjugate of $z = a+bi$, denoted $\overline{z}$, is the complex number $a - bi$. The modulus (also called the absolute value, magnitude, or length) of $z = a+bi$, denoted $|z|$, is the real number $\sqrt{a^2+b^2}$.
  - The notation for conjugate varies among disciplines. The notation z^* is often used in physics and computer programming to denote the complex conjugate (in place of $\overline{z}$) since it is easier to type on a standard keyboard.
  - Example: For $z = 3+4i$ we have $\overline{z} = 3-4i$ and $|z| = \sqrt{3^2+4^2} = 5$.

- Two complex numbers are added (or subtracted) simply by adding (or subtracting) their real and imaginary parts: $(a+bi)+(c+di) = (a+c)+(b+d)i$.
  - Example: The sum of $1+2i$ and $3-4i$ is $\boxed{4-2i}$. The difference is $(1+2i)-(3-4i) = \boxed{-2+6i}$.

- Two complex numbers are multiplied using the distributive law and the fact that $i^2 = -1$: $(a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$.
  - Example: The product of $1+2i$ and $3-4i$ is $(1+2i)(3-4i) = 3+6i-4i-8i^2 = \boxed{11+2i}$.
  - Observe in particular that for $z = a+bi$, we have $|z|^2 = a^2+b^2 = z\cdot\overline{z}$.

- For division, we rationalize the denominator using the conjugate: $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$.
  - Example: The quotient of $2i$ by $1-i$ is $\dfrac{2i}{1-i} = \dfrac{2i(1+i)}{(1-i)(1+i)} = \dfrac{-2+2i}{2} = \boxed{-1+i}$.

- Here are a few more simple properties of complex number arithmetic:
- Proposition (Complex Arithmetic): Suppose z and w are complex numbers.
  1. We have $\mathrm{Re}(z) = (z+\overline{z})/2$ and $\mathrm{Im}(z) = (z-\overline{z})/(2i)$.
  2. We have $\overline{z+w} = \overline{z}+\overline{w}$, $\overline{zw} = \overline{z}\cdot\overline{w}$, and $\overline{\overline{z}} = z$.
  3. We have $|\overline{z}| = |z|$ and $|zw| = |z|\cdot|w|$.
  4. We have $z = \overline{z}$ if and only if z is real, while $\overline{z} = -\overline{z}$ if and only if z is purely imaginary (of the form ri where r is real).
  5. We have $\mathrm{Re}(z) \le |z|$ and $\mathrm{Im}(z) \le |z|$.
  6. (Triangle Inequality) We have $|z+w| \le |z|+|w|$.
     - Proofs: (1)-(5) are easy algebraic calculations.
     - For (6), use (1) and (2) to observe $z\overline{w}+w\overline{z} = 2\mathrm{Re}(z\overline{w})$, and (5) and (3) to observe $2\mathrm{Re}(z\overline{w}) \le 2\,|z\overline{w}| = 2\,|z|\,|w|$.
     - Then $|z+w|^2 = (z+w)(\overline{z+w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = |z|^2+|w|^2+2\mathrm{Re}(z\overline{w}) \le |z|^2+|w|^2+2\,|z|\,|w| = (|z|+|w|)^2$. Since both $|z+w|$ and $|z|+|w|$ are nonnegative, taking the square root yields the desired $|z+w| \le |z|+|w|$.

- We emphasize that (2) above shows that the conjugate is both additive and multiplicative.
  - Example: If $z = 1+2i$ and $w = 3-i$, then $\overline{z} = 1-2i$ and $\overline{w} = 3+i$. We compute $z+w = 4+i$, $\overline{z+w} = 4-i$, $zw = 5+5i$ and $\overline{z}\cdot\overline{w} = 5-5i$, so indeed $\overline{z+w} = \overline{z}+\overline{w}$ and $\overline{zw} = \overline{z}\cdot\overline{w}$.

- The multiplicativity of the conjugate explains the procedure for performing division: we write $\dfrac{z}{w} = \dfrac{z \cdot \overline{w}}{w \cdot \overline{w}} = \dfrac{z \cdot \overline{w}}{|w|^2}$, where the denominator is now the real number $|w|^2$.

- The real numbers and the complex numbers are both examples of fields: sets of numbers that can be added, subtracted, multiplied, and divided (except by zero) and possess various algebraic relations involving these operations. More formally:

- Definition: A field is an ordered triple $(F, +, \cdot)$ consisting of a set of numbers F together with two binary operations[1], $+$ (addition) and $\cdot$ (multiplication), satisfying the following axioms:

  [F1] Addition is associative: $a + (b + c) = (a + b) + c$ for any elements a, b, c in F.

  [F2] Addition is commutative: $a + b = b + a$ for any elements a, b in F.

  [F3] There is an additive identity 0 satisfying $a + 0 = a$ for all a in F.

  [F4] Every element a in F has an additive inverse $-a$ satisfying $a + (-a) = 0$.

  [F5] Multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any elements a, b, c in F.

  [F6] Multiplication is commutative: $a \cdot b = b \cdot a$ for any elements a, b in F.

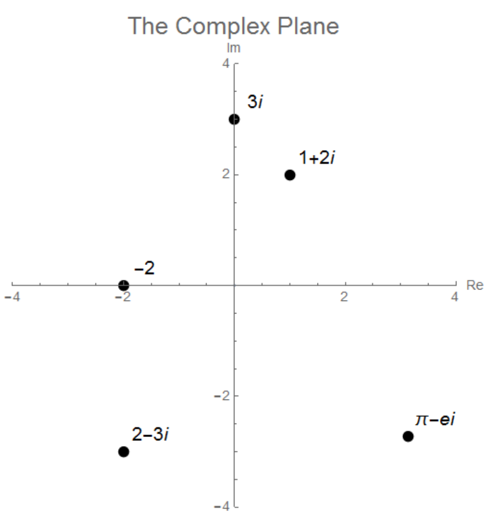
  [F7] There is a multiplicative identity $1 \neq 0$, satisfying $1 \cdot a = a$ for all a in F.

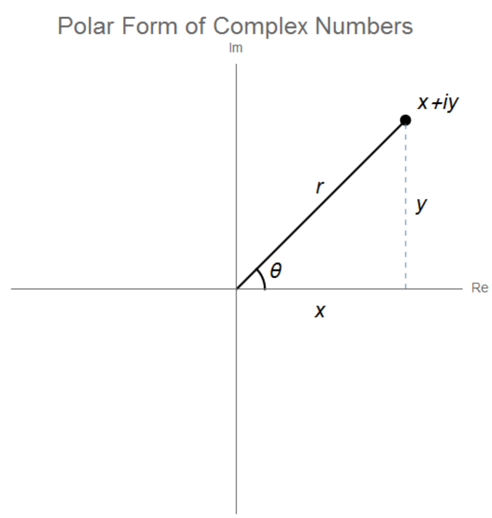
  [F8] Every nonzero a in F has a multiplicative inverse a^{-1} satisfying $a \cdot a^{-1} = 1$.

  [F9] Multiplication distributes over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ for any elements a, b, c in F.

1.1.2 Polar Form, Complex Exponentials, Powers, and Roots

- We often think of the real numbers geometrically, as a line. The natural way to think of the complex numbers is as a plane, with the x-coordinate denoting the real part and the y-coordinate denoting the imaginary part.





  - Once we do this, there is a natural connection to polar coordinates: namely, if $z = x + yi$ is a complex number which we identify with the point (x, y) in the complex plane, then the modulus $|z| = \sqrt{x^2 + y^2}$ is simply the coordinate r when we convert (x, y) from Cartesian to polar coordinates.
  - Furthermore, if we are given that $|z| = r$, we can uniquely identify z given the angle θ that the line connecting z to the origin makes with the positive real axis. (This is the same θ from polar coordinates.)

- From polar coordinates (or simple trigonometry), we see that we can write z in the form $z = r[\cos(\theta) + i\sin(\theta)]$, which is called the polar form of z.

[1]The result of applying these operations to elements a and b is denoted by $a + b$ and $a \cdot b$ (or simply ab), respectively. The definition of "binary operation" means that $a + b$ and $a \cdot b$ are also numbers in F.

- The length r is simply the modulus of z, while the angle θ is called the <u>argument</u> of z and sometimes denoted $\theta = \arg(z)$.
- We will emphasize that although r is unique, θ is not: since the sine and cosine are periodic with period 2π, any θ that differs by an integral multiple of 2π yields an equivalent polar form. We will implicitly identify polar forms yielding the same complex number.
- <u>Example</u>: For $r = 2$ and $\theta = 0$ we obtain $z = 2[\cos 0 + i \sin 0] = 2$. Taking $r = 2$ and $\theta = 2\pi$ also yields $z = 2[\cos 2\pi + i \sin 2\pi] = 2$.
- Conversely, if we know $z = x + iy$ then we can compute the (r, θ) form fairly easily by solving $x = r\cos\theta$ and $y = r\sin\theta$ for r and θ.
- Explicitly, we have $r = \sqrt{x^2 + y^2} = |z|$ and we can take $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ if $x > 0$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ if $x < 0$. (The extra $+\pi$ is needed when $x < 0$ because of the fact that the principal arctangent function only has range $(-\pi/2, \pi/2)$, so we would otherwise get the wrong value for θ if z lies in the second or third quadrants.)
- <u>Example</u>: If $z = 1 + i$, then the corresponding values of r and θ above are $r = |z| = \sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$, so we can write z in polar form as $z = \boxed{\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]}$. Indeed, we may check that $\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 1 + i$, as it should be.
- <u>Example</u>: If $z = -1 + i\sqrt{3}$, then the corresponding values of r and θ above are $r = |z| = 2$ and $\theta = \pi + \tan^{-1}(-\sqrt{3}) = 2\pi/3$, so we can write z in polar form as $z = \boxed{2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]}$.

- We can also repackage the polar form using complex exponentials.
  - We will later give a more precise approach to complex exponentials using power series, but for now, we will simply give a definition using the familiar real exponential and real sine and cosine functions.

- <u>Definition</u>: If $z = x + iy$ is a complex number, we define the <u>complex exponential</u> $e^z = e^x(\cos y + i \sin y)$.
  - <u>Examples</u>: We have $e^{i\pi/2} = e^0(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = \boxed{i}$ and $e^{1+i\pi} = e^1(\cos\pi + i\sin\pi) = \boxed{-e}$.
  - The motivation here is that we want the complex exponential to obey the familiar rules for the real exponential function, and so in particular we want $e^{x+iy} = e^x e^{iy}$.
  - It therefore suffices just to define e^{iy}, which we do via <u>Euler's identity</u> $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
  - Euler's identity encodes a lot of information: for example, we claim that $e^{i\theta} \cdot e^{i\varphi} = e^{i(\theta+\varphi)}$. Expanding both sides with Euler's identity yields

$$\begin{aligned} e^{i\theta} \cdot e^{i\varphi} &= [\cos\theta + i\sin\theta]\,[\cos\varphi + i\sin\varphi] \\ &= [\cos\theta\cos\varphi - \sin\theta\sin\varphi] + i\,[\sin\theta\cos\varphi + \cos\theta\sin\varphi] \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi) = e^{i(\theta+\varphi)} \end{aligned}$$

    where the key step in the middle uses the usual addition identities $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$ and $\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$ for sine and cosine.
  - Another way of interpreting this calculation is that the (initially rather arbitrary-seeming) sine and cosine addition formulas actually just reflect the natural structure of the multiplication of complex numbers.

- Another convenient result follows by applying Euler's identity to the simple relation $e^{i(n\theta)} = (e^{i\theta})^n$, which when written out yields <u>De Moivre's identity</u> $\cos(n\theta) + i\sin(n\theta) = [\cos\theta + i\sin\theta]^n$.
  - By expanding the right-hand side using the binomial theorem we can obtain identities for $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin\theta$ and $\cos\theta$.
  - <u>Example</u>: Setting $n = 2$ produces $\cos(2\theta) + i\sin(2\theta) = [\cos\theta + i\sin\theta]^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$, and so we recover the double-angle formulas $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

- Example: Setting $n = -1$ produces $\cos(-\theta) + i\sin(-\theta) = [\cos\theta + i\sin\theta]^{-1} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$, and so we recover the standard identities $\cos^2\theta + \sin^2\theta = 1$, $\cos(-\theta) = \cos\theta$, and $\sin(-\theta) = -\sin\theta$.

- Using Euler's identity and the polar form of complex numbers above, we see that every complex number can be written in exponential form as $z = r \cdot e^{i\theta}$ for the same r and θ we described above.
  - Example: We can draw $1 + i$ in the complex plane, or use the formulas, to see that $|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \dfrac{\pi}{4}$, and so we see that $1 + i = \boxed{\sqrt{2} \cdot e^{i\pi/4}}$.
  - Example: Either by geometry or trigonometry, we see that $\left|1 - i\sqrt{3}\right| = 2$ and $\arg(1 - i\sqrt{3}) = -\dfrac{\pi}{3}$, hence $1 - i\sqrt{3} = \boxed{2 \cdot e^{-i\pi/3}}$.
  - Example: Using the formulas for r and θ above, we have $3 + 2i = \boxed{\sqrt{13} \cdot e^{i \cdot \arctan(2/3)}}$.

- The rectangular $a + bi$ form of a complex number is more convenient for addition, while the polar $re^{i\theta}$ form is more convenient for multiplication, since we may easily multiply $(re^{i\theta})(se^{i\varphi}) = (rs)e^{i(\theta+\varphi)}$. (This calculation is often summarized as "lengths multiply, angles add".)
  - In particular, it is very easy to take powers of complex numbers when they are in exponential form: we have $(r \cdot e^{i\theta})^n = r^n \cdot e^{i(n\theta)}$.

- Example: Compute $(1 + i)^8$.
  - From above we have $1 + i = \sqrt{2} \cdot e^{i\pi/4}$, so $(1 + i)^8 = \left(\sqrt{2} \cdot e^{i\pi/4}\right)^8 = (\sqrt{2})^8 \cdot e^{8i\pi/4} = 2^4 \cdot e^{2i\pi} = \boxed{16}$. (Note how much easier this is compared to multiplying $1 + i$ by itself eight times.)

- Example: Compute $(1 - i\sqrt{3})^9$.
  - From above we have $1 - i\sqrt{3} = 2 \cdot e^{-i\pi/3}$, so $(1 - i\sqrt{3})^9 = 2^9 \cdot e^{-9i\pi/3} = 512 \cdot e^{-3i\pi} = \boxed{-512}$.

- Taking roots of complex numbers is also fairly straightforward using the polar form, since we may interpret roots as fractional powers. We do need to be slightly careful, since there are in general n different nth roots of any nonzero complex number when n is a positive integer.
  - Explicitly, suppose we wish to solve $z^n = re^{i\theta}$. If we write z in polar form as $z = se^{i\varphi}$, then $z^n = s^n e^{in\varphi}$. By the uniqueness of the modulus we have $s^n = r$ and also $e^{i\theta} = e^{in\varphi}$, which is equivalent to having $\theta = n\varphi + 2k\pi$ for some integer k.
  - Solving for s and φ yields $s = r^{1/n}$ (the real nth root of the nonnegative real number r), and $\varphi = \dfrac{\theta}{n} + \dfrac{2k\pi}{n}$ for some integer k. We can see that for $k = 0, 1, \dots, n-1$, these values of φ yield distinct complex numbers but that any other value of k will simply repeat one of these n values, since its argument will differ from one of these by an integer multiple of 2π.
  - Thus, we see that the n possible nth roots of $z = re^{i\theta}$ are $\sqrt[n]{r}e^{(\theta+2\pi k)i/n}$ for integers $k = 0, 1, \dots, n - 1$.

- Example: Find all complex square roots of $2i$.
  - We are looking for square roots of $2i = 2e^{i\pi/2}$. By the formula, the two square roots are $\sqrt{2}e^{i[\pi/4+k\pi]}$ for $k = 0, 1$, which are $\boxed{\sqrt{2}e^{i\pi/4} \text{ and } \sqrt{2}e^{5i\pi/4}}$.
  - Converting from exponential to rectangular form using Euler's formula gives the two square roots in $a + bi$ form as $\boxed{1 + i \text{ and } -1 - i}$.
  - Indeed, we can easily multiply out to verify that $(1 + i)^2 = (-1 - i)^2 = 2i$, as it should be.

- Example: Find all complex numbers $z = a + bi$ with $z^3 = 1$.
  - We are looking for cube roots of $1 = 1 \cdot e^0$. By the formula, the three cube roots of 1 are $1 \cdot e^{2ki\pi/3}$, for $k = 0, 1, 2$, which are $\boxed{e^0,\ e^{2\pi i/3},\ e^{4\pi i/3}}$.

- Converting to rectangular form using Euler's formula gives the roots as $\boxed{1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i}$ in $a+bi$ form.

- The n solutions to the equation $z^n = 1$ are called the <u>nth roots of unity</u>, and are given explicitly by $1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}$ where $\zeta_n = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$.
    - Since their arguments are the multiples of $2\pi/n$, the nth roots of unity form the vertices of a regular n-gon on the unit circle.
    - Additionally, we can use the nth roots of unity to give the factorization of the polynomial $z^n - 1$: specifically, we have $z^n - 1 = (z-1)(z-\zeta_n)\cdots(z-\zeta_n^{n-1})$.
    - The set $G = \{1, \zeta_n, \dots, \zeta_n^{n-1}\}$ is closed under multiplication hence is a finite multiplicative group; indeed, G is a cyclic group of order n that is generated by ζ_n since all of the elements are powers of ζ_n.
    - The roots of unity have many interesting properties and show up often in algebra, number theory, and analysis (particularly Fourier analysis).

- Although we can always extract roots by converting to exponential form, if the argument is irrational then it can be difficult to simplify the resulting expressions back into $a+bi$ form, even when the results turn out to be pleasant.
    - For example, if we wish to find the complex square roots of $15+8i$, converting to exponential form yields $15-8i = 17e^{i\cdot\arctan(8/15)}$, which yields square roots $\pm\sqrt{17}e^{i\cdot\arctan(8/15)/2}$; this expression is not particularly easy to evaluate.
    - We can give an explicit formula for evaluating square roots directly in $a+bi$ form: explicitly, the two square roots of $z = a+bi$ are given by $\pm(c+di)$ where c is the positive square root of $\dfrac{a+\sqrt{a^2+b^2}}{2}$ and $d = \dfrac{b}{2c}$.
    - To verify this formula we simply observe that $c^2 - d^2 = \dfrac{a+\sqrt{a^2+b^2}}{2} + \dfrac{b^2}{a+\sqrt{a^2+b^2}} = a$ and $2cd = b$, so that $(c+di)^2 = (c^2-d^2) + 2cdi = a+bi$ as claimed.

- <u>Example</u>: Find the complex square roots of $15+8i$.
    - Per the formula with $a = 15$ and $b = 8$, we obtain $c = \sqrt{\dfrac{15+\sqrt{15^2+8^2}}{2}} = 4$ and $d = \dfrac{8}{2\cdot 4} = 1$, so the square roots are $\boxed{\pm(4+i)}$.

- Many textbooks introduce complex numbers as a tool for giving meaning to the formal symbols obtained when using the quadratic formula to "solve" quadratic equations that do not have real solutions.
    - Explicitly, if a, b, c are real numbers and $a \neq 0$, then we may complete the square in the expression $az^2 + bz + c$ and write it as $a(z + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$.
    - We may then obtain the usual quadratic formula for the roots of the polynomial $az^2+bz+c = 0$; namely, $z = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.
    - In the situation where $b^2 - 4ac < 0$, the solutions are not real numbers but rather complex numbers. For example, it indicates that the solutions to $z^2 + 2z + 2 = 0$ are $z = \dfrac{-2\pm\sqrt{-4}}{2} = -1 \pm i$.
    - Indeed, we can check that if we evaluate the expression $z^2 + 2z + 2$ when $z = -1+i$ or $-1-i$, we obtain 0.
    - We can continue further by factoring the polynomial as $az^2 + bz + c = a(z-r_1)(z-r_2)$ where $r_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ are the two roots.

- Using the formula for simplifying square roots given above, we may find the roots of any quadratic polynomial with complex coefficients.

- Example: Solve the quadratic $z^2 - (2+i)z + (6+6i) = 0$.
  - The quadratic formula yields $z = \dfrac{(2+i) \pm \sqrt{(2+i)^2 - 4(6+6i)}}{2} = \dfrac{(2+i) \pm \sqrt{-21-20i}}{2}$ so we must compute the square roots of $-21-20i$.
  - Per the formula with $a = -21$ and $b = -20$, we obtain $c = \sqrt{\dfrac{-21+29}{2}} = 2$ and $d = \dfrac{-20}{2 \cdot 2} = -5$, so the square roots are $\pm(2-5i)$.
  - Thus we obtain the roots $z = \dfrac{(2+i) \pm (2-5i)}{2} = \boxed{2-2i,\ 3i}$.

- More generally, the Fundamental Theorem of Algebra, which we will prove later, says that any polynomial equation $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ can be completely factored as a product of linear terms over the complex numbers. (This a foundational result in algebra and the first complete and correct proof was given by Argand and Gauss in the early 1800s.)

1.1.3 Basic Topology of $\mathbb{C}$

- We now introduce some useful topological properties of subsets of $\mathbb{C}$ that are primarily for our convenience later.

- Definition: If $a \in \mathbb{C}$, the open disc with radius $r > 0$ centered at z is the set $D_r(a) = \{z \in \mathbb{C} : |z-a| < r\}$.

- If S is a subset of $\mathbb{C}$, we may distinguish three different classes of points in $\mathbb{C}$ relative to S, based on their behaviors when we draw balls around them.
  1. If we can draw an open disc of positive radius around a that is entirely contained in S, then we call a an interior point of S. Interior points are necessarily contained in S.
  2. If we can draw an open disc of positive radius around a that is entirely contained in S^c, the complement of S, then we call a an exterior point point of S. (Equivalently, it is an interior point of S^c, and thus is necessarily contained in S^c.)
  3. Otherwise, no matter what open disc we draw centered at a, it will always contain some points in S and some points in S^c, in which case we call a a boundary point of S. Boundary points may be in S or in S^c.
     - Intuitively, the interior points of S are a positive distance away from S^c, the exterior points of S are a positive distance away from S, and the boundary points of S are an arbitrarily small distance away from both S and S^c.

- Definition: If S is a subset of $\mathbb{C}$, the interior of S, denoted $\text{int}(S)$, is the set of its interior points. A set S is open if all its points are interior points, which is equivalent to saying it contains none of its boundary points. The boundary of S, denoted $\partial(S)$, is the set of its boundary points. A set S is closed if it contains all its boundary points, which is equivalent to saying its complement is open.
  - There are various equivalent definitions and properties of open sets: for example, S is open if and only if it is a union (not necessarily finite) of open discs.
  - Example: For any $a \in \mathbb{C}$ and any $r > 0$, if S is the open disc $|z-a| < r$, then S actually an open set, since all its points are interior points. Explicitly, for any $z \in S$, if $r - |z-a| = s > 0$, then by the triangle inequality, the open disc $D_{s/2}(z)$ is contained in the set. Alternatively, the set of boundary points is the circle $|z-a| = r$, and none of these points are in S.
  - Example: For any $a \in \mathbb{C}$ and any $r > 0$, if S is the closed disc $|z-a| \le r$, then S actually is a closed set, since its boundary is the circle $|z-a| = r$ and all of these points are in S.
  - Example: Any finite subset of $\mathbb{C}$ is closed, since all of the finitely many points are boundary points. As a consequence, the complement of any finite subset of $\mathbb{C}$ is open.

- Example: The sets $|z| > 1$ and $0 < \text{Re}(z) < 1$ are both open. Their boundaries are respectively the unit circle $|z| = 1$ and the pair of lines $\text{Re}(z) = 0, 1$.
- Example: The sets $|z| \geq 1$ and $0 \leq \text{Re}(z) \leq 1$ are both closed. Their boundaries are respectively the unit circle $|z| = 1$ and the pair of lines $\text{Re}(z) = 0, 1$.
- Example: The unit circle $|z| = 1$ is closed. This set is its own boundary.
- Example: The empty set is both open and closed, as is $\mathbb{C}$ itself. (In fact, these are the only two subsets of $\mathbb{C}$ that are both closed and open, though this is harder to prove than it might seem[2].)

- Although there is much more to say about the topological properties of arbitrary subsets of $\mathbb{C}$ (which can have quite pathological properties!), we are primarily interested in open regions and their associated boundaries, which tend to behave much more nicely.

- Definition: A region in $\mathbb{C}$ consists of an open set along with a subset (possibly empty, possibly all) of its boundary points.
  - Examples: The sets $|z| < 1$, $|z| \leq 1$, $|z| > 1$, $|z| \geq 1$, and $\{z : z < 1\} \cup \{-1, i\}$ are all regions in $\mathbb{C}$.
  - Examples: The sets $\mathbb{C}$ and $0 < \text{Re}(z) \leq 1$ are regions in $\mathbb{C}$.

- Another important notion is that of connectedness.
  - Intuitively, a space is connected if it cannot be decomposed into "separate pieces", in the sense that the pieces do not overlap one another's closures.
  - We also have another way to phrase this idea of connectedness: namely, if any two points in the set can be joined by a continuous path inside the set.

- Definition: If S is a subset of $\mathbb{C}$, S is connected if it cannot be written as the union of two nonempty subsets with disjoint closures, and S is path-connected if for any two points z_1 and z_2 in S, there exists a continuous function $f : [0, 1] \to \mathbb{C}$ whose image is contained in S and such that $f(0) = z_1$ and $f(1) = z_2$.
  - There are various equivalent formulations of these conditions: for example, a connected set is equivalently one that cannot be written as the union of two nonempty disjoint proper open subsets.
  - In general any path-connected subset is necessarily connected (the proof follows by the same argument used to show that $\emptyset$ and $\mathbb{C}$ are the only subsets of $\mathbb{C}$ that are both closed and open), but the converse is not true in general[3].
  - However, connectedness and path-connectedness are equivalent for open sets, and thus also for regions[4].
  - Since path-connectedness is much more concrete and easier to visualize, we will generally phrase things in terms of path-connectedness.
  - Example: The regions $\mathbb{C}$, $|z| < 1$, $|z| \leq 1$, $|z| > 1$, $|z| \geq 1$, $\{z : z < 1\} \cup \{-1, i\}$, and $0 < \text{Re}(z) \leq 1$ are all path-connected, since in fact any two points in these regions are joined by a line inside the region.
  - Example: The region with $|z| < 1$ or $|z| > 2$ is not path-connected, since there is no way to draw a continuous path from a point in the first component to a point in the second one without passing through points with $1 \leq |z| \leq 2$, which are not in the region. (Here, we can see that this region consists of two connected components, namely $|z| < 1$ and $|z| > 2$.)

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[2]For completeness: if S or S^c is empty the result is trivial so suppose S is both open and closed in $\mathbb{C}$ and let $z \in S$ and $w \in S^c$. Consider the real-valued function $f : [0, 1] \to \mathbb{C}$ given by $f(t) = tz + (1 - t)w$, which is simply the graph of the line segment from z to w, and define T to be the subset of elements $t \in [0, 1]$ such that $f(t) \in S$. Then since $f(1) = w \in S^c$, T is a nonempty proper subset of S. If α is the least upper bound of T, then by definition there exist elements of T and of T^c that are arbitrarily close to α (otherwise, we could decrease or increase α respectively). Applying f shows that there are elements of S and of S^c that are arbitrarily close to $f(\alpha)$, and so $f(\alpha)$ is a boundary point of S. But if $f(\alpha) \in S$ then S is not open, while if $f(\alpha) \in S^c$ then S is not closed, contradiction.

[3]A standard counterexample example is the infamous "topologist's sine curve" given by the graph of $y = \sin(1/x)$ for $x \neq 0$ along with the origin. This set is not path-connected (there is no way to draw a path from $(0, 0)$ to any other point on the graph) but it is connected (there is no way to separate the graph into two pieces without causing the two pieces' closures to intersect).

[4]For this, suppose that S is a nonempty connected open set and let $z \in S$. Define U to be the set of points in S that can be joined by a path to z and define V to be the subset of points in S that cannot be joined by a path to z. For any $w \in U$, since S is open there exists an open disc of radius $r > 0$ such that $D_r(w) \subseteq S$. But for every point in this disc, there is a path joining it to w (namely, along an appropriate radius) and then following this path from w to z shows that this point is also connected to z. Thus in fact $D_r(w) \subseteq U$ so U is open. Similarly, V is open: for $w' \in V$ we obtain a similar disc $D_r(w')$ inside S, and then any point in the disc $D_r(w')$ cannot be connected to z, since otherwise w' would be. But then S is the union of two disjoint open subsets, contradicting connectedness.

- There is one other useful notion regarding regions: that of boundedness.
- Definition: If S is a subset of $\mathbb{C}$, S is bounded if there exists some positive radius R such that S is contained in the open disc of radius R centered at 0, and otherwise S is unbounded.
  - Equivalently, S is bounded if there exists some $R > 0$ with $|z| < R$ for all $z \in S$. Inversely, S is unbounded when there exists a sequence of elements $z_1, z_2, \dots, z_n, \dots$ in S such that $|z_n| \to \infty$ as $n \to \infty$.
  - Example: The regions $|z| < 1$, $|z| \leq 1$, and $\{z : z < 1\} \cup \{-1, i\}$ are all bounded, since they all lie inside the disc of radius 2 centered at 0.
  - Example: The regions $\mathbb{C}$, $|z| > 1$, $|z| \geq 1$, and $0 < \mathrm{Re}(z) \leq 1$ are all unbounded, since they contain elements of arbitrarily large absolute value.

1.2 The Complex Derivative and Holomorphic Functions

- Now that we have discussed the arithmetic of the complex numbers, we begin our study of complex-valued functions $f : \mathbb{C} \to \mathbb{C}$.
- Our first main goal is to generalize the notion of the derivative of a function to the complex case. There is a very natural way to try to do this; namely, by defining the complex derivative of a function as a limit in the same way as with a real-valued function.
  - Explicitly, if $f : \mathbb{C} \to \mathbb{C}$, we would like to try defining the complex derivative at a point $z_0 \in \mathbb{C}$ as $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.
  - This is in fact the definition we will eventually use, but we must first review some properties regarding limits of complex-valued functions.

1.2.1 Limits

- We begin by defining limits of complex-valued functions. We can use essentially the same definition as in the case of real-valued functions:
- Definition: A function $f : \mathbb{C} \to \mathbb{C}$ has the limit L as $z \to a$, written as $\lim_{z \to a} f(z) = L$ if for any $\epsilon > 0$ (no matter how small) there exists a $\delta > 0$ (depending on ϵ) such that for all $z \in \mathbb{C}$ with $0 < |z - a| < \delta$, we have $|f(z) - L| < \epsilon$.
  - This is simply the usual ϵ-δ definition of limit but with the variables having domain $\mathbb{C}$ rather than $\mathbb{R}$.
  - The usual intuition is as follows: suppose you claim that the function $f(x)$ has a limit L, as x gets close to a. In order to prove to me that the function really does have that limit, I challenge you by handing you some value $\epsilon > 0$, and I want you to give me some punctured disc $0 < |z - a| < \delta$ (a disc centered at $z = a$ with radius δ, excluding the center) with the property that $f(z)$ is always within ϵ for z in that disc, except possibly at a.
  - If $f(z)$ really does stay close to the limit value L as z gets close to a, then, no matter what value of ϵ I picked, you should always be able to answer my challenge with some punctured disc, because the values of $f(z)$ should stay near L when z is near a.
- We can mostly avoid using the formal definition as a practical matter by instead establishing a number of basic limit evaluations and limit rules, as follows:
- Proposition (Basic Limits): Let $a \in \mathbb{C}$ and suppose f and g have $\lim_{z \to a} f(z) = L_f$ and $\lim_{z \to a} g(z) = L_g$.
  1. We have $\lim_{z \to a} z = a$ and for any $c \in \mathbb{C}$ we have $\lim_{z \to a} c = c$.
  2. For any $c \in \mathbb{C}$ we have $\lim_{z \to a} cf(z) = cL_f$, $\lim_{z \to a} |f(z)| = |L_f|$, and $\lim_{z \to a} \overline{f(z)} = \overline{L_f}$.
  3. We have $\lim_{z \to a} [f(z) + g(z)] = L_f + L_g$, $\lim_{z \to a} [f(z) - g(z)] = L_f - L_g$, and $\lim_{z \to a} f(z)g(z) = L_f L_g$.

4. If $L_g \neq 0$, we have $\lim_{z \to a} \dfrac{f(z)}{g(z)} = \dfrac{L_f}{L_g}$.
5. If $L = x + yi$, then $\lim_{z \to a} f(z) = L$ if and only if $\lim_{z \to a} \mathrm{Re}[f(z)] = x$ and $\lim_{z \to a} \mathrm{Im}[f(z)] = y$.
   - These results are all standard applications of the limit definition, and the proofs are essentially the same as the corresponding results for real-valued functions.
   - For example, for the first part of (2), let $\epsilon > 0$. If $c = 0$ the result is trivial since $cf(z) = 0 = cL_f$ so we may choose any positive δ. Otherwise with $c \neq 0$, since $\lim_{z \to a} f(z) = L_f$ there exists $\delta > 0$ such that $|f(z) - L_f| < \epsilon / |c|$ for $0 < |z - a| < \delta$. Then we have $|cf(z) - cL_f| = |c| \cdot |f(z) - L_f| < \epsilon$ as desired.
   - We remark that (5), in particular, allows us to reduce any question about limits of complex functions to the corresponding real-valued limits of its real and imaginary parts.

- Example: Find $\lim_{z \to 2-i} \dfrac{5 - z}{2|z|}$.
  - Using the limit rules we have $\lim_{z \to 2-i} \dfrac{5-z}{2|z|} = \dfrac{\lim_{z \to 2-i}(5-z)}{\lim_{z \to 2-i} 2|z|} = \dfrac{\lim_{z \to 2-i} 5 - \lim_{z \to 2-i} z}{2|\lim_{z \to 2-i} z|} = \dfrac{5 - (2-i)}{2|2-i|} = \boxed{\dfrac{3+i}{2\sqrt{5}}}$.

- We also have the natural notion of continuity:

- Definition: If $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued function, we say f is continuous at $a \in \mathbb{C}$ if $\lim_{z \to a} f(z) = f(a)$. If f is continuous on its entire domain, we say f is continuous everywhere (or often, just continuous).
  - In other words, a continuous function is one whose limit as $z \to a$ is simply the value of the function at a.
  - Per limit properties (1)-(3), we see that sums, differences, products, and quotients (with nonzero denominator) of continuous functions are continuous.
  - Example: Since polynomials are constructed from constants and the variable z using addition, subtraction, and multiplication, by the above we see that any polynomial $p(z)$ is continuous everywhere. More generally, by property (4) we see that any rational function $\dfrac{p(z)}{q(z)}$ is continuous whenever $q(z) \neq 0$.

- Even more generally, by (5), for any continuous real-valued functions $p(x, y)$ and $q(x, y)$ we see that the function $f(x + iy) = p(x, y) + iq(x, y)$ is a continuous complex-valued function.
  - Example: The complex-conjugation function $f(z) = \overline{z}$ is continuous, since it is given by $f(x+iy) = x - iy$ and the real and imaginary components are both continuous.
  - Example: The modulus function $f(z) = |z|$ is continuous, since it is given by $f(x + iy) = \sqrt{x^2 + y^2}$ and the real and imaginary components are both continuous.
  - Example: The complex exponential function $f(z) = e^z$ is continuous, since it is given by $f(x + iy) = e^x \cos y + ie^x \sin y$ and the real and imaginary components are both continuous.

- We also have various substitution results for continuous functions:

- Proposition (Substitution and Continuity): Let $f : \mathbb{C} \to \mathbb{C}$.
  1. For any $g : \mathbb{C} \to \mathbb{C}$, if $\lim_{z \to a} g(z) = L$ exists and f is continuous at L, then $\lim_{z \to a} f(g(z)) = f(L)$.
     - Proof: Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta_1 > 0$ such that $|w - L| < \delta_1$ implies $|f(w) - f(L)| < \epsilon$. (Note that we can include $w = L$ since f is continuous.)
     - Additionally, since $\lim_{z \to a} g(z) = L$, there exists $\delta > 0$ such that $0 < |z - a| < \delta$ implies $|g(z) - L| < \delta_1$.
     - Thus, taking $w = g(z)$, we see that if $0 < |z - a| < \delta$ then $|g(z) - L| < \delta_1$ hence $|f(g(z)) - f(L)| < \epsilon$, so $\lim_{z \to a} f(g(z)) = f(L)$ as desired.

2. If g is continuous at $z = a$ and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at $z = a$. Thus, the composition of continuous functions is continuous.
   - Proof: Immediate from (1).
3. (Limits Along Curves) If $\lim_{z \to a} f(z)$ exists and $c : \mathbb{R} \to \mathbb{C}$ is continuous with $c(0) = a$, then $\lim_{z \to a} f(z) = \lim_{t \to 0} f(c(t))$.
   - In other words, if c is a continuous curve passing through $z = a$, then if $f(x, y)$ has a limit at $z = a$, we may compute the limit by approaching a along the curve c.
   - Proof: The same as (1) with $g : \mathbb{C} \to \mathbb{C}$ changed to $c : \mathbb{R} \to \mathbb{C}$.
4. ("Two-Paths Test") If $c : \mathbb{R} \to \mathbb{C}$ and $d : \mathbb{R} \to \mathbb{C}$ are continuous with $c(0) = d(0) = a$, and $\lim_{t \to 0} f(c(t)) \neq \lim_{t \to 0} f(d(t))$, then $\lim_{z \to a} f(z)$ does not exist.
   - In other words, if f has two different limits along the paths c, d as $z \to a$, then f has no limit at $z = a$.
   - Proof: By (3), if $\lim_{z \to a} f(z)$ exists, then $\lim_{t \to 0} f(c(t))$ and $\lim_{t \to 0} f(d(t))$ would both equal $\lim_{z \to a} f(z)$. Since they are not equal, the limit cannot exist.

- We can use (4) above to give a convenient way of establishing that a limit $\lim_{z \to a} f(z)$ does not exist: namely, by finding two different paths approaching a along which $f(z)$ has different limits.

- Example: Show that $\lim_{z \to 0} \dfrac{z^2}{|z|^2} = \lim_{z \to 0} \dfrac{z}{\overline{z}}$ does not exist.
   - We try various paths approaching zero. Along the real axis, with $c(t) = t$ as the real parameter $t \to 0$, we obtain the limit $\lim_{t \to 0} \dfrac{t^2}{|t|^2} = \lim_{t \to 0} 1 = 1$.
   - Along the imaginary axis, with $c(t) = it$ as $t \to 0$, we obtain the limit $\lim_{t \to 0} \dfrac{(it)^2}{|it|^2} = \lim_{t \to 0} -1 = -1$.
   - Since the limits along these paths are different, the original limit does not exist.
   - Remark: In terms of real and imaginary parts, this limit is equivalent to $\lim_{(x,y) \to (0,0)} \dfrac{(x^2 - y^2) + 2xyi}{x^2 + y^2}$ where $x = x + iy$. To establish the nonexistence of the given limit, we could equivalently have shown that the real limit $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

- Example: Show that $\lim_{z \to 0} \dfrac{z^2 - \overline{z}^2}{z\overline{z}}$ does not exist.
   - We try various paths approaching zero. Along the real axis, with $c(t) = t$ as the real parameter $t \to 0$, we obtain the limit $\lim_{t \to 0} \dfrac{t^2 - t^2}{t \cdot t} = \lim_{t \to 0} 0 = 0$.
   - Along the imaginary axis, with $c(t) = it$ as $t \to 0$, we obtain the limit $\lim_{t \to 0} \dfrac{(it)^2 - (-it)^2}{it \cdot (-it)} = \lim_{t \to 0} 0 = 0$.
   - Although the limits along these paths are equal, this does not mean the original limit exists.
   - Let us try along the line $c(t) = t + it$ as $t \to 0$. We obtain the limit $\lim_{t \to 0} \dfrac{(t + it)^2 - (t - it)^2}{(t + it)(t - it)} = \lim_{t \to 0} \dfrac{4it^2}{2t^2} = 2i$.
   - Since the limit along this path is different from the limit along the other paths, the original limit does not exist.

1.2.2 The Complex Derivative

- We now have enough background on limits to define the derivative. We can use the same definition as the usual one for real-valued functions:

- Definition: If $f : \mathbb{C} \to \mathbb{C}$ is a complex-valued function, the complex derivative $f'(a)$ at a point $z_0 \in \mathbb{C}$ is the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, assuming it exists.
  - As with the derivative of a real-valued function, we can equivalently state this definition as $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$.
  - In some cases the first definition is easier to use, while in others the second definition is easier to use.

- Example: Verify that the complex derivative of $f(z) = z^2$ exists everywhere and compute it.
  - We compute $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0$.
  - Since the limit always exists, we can say that $f'(z_0) = \boxed{2z_0}$ everywhere.
  - Note that this agrees with, and in fact extends, the (ordinary) real-valued derivative of the function $f(x) = x^2$ on the real line.

- Example: Verify that the complex derivative of $f(z) = 2z^3 + 5$ exists everywhere and compute it.
  - We compute $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{(2z^3 + 5) - (2z_0^3 + 5)}{z - z_0} = \lim_{z \to z_0} (2z^2 + 2zz_0 + 2z_0^3) = 3z_0^2$.
  - Since the limit always exists, we can say that $f'(z_0) = \boxed{3z_0^2}$ everywhere.
  - Notice again that this agrees with the real-valued derivative of the function $f(x) = 2x^3 + 5$.

- Example: Determine at which points the complex derivative of $f(z) = \overline{z}$ exists.
  - The required limit is $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0}$. There does not seem to be a natural way to simplify this expression further.
  - Let us try to compute the limit along different paths approaching z_0.
  - Along the horizontal line $z = z_0 + t$, for a real parameter $t \to 0$, the limit becomes $\lim_{t \to 0} \frac{\overline{(z_0 + t)} - \overline{z_0}}{(z_0 + t) - z_0} = \lim_{t \to 0} \frac{t}{t} = 1$.
  - Along the vertical line $z = z_0 + it$ for a real parameter $t \to 0$, the limit becomes $\lim_{t \to 0} \frac{\overline{(z_0 + it)} - \overline{z_0}}{(z_0 + it) - z_0} = \lim_{t \to 0} \frac{-it}{it} = -1$.
  - Along these two paths the limit has different values, so the overall limit does not exist at any point z_0. This means the derivative $\boxed{\text{does not exist}}$ at any point.

- Example: Determine at which points the complex derivative of $f(z) = z\overline{z}$ exists.
  - The required limit is $\lim_{h \to 0} \frac{(z+h)\overline{(z+h)} - z\overline{z}}{h} = \lim_{h \to 0} \frac{h\overline{z} + \overline{h}z + h\overline{h}}{h} = \lim_{h \to 0} \left[\overline{z} + \overline{h} + \frac{\overline{h}}{h} z\right] = \overline{z} + \lim_{h \to 0} \frac{\overline{h}}{h} z$.
  - We can see that if $z = 0$ then the limit $\lim_{h \to 0} \frac{\overline{h}}{h} z$ is simply zero (since the expression itself is zero), so for $z = 0$ the derivative exists and is zero.
  - Otherwise, if $z \neq 0$, then the limit $\lim_{h \to 0} \frac{\overline{h}}{h} z$ does not exist since the limit along the real axis is z while the value on the imaginary axis is $-z$.
  - We conclude that $f(z)$ is $\boxed{\text{only differentiable at } z = 0}$, and its derivative there is 0.

- Since our definition of the complex derivative is exactly the same as that of the derivative of a real-valued function, it is quite reasonable to expect that all of the usual differentiation rules will apply. This is indeed the case.

- Proposition (Differentiation Rules): Suppose f and g are both complex-differentiable at $z = a$. Then the following hold:

  1. If f is differentiable at a then f is continuous at a.
  2. (Sums/Differences) We have $(f+g)'(a) = f'(a) + g'(a)$ and $(f-g)'(a) = f'(a) - g'(a)$.
  3. (Product Rule) We have $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.
  4. (Quotient Rule) If $g(a) \neq 0$, we have $(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.
  5. (Chain Rule) If g is differentiable at $f(a)$ then $(f \circ g)'(a) = f'(g(a))g'(a)$.
     - The proofs of all of these are exactly the same as the usual proofs in single-variable calculus.
     - For example, for (1) we have $\lim_{z\to a}[f(z) - f(a)] = \lim_{z\to a} \dfrac{f(z)-f(a)}{z-a} \cdot (z-a) = f'(a) \cdot 0 = 0$ hence $\lim_{z\to a} f(z) = f(a)$ as claimed.
  6. (Polynomials) Any complex polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is differentiable with complex derivative $p'(z) = n a_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$.
     - Example: The complex derivative of $p(z) = iz^4 - (3+i)z + 5$ is $p'(z) = 4iz^3 + (3+i)$.
     - Observe in particular that this is exactly the same formula for differentiating a real polynomial.
     - Proof: Apply (1)-(4) and the fact that the complex derivative of z is 1.

- As a consequence of (6) above we also see that rational functions of z are complex-differentiable on large regions (namely, their entire domains, which consist of the entire complex plane except the finite number of points where their denominators are zero).
  - Our primary objects of study will be functions that are complex-differentiable on such large regions, rather than functions like $f(z) = z\overline{z}$ that are only differentiable on a small set, so we give them a simpler name.

- Definition: A function whose complex derivative exists on a region U is said to be holomorphic on U.
  - Although being holomorphic seems to be a relatively mild condition, it actually turns out to be quite restrictive. As we will see in later chapters, holomorphic functions have a large number of unexpectedly pleasant properties.
  - For example, even though by definition a holomorphic function only possesses a first derivative, in fact a holomorphic function necessarily has derivatives of *all* orders (compare with the situation with real-differentiable functions, which may not even have a second derivative).
  - Furthermore, as essentially an immediate consequence of having derivatives of all orders, holomorphic functions may be represented locally by power series which (as we will show) necessarily have a positive radius of convergence on which they converge to the value of the original function.

- We would like to be able to establish the holomorphicity of most "nice" functions, such as e^z.
  - However, differentiating more complicated functions directly from the definition is quite a bit more painful. (Try differentiating e^z using the definition, for example!)

- We can also attempt to compute derivatives for functions written in terms of real and imaginary parts. However, when we try to compute the limit using the definition, we will end up with a mess, since it requires writing everything in terms of the original limit variable z.
  - We can always convert any function of the form $p(x, y)$ into one in terms of $z = x + iy$ and $\overline{z} = x - iy$ by substituting $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$.
  - Example: The function $f(x+iy) = 4x^2 - 6iy$ is equivalent to $f(z) = 4[\frac{z+\overline{z}}{2}]^2 - 6i[\frac{z-\overline{z}}{2i}] = (z+\overline{z})^2 - 3(z-\overline{z})$.
  - In some cases we can still use the two-paths approach directly to see that the derivative does not exist.

- Example: Show that the complex derivative of $f(x+iy) = (x^2+y)+(2x+y^2)i$ does not exist at any point $z = x+iy$.
  - Let us try to compute the limit along different paths approaching $z_0 = x_0 + y_0 i$.
  - Along the horizontal line $z = z_0 + t$ (i.e., $x = x_0 + t$, $y = y_0$) for a real parameter $t \to 0$, the limit becomes $\lim_{t\to 0} \dfrac{(x_0+t)^2 + 2i(x_0+t) - (x_0^2 + 2ix_0)}{t} = \lim_{t\to 0} \dfrac{2x_0 t + t^2 + 2it}{t} = 2x_0 + 2i$.
  - Along the vertical line $z = z_0 + it$ (i.e., $x = x_0$, $y = y_0 + it$) for a real parameter $t \to 0$, the limit becomes $\lim_{t\to 0} \dfrac{(y_0+t) + i(y_0+t)^2 - (y_0 + iy_0^2)}{it} = \lim_{t\to 0} \dfrac{t + i(2y_0 t + t^2)}{it} = -i + 2y_0$.
  - Along these two paths the limits are always different (since x_0 and y_0 are real), so the complex derivative $\boxed{\text{does not exist}}$.

- It is much harder to show that a complex derivative *does* exist when the function is described in terms of real and imaginary parts. One way to do this is to convert back into an expression involving z and $\overline{z}$.

- Example: Show that $f(x+iy) = \dfrac{x-iy}{x^2+y^2}$ has a complex derivative for $x+iy \neq 0$, and compute it.
  - We rewrite f in terms of z and $\overline{z}$: noting that $x - iy = \overline{z}$ and $x^2 + y^2 = z\overline{z}$, we see that $f(z) = \dfrac{\overline{z}}{z\overline{z}} = \dfrac{1}{z}$.
  - Thus, $f(z) = \dfrac{1}{z}$, so by the usual differentiation rules, we have $f'(z) = \boxed{-\dfrac{1}{z^2}}$ for $z \neq 0$.

1.2.3 Partial Derivatives and the Cauchy-Riemann Equations

- There is another possible approach to complex differentiation that is motivated by multivariable calculus.
  - As we have already been discussing, we can equivalently consider any complex-valued function $f(z) = f(x+iy)$ as a function of the two variables x and y representing the real and imaginary parts of z.
  - This separation of the real and imaginary parts allows us to calculate partial derivatives of f with respect to x and y.

- Definition: If $f(z) = f(x+iy)$ is a complex-valued function, we define the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at a point $z = z+bi$ to be $\dfrac{\partial f}{\partial x}(a+bi) = \lim_{h\to 0} \dfrac{f(a+h+bi) - f(a+bi)}{h}$ and $\dfrac{\partial f}{\partial y}(a+bi) = \lim_{h\to 0} \dfrac{f(a+hi+bi) - f(a+bi)}{h}$, where both limits are single-variable real limits.
  - These complex partial derivatives have the same interpretation as in multivariable calculus: $\dfrac{\partial f}{\partial x}$ represents the rate of change of f as the variable x changes but y remains constant (i.e., the rate of change in the "real direction"), while $\dfrac{\partial f}{\partial x}$ represents the rate of change of f as the variable y changes but x remains constant (i.e., the rate of change in the "imaginary direction").
  - When computing partial derivatives, we may simply differentiate the real and imaginary parts separately with respect to the appropriate variable, and all of the other familiar properties of partial derivatives also apply here (e.g., the product rule, the quotient rule, and the chain rule).

- Example: For $f(x+iy) = (x^2+y) + (2x+y^2)i$ we have $\dfrac{\partial f}{\partial x} = 2x + 2i$ and $\dfrac{\partial f}{\partial y} = 1 + 2yi$.

- Example: For $f(x+iy) = (ye^{2x} + x^2 y) + (2\sin(x+y) - 4y^2)i$, find the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
  - Differentiating yields $\dfrac{\partial f}{\partial x} = \boxed{(2ye^{2x} + 2xy) + (2\cos(x+y))i}$ and $\dfrac{\partial f}{\partial y} = \boxed{(e^{2x} + x^2) + (2\cos(x+y) - 8y)i}$.

- Example: For $f(z) = z^2$, find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
  - We first write f write out explicitly in terms of real and imaginary parts as $f(x+iy) = (x^2-y^2)+2xyi$.
  - Now evaluating the partial derivatives yields $\frac{\partial f}{\partial x} = \boxed{2x+2yi}$ and $\frac{\partial f}{\partial y} = \boxed{-2y+2xi}$.
- More fruitfully, instead of working with the variables x and y, we could apply the change of variables $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/(2i)$ to write the original function in terms of z and $\overline{z}$.
  - By doing this, we can now view our complex-valued function f as a function of the two variables z and $\overline{z}$, and take partial derivatives with respect to z and $\overline{z}$.
  - When f is written in terms of z and $\overline{z}$ already, these partial derivatives are easy to evaluate (we just view f as a two-variable function of z and $\overline{z}$ and evaluate partial derivatives, just as above):
  - Example: For $f(z) = z^2$, we have $\frac{\partial f}{\partial z} = 2z$ and $\frac{\partial f}{\partial \overline{z}} = 0$.
  - Example: For $f(z) = \overline{z}$, we have $\frac{\partial f}{\partial z} = 0$ and $\frac{\partial f}{\partial \overline{z}} = 1$.
  - Example: For $f(z) = z\overline{z}$, we have $\frac{\partial f}{\partial z} = \overline{z}$ and $\frac{\partial f}{\partial \overline{z}} = z$.
- When f is written in terms of x and y instead, we can evaluate the partial derivatives by direct substitution, but we can avoid some of the work by using the multivariable chain rule.
  - Explicitly, since $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/(2i)$, by the multivariable chain rule we have $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right]$ and $\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]$.
  - Remark: Formally, these are actually the definitions of the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$.
- Example: For $f(x+iy) = (x^2+y)+i(2x+y^2)$, find $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$.
  - First, we compute $\frac{\partial f}{\partial x} = 2x+2i$ and $\frac{\partial f}{\partial y} = 1+2yi$.
  - Thus, $\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] = \frac{1}{2}\left[(2x+2i)-i(1+2yi)\right] = \frac{1}{2}\left[(2x-2y)+3i\right] = (\frac{1}{2}z+\frac{1}{2}\overline{z})+(-\frac{1}{2}z+\frac{1}{2}\overline{z}+\frac{1}{2})i$.
  - Likewise, $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] = \frac{1}{2}\left[(2x+2i)+i(1+2yi)\right] = \frac{1}{2}\left[(2x-2y)+3i\right] = (\frac{1}{2}z+\frac{1}{2}\overline{z})+(\frac{1}{2}z-\frac{1}{2}\overline{z}+\frac{3}{2})i$.
  - We may check this by substituting in $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$ and expanding to obtain $f(z) = (\frac{1}{4}z^2+\frac{1}{2}z\overline{z}+\frac{1}{4}\overline{z}^2)+(\frac{1}{2}z-\frac{1}{4}z^2+\frac{3}{2}\overline{z}+\frac{1}{2}z\overline{z}-\frac{1}{4}\overline{z}^2)i$, and then differentiate explicitly.
  - This yields $\frac{\partial f}{\partial z} = (\frac{1}{2}z+\frac{1}{2}\overline{z})+(\frac{1}{2}-\frac{1}{2}z+\frac{1}{2}\overline{z})i$ and $\frac{\partial f}{\partial \overline{z}} = (\frac{1}{2}z+\frac{1}{2}\overline{z})+(\frac{3}{2}+\frac{1}{2}z-\frac{1}{2}\overline{z})i$, which (of course) agrees with the calculations above.
- Example: For $f(x+iy) = e^x(\cos y + i\sin y)$, find $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$.
  - Notice that this function is simply the complex exponential $f(z) = e^z$, and it is very tempting simply to differentiate formally to obtain the requested derivatives. However, we have not established the usual differentiation rule for the complex exponential yet, so we must instead revert to using real and imaginary parts.

- We have $\dfrac{\partial f}{\partial x} = e^x(\cos y + i \sin y) = e^z$ and $\dfrac{\partial f}{\partial y} = e^x(-\sin y + i\cos y) = ie^z$.
- Thus $\dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right] = \dfrac{1}{2}\left[e^z - i(ie^z)\right] = e^z$ and $\dfrac{\partial f}{\partial \overline{z}} = \dfrac{1}{2}\left[\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right] = \dfrac{1}{2}\left[e^z + i(ie^z)\right] = 0$.
- Of course, these do agree with the intuitively-sensible partial derivatives for the complex exponential!

- Let us now investigate the relationship between the complex derivative $f'(z)$ and the partial derivatives $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \overline{z}}$.
  - First, observe that possessing a complex derivative is a much stronger property than simply possessing partial derivatives.
  - For example, the partial derivatives of $f(z) = \overline{z}$ are certainly defined, since $\dfrac{\partial f}{\partial z} = 0$ and $\dfrac{\partial f}{\partial \overline{z}} = 1$, but this function is not complex-differentiable anywhere.
  - Likewise, for $f(z) = z\overline{z}$, the partial derivatives $\dfrac{\partial f}{\partial z} = \overline{z}$ and $\dfrac{\partial f}{\partial \overline{z}} = z$ are defined, but the complex derivative is only defined when $z = 0$.
  - Similarly, for $f(x+iy) = (x^2+y) + i(2x+y^2)$, the complex derivative does not exist anywhere while the partial derivatives are $\dfrac{\partial f}{\partial z} = (\frac{1}{2}z + \frac{1}{2}\overline{z}) + (\frac{1}{2} - \frac{1}{2}z + \frac{1}{2}\overline{z})i$ and $\dfrac{\partial f}{\partial \overline{z}} = (\frac{1}{2}z + \frac{1}{2}\overline{z}) + (\frac{3}{2} + \frac{1}{2}z - \frac{1}{2}\overline{z})i$.
  - On the other hand, if $f(z)$ is a polynomial in z, say $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then it is complex-differentiable everywhere with derivative $f'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$, while its partial derivatives are $\dfrac{\partial f}{\partial z} = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$ and $\dfrac{\partial f}{\partial \overline{z}} = 0$.
  - Additionally, for the complex exponential $f(z) = e^z$, we computed $\dfrac{\partial f}{\partial z} = e^z$ and $\dfrac{\partial f}{\partial \overline{z}} = 0$. We did not compute the complex derivative using the definition, but it seems reasonable to expect that it should exist and equal e^z (which in fact it does).
  - Notice that in all of the examples we have listed where the function is complex-differentiable (or where we expect it to be), the partial derivative with respect to $\overline{z}$ is zero and the partial derivative with respect to z is equal to the actual complex derivative.
  - Inversely, in all of the situations where the function is not complex-differentiable, the partial derivative with respect to $\overline{z}$ is not zero.

- In fact, we can show that if the complex derivative exists, then in fact it equals the partial derivative $\dfrac{\partial f}{\partial z}$, and also the partial derivative $\dfrac{\partial f}{\partial \overline{z}}$ must equal zero.
  - Informally, what this means is that in order for f to be complex-differentiable, it must be a function "of z alone", not involving any $\overline{z}$ terms.

- Theorem (Cauchy): Suppose that f is complex-differentiable at $z = z_0$. Then $\dfrac{\partial f}{\partial z}(z_0) = f'(z_0)$ and $\dfrac{\partial f}{\partial \overline{z}}(z_0) = 0$.
  - Proof: Let us explicitly write $f(x+iy) = u(x,y) + iv(x,y)$ where u and v are real-valued, and suppose that f is complex-differentiable at $z_0 = x_0 + y_0 i$.
  - Consider the values of the limit $f'(z_0) = L = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ along a horizontal and vertical line through z_0.
  - Along the horizontal line the limit is

$$\begin{aligned} L &= \lim_{t \to 0} \frac{u(x_0+t,y_0) + iv(x_0+t,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{t} \\ &= \lim_{t\to 0} \frac{u(x_0+t,y_0) - u(x_0,y_0)}{t} + i \lim_{t \to 0} \frac{v(x_0+t,y_0) - v(x_0,y_0)}{t} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

while along the vertical line the limit is

$$\begin{aligned} L &= \lim_{t\to 0} \frac{u(x_0, y_0+t) + iv(x_0, y_0+t) - u(x_0, y_0) - iv(x_0, y_0)}{it} \\ &= -i \lim_{t\to 0} \frac{u(x_0, y_0+t) - u(x_0, y_0)}{t} + \lim_{t\to 0} \frac{v(x_0, y_0+t) - v(x_0, y_0)}{t} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned}$$

- Since these expressions must be equal and u, v are real-valued, we require $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.
- But now since $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$, we see that the partial derivative $\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = L = f'(z_0)$, while the partial derivative $\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] = 0$, as claimed.

- These relations on the real and imaginary parts of f that we identified above are called the Cauchy-Riemann equations. It turns out that having a complex derivative on a region is actually equivalent to satisfying the Cauchy-Riemann equations:

- Theorem (Looman-Menchoff): Suppose that $f(x+iy) = u(x,y)+iv(x,y)$. Then f has a complex derivative at every point z_0 in an open region U if and only if f is continuous on U and u and v satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, or equivalently when $\frac{\partial f}{\partial \overline{z}} = 0$, at all points in U, and in such a case the complex derivative is given by $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial f}{\partial z}$.
  - This theorem is a fundamental result in complex analysis, and allows us to rapidly determine whether a function is holomorphic.
  - We showed the forward direction already, but we will defer the proof of the reverse direction for now, since the most natural approach requires integration in the complex plane. (Our proof later will also assume that the partial derivatives of f are continuous, rather than just f itself, which substantially simplifies the argument.)
  - From the result above, we can immediately see that all of the usual properties of the derivative of a real-valued function (such as the product rule and the chain rule) extend to the complex derivative, since the complex derivative can be written in terms of real-valued partial derivatives.

- Example: Show that $f(x+iy) = e^x(\cos y + i\sin y)$ is holomorphic and that it equals its own derivative.
  - We simply check the Cauchy-Riemann equations: we have $u = e^x \cos y$ and $v = e^x \sin y$.
  - Then $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$.
  - The equations are satisfied, so f is $\boxed{\text{holomorphic}}$ and its derivative is $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$, which is indeed equal to the original function.
  - Thus, we see that the complex exponential $f(z) = e^z$ is indeed holomorphic and its derivative is itself.

- Example: Determine whether $f(x+iy) = \frac{y+ix}{x^2+y^2}$ is holomorphic, and if so find its derivative.
  - We simply check the Cauchy-Riemann equations: we have $u = \frac{y}{x^2+y^2}$ and $v = \frac{x}{x^2+y^2}$.
  - Then $\frac{\partial u}{\partial x} = \frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$.
  - The equations are satisfied, so f is $\boxed{\text{holomorphic}}$ and its derivative is $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{2xy + i(x^2-y^2)}{(x^2+y^2)^2}$.

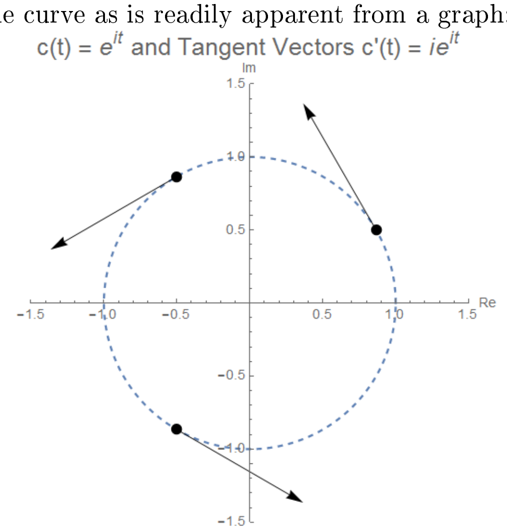
- Remark: Note in fact that $f(z) = \dfrac{i}{z}$ from which we can easily compute $f'(z) = \dfrac{-i}{z^2}$.

- Example: Determine whether $f(x+iy) = e^y(\cos x + i\sin x)$ is holomorphic, and if so find its derivative.
  - We simply check the Cauchy-Riemann equations: we have $u = e^y \cos x$ and $v = e^y \sin x$.
  - Then $\dfrac{\partial u}{\partial x} = -e^y \sin x$ whereas $\dfrac{\partial v}{\partial y} = e^y \sin x$. Since these are not equal, we see that f is $\boxed{\text{not holomorphic}}$.

- Example: Determine whether $f(z) = \dfrac{z^3}{\overline{z}^2}$ is holomorphic, and if so find its derivative.
  - Since f is given in terms of z and $\overline{z}$ we simply compute its partial derivative with respect to $\overline{z}$: this yields $\dfrac{\partial f}{\partial \overline{z}} = -2\dfrac{z^3}{\overline{z}^3}$. Since this quantity is nonzero, we see that f is $\boxed{\text{not holomorphic}}$.
  - Observe that if we define $f(0) = 0$, then we have $\lim_{z\to 0}|f(z)| = \lim_{z\to 0}|z| = 0$ and hence $\lim_{z\to 0} f(z) = 0 = f(0)$, so f is continuous at 0.
  - Also, $\frac{\partial f}{\partial x}(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h}{h} = 1$ and $\frac{\partial f}{\partial y}(0) = \lim_{h\to 0}\frac{f(ih)-f(0)}{h} = \lim_{h\to 0}\frac{ih}{h} = i$, so $\frac{\partial f}{\partial \overline{z}}(0) = \frac{1}{2}\left[\frac{\partial f}{\partial x}(0) - i\frac{\partial f}{\partial y}(0)\right] = 0$. Thus $\frac{\partial f}{\partial \overline{z}}$ is also zero at $z = 0$, meaning that f satisfies the Cauchy-Riemann equations at 0.
  - However, the complex derivative of f does not exist at $z = 0$ since the limit $\lim_{z\to 0}\frac{f(z)}{z} = \lim_{z\to 0}\frac{z^2}{\overline{z}^2}$ does not exist: its value along the real axis is 1 while its value along the line $\theta = \pi/4$ is -1.
  - Thus, this example illustrates that we *cannot* simply check the Cauchy-Riemann equations at a single point to determine complex differentiability at that point.

1.2.4 Holomorphic Functions and Angles

- Now that we have studied holomorphic functions in general, we will now discuss one very interesting geometric property of holomorphic functions: namely, that they preserve angles.
  - To make this more precise, suppose that we have a continuous curve $c : \mathbb{R} \to \mathbb{C}$ with $c(t) = x(t) + iy(t)$.
  - If we further assume that c is a differentiable function (i.e., that both $x(t)$ and $y(t)$ are differentiable real-valued function), then we have a well-defined notion of the tangent vector to the curve c given by $c'(t) = x'(t) + iy'(t)$.
  - Example: For $c(t) = e^{it} = \cos t + i\sin t$, we have $c'(t) = ie^{it} = -\sin t + i\cos t$, which indeed yields complex vectors tangent to the curve as is readily apparent from a graph:



  - If we have two differentiable curves c_1 and c_2 passing through the same point $z_0 \in \mathbb{C}$, we define the angle between c_1 and c_2 to be the angle between their tangent vectors at z_0.

- Our claim is that if $f : R \to \mathbb{C}$ is holomorphic on a region R and $z_0 \in R$, then the angle between two curves c_1 and c_2 is preserved upon applying f, which is to say, the angle between c_1 and c_2 is the same as the angle between $f \circ c_1$ and $f \circ c_2$.

- Theorem (Holomorphic Functions Preserve Angles): Suppose that $f : R \to \mathbb{C}$ is holomorphic on a region R and that two differentiable curves c_1 and c_2 pass through the point $z_0 \in R$. Then if $f'(z_0) \neq 0$, the angle between c_1 and c_2 at z_0 is the same as the angle between $f \circ c_1$ and $f \circ c_2$ at $f(z_0)$: in other words, f preserves angles.
  - The main idea of the proof is simply to use the chain rule, which in this situation is as follows: for a differentiable curve c and a holomorphic function $f : R \to \mathbb{C}$ on a region R, we have $\frac{d}{dt}[(f \circ c)(t)] = f'(c(t)) \cdot c'(t)$.
  - Proof: By the chain rule, the tangent vector to $f \circ c$ at $f(z_0)$, as a complex number, is simply the tangent vector to c at z_0 multiplied by the nonzero value $f'(z_0)$.
  - By expressing this multiplication in polar (or exponential) form we see that it simply dilates by a fixed factor and rotates by a fixed angle, so it preserves the relative angle between any two vectors.
  - In particular, scaling by $f'(z_0)$ preserves the angle between the tangent vectors to c_1 and c_2, as claimed.
  - Remark: The argument here (that complex scalings preserve angles) is informal. We can formalize it using the language of inner products.
  - Explicitly, for $z, w \in \mathbb{C}$ define the pairing $\langle z, w \rangle = \operatorname{Re}(z\overline{w})$, which for $z = a + bi$ and $w = c + di$ is simply $\langle z, w \rangle = ac + bd$, the usual dot product of two vectors in $\mathbb{R}^2$.
  - Then by an application of the law of cosines, one may show that $\langle z, w \rangle = |z|\,|w| \cos\theta$, where θ is the directed angle between z and w, and so for nonzero z, w we have $\cos\theta = \frac{\langle z, w \rangle}{|z|\,|w|}$.
  - For any nonzero $\alpha \in \mathbb{C}$, if φ is the directed angle between αz and αw we see $\cos\varphi = \frac{\langle \alpha z, \alpha w \rangle}{|\alpha z|\,|\alpha w|} = \frac{\operatorname{Re}(|\alpha|^2 z\overline{w})}{|\alpha|^2 |z|\,|w|} = \frac{\operatorname{Re}(z\overline{w})}{|z|\,|w|} = \cos\theta$, and thus we have $\varphi = \theta$ as claimed.

- It is very useful to work with maps preserving angles, since they tend to preserve other convenient physical properties. We give such maps a name:

- Definition: A conformal map is a function $f : R \to S$ that preserves angles, in the sense for any differentiable curves c_1 and c_2 in R, the angle between c_1 and c_2 at $z_0 \in R$ is the same as the angle between $f \circ c_1$ and $f \circ c_2$ at $f(z_0) \in S$.
  - Our result above shows that holomorphic functions with nonzero derivatives are always conformal.
  - Conformal maps (particularly, conformal maps whose inverses exist and are also conformal) are useful because they can allow us to transport questions back and forth between two regions, one example of which is studying solutions boundary-value problems arising in physics.
  - Another common application of conformal maps is in constructing (literal!) maps in cartography: although a conformal function may distort distances, it will still faithfully preserve angles.
  - We will return to discuss various methods for constructing conformal maps on particular regions later.

Well, you're at the end of my handout. Hope it was helpful.