Calculus II (part 5s): Second-Order Differential Equations (by Evan Dummit, 2015, v. 1.50)

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# 9 Second-Order Differential Equations

- In this supplement, we will discuss second-order differential equations and some of their applications.
- As with first-order equations, it is impossible to give a method for solving a general second-order equation. Therefore, we will only treat the simplest class of second-order equations: linear equations with constant coefficients. Such equations have the form $ay'' + by' + cy = Q(x)$, for constants $a$, $b$, $c$, and an arbitrary function $Q(x)$.
- We start by examining homogeneous equations, with the simpler form $ay'' + by' + cy = 0$, and then show how to use these methods to solve non-homogeneous equations. Then we discuss a few applications.

## 9.8 Second-Order: Existence-Uniqueness Theorem

- Like with first-order equations, there is also an existence-uniqueness theorem for second-order equations.
- Theorem (Existence-Uniqueness): If $P_1(x)$, $P_0(x)$, and $Q(x)$ are functions continuous on an interval containing $a$, then there is a unique solution (possibly on a smaller interval) to the initial value problem $y'' + P_1(x)\, y' + P_0(x)\, y = Q(x)$, for any initial condition $y(a) = b_1$, $y'(a) = b_2$. Additionally, every solution $y_{gen}$ to the general equation may be written as $y_{gen} = y_{par} + y_{hom}$, where $y_{par}$ is any one particular solution to the equation, and $y_{hom}$ is a solution to the homogeneous equation $y'' + P_1(x)\, y' + P_0(x)\, y = 0$.
  - What this theorem says is: in order to solve the general equation $y'' + P_1(x)\, y' + P_0(x)\, y = Q(x)$, it is enough to find one solution to this equation along with the general solution to the homogeneous equation.
  - The existence-uniqueness part of the theorem is hard, but the second part is fairly simple to show: $y_1$ and $y_2$ are solutions to the general equation, then their difference $y_1 - y_2$ is a solution to the homogeneous equation – to see this, just subtract the resulting equations and apply derivatives rules.
- Example: In order to solve the equation $y''(x) = e^x$, the theorem says we only need to find one function which is a solution, and then solve the homogeneous equation $y''(x) = 0$.
  - We can just try simple functions until we discover that $y(x) = e^x$ has $y''(x) = e^x$.
  - Then we need only solve the homogeneous equation $y''(x) = 0$. We can just integrate both sides twice to see that the solutions are $y = Ax + B$, for any constants $A$ and $B$.
  - Thus the general solution to the general equation $y''(x) = e^x$ is $\boxed{y(x) = e^x + Ax + B}$.
  - We can also verify that if we impose the initial conditions $y(0) = c_1$ and $y'(0) = c_2$, then (as the theorem dictates) there is the unique solution $y = e^x + (c_2 - 1)x + (c_1 - 1)$.

## 9.9 Second-Order: Linear, Constant Coefficients, Homogeneous

- The general second-order linear homogeneous differential equation with constant coefficients is of the form $ay'' + by' + cy = 0$, where $a$, $b$, and $c$ are some constants.

- Theorem: There exist two linearly-independent functions $y_1(x)$ and $y_2(x)$ such that every solution to the equation $ay'' + by' + cy = 0$ is of the form $C_1 y_1 + C_2 y_2$ for some constants $C_1$ and $C_2$.

  - The term "linearly-independent" means that $y_1(x)$ is not a constant multiple of $y_2(x)$, as a function of $x$. For example, 1 and $x$ are linearly independent, as are $x^3$ and $e^x$, but $e^x$ and $\pi \cdot e^x$ are not.
  - In essence this theorem says that there are "two" different solutions to this second-order equation, and all other solutions are just a simple combination of these two.
  - The proof of this theorem is more advanced than the things we will cover in this course. (But you'll learn about it if you take a course on differential equations or linear algebra.)

- Based on solving first-order linear homogeneous equations (like $y' = ky$), we expect the solutions to involve exponentials. If we try setting $y = e^{rx}$ then after some arithmetic we end up with $ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$. Multiplying both sides by $e^{-rx}$ and cancelling yields the characteristic equation $\boxed{ar^2 + br + c = 0}$. So if we can find two values of $r$ satisfying this much easier quadratic equation – e.g., by using the quadratic formula which says we get $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ – we will get solutions to the original differential equation.

- There are three kinds of behavior to the values of $r$ we get, based on the discriminant $D = b^2 - 4ac$ of the quadratic:

  - Case $D > 0$. In this case we get the two unequal real numbers $r_1 = \dfrac{-b - \sqrt{D}}{2a}$ and $r_2 = \dfrac{-b + \sqrt{D}}{2a}$, and the general solution is $\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$.
  - Case $D = 0$. In this case both roots are equal, so we only get one value $r = \dfrac{-b}{2a}$ yielding the solution $y = e^{rx}$. We know there must be another solution, and (based on what we see occurs in the simple example of $y'' = 0$) we can verify that $y = xe^{rx}$ also works. Therefore we have a general solution of $\boxed{y = C_1 e^{rx} + C_2 x e^{rx}}$.
  - Case $D < 0$. In this case we get two complex conjugate values of $r$, namely $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ with $\alpha = -\dfrac{b}{2a}$ and $\beta = \dfrac{\sqrt{|D|}}{2a}$. As in case 1, we could just write down $e^{r_1 x}$ and $e^{r_2 x}$ as our solutions, but we really want real-valued solutions, and $e^{r_1 x}$ and $e^{r_2 x}$ have complex numbers in the exponents. Using the identities $e^{\alpha x}\sin(\beta x) = \dfrac{1}{2i}\left[e^{r_1 x} - e^{r_2 x}\right]$ and $e^{\alpha x}\cos(\beta x) = \dfrac{1}{2}\left[e^{r_1 x} + e^{r_2 x}\right]$, we can show that the general real-valued solution is $\boxed{y = C_1 e^{\alpha x}\sin(\beta x) + C_2 e^{\alpha x}\cos(\beta x)}$.

- Therefore, to solve the general second-order linear homogeneous differential equation with constant coefficients, follow these steps:

  - Step 1: Rewrite the differential equation in the form $ay'' + by' + cy = 0$.
  - Step 2: Solve the characteristic equation $ar^2 + br + c = 0$.
  - Step 3: Determine which of the 3 cases applies, and write down the general solution.
  - Step 4: If given additional conditions, solve for the constants $C_1$ and $C_2$ using the conditions.

- Example: Find all functions $y$ such that $y'' + y' - 6 = 0$.

  - Step 2: The characteristic equation is $r^2 + r - 6 = 0$ which has roots $r = 2$ and $r = -3$.
  - Step 3: We are in Case $D > 0$ since there are unequal real roots. So the general solution is $\boxed{y = C_1 e^{2x} + C_2 e^{-3x}}$.

- Example: Find all functions $y$ such that $y'' - 2y' + 1 = 0$, with $y(0) = 1$ and $y'(0) = 2$.

- Step 2: The characteristic equation is $r^2 - 2r + 1 = 0$ which has the single root $r = 1$.
- Step 3: We are in Case $D = 0$ since there is a repeated real root. So the general solution is $y = C_1 e^x + C_2 x e^x$.
- Step 4: Plugging in the two conditions gives $1 = C_1 \cdot e^0 + C_2 \cdot 0$, and $2 = C_1 e^0 + C_2 \left[(0+1)e^0\right]$ from which $C_1 = 1$ and $C_2 = 1$. Hence the particular solution requested is $\boxed{y = e^x + xe^x}$.

- Example: Find all real-valued functions $y$ such that $y'' = -4y$.
  - Step 1: The standard form here is $y'' + 4y = 0$.
  - Step 2: The characteristic equation is $r^2 + 4 = 0$ which has roots $r = 2i$ and $r = -2i$.
  - Step 3: We are in Case $D < 0$ since there are two nonreal roots. Since the problem asks for real-valued functions we write $e^{r_1 x} = \cos(2x) + i\sin(2x)$ and $e^{r_2 x} = \cos(2x) - i\sin(2x)$ to see that the general solution is $\boxed{y = C_1 \cos(2x) + C_2 \sin(2x)}$.

## 9.10 Second-Order: Linear, Constant Coefficients, Non-Homogeneous

- The general second-order linear nonhomogeneous differential equation with constant coefficients is of the form $ay'' + by' + cy = R(x)$, where $a$, $b$, and $c$ are some constants, and $Q(x)$ is some function of $x$.
- From the Existence-Uniqueness Theorem, all we need to do is find one solution to the general equation, and find all solutions to the homogeneous equation. Since we know how to solve the homogeneous equation in full generality, we just need to develop some techniques for finding one solution to the general equation.
- There are essentially two ways of doing this.
  - The Method of Undetermined Coefficients is just a fancy way of making an an educated guess about what the form of the solution will be and then checking if it works. It will work whenever the function $Q(x)$ is a sum of terms of the form $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$, where $k$ is an integer and $\alpha$ and $\beta$ are real numbers. Thus, for example, we could use the method for a function like $Q(x) = x^3 e^{8x}\cos(x) - 4\sin(x) + x^{10}$, but not a function like $Q(x) = \tan(x)$.
  - Variation of Parameters is a more complicated method which uses some linear algebra and cleverness to use the solutions of the homogeneous equation to find a solution to the non-homogeneous equation. It will always work, for any function $Q(x)$, but generally requires more setup and computation.

### 9.10.1 Undetermined Coefficients

- The idea behind the method of undetermined coefficients is that we can guess what our solution should look like (up to some coefficients we have to solve for), if $Q(x)$ involves sums and products of polynomials, exponentials, and trigonometric functions. Specifically, we try a solution $y =$ [stuff], where the 'stuff' is a sum of things similar to the terms in $Q(x)$.
- Note that the method of undetermined coefficients really does not care whether the equation is second-order or not, as long as it has constant coefficients. It's possible to use the same ideas to solve differential equations of higher order (as long as they have constant coefficients), too.
- Here is the procedure for generating the trial solution:
  - Step 1: Generate the "first guess" for the trial solution as follows:
    - Replace all numerical coefficients of terms in $Q(x)$ with variable coefficients. If there is a sine (or cosine) term, add in the companion cosine (or sine) terms, if they are missing. Then group terms of $R(x)$ into "blocks" of terms which are the same up to a power of $x$, and add in any missing lower-degree terms in each "block".
    - Thus, if a term of the form $x^n e^{rx}$ appears in $Q(x)$, fill in the terms of the form $e^{rx} \cdot [A_0 + A_1 x + \cdots + A_n x^n]$, and if a term of the form $x^n e^{\alpha x}\sin(\beta x)$ or $x^n e^{\alpha x}\cos(\beta x)$ appears in $Q(x)$, fill in the terms of the form $e^{\alpha x}\cos(\beta x) \cdot [D_0 + D_1 x + \cdots + D_n x^n] + e^{\alpha x}\sin(\beta x)\,[E_0 + E_1 x + \cdots + E_n x^n]$.

- Step 2: Solve the homogeneous equation, and write down the general solution.
- Step 3: Compare the "first guess" for the trial solution with the solutions to the homogeneous equation. If any terms overlap, multiply all terms in the overlapping "block" by the appropriate power of $x$ which will remove the duplication.

- Here is a series of examples demonstrating the procedure for generating the trial solution:
  - Example: $y'' - y = x$.
    - Step 1: We fill in the missing constant term in $Q(x)$ to get $D_0 + D_1 x$.
    - Step 2: The homogeneous solution is $A_1 e^x + A_2 e^{-x}$.
    - Step 3: There is no overlap, so the trial solution is $\boxed{D_0 + D_1 x}$.
  - Example: $y'' + y' = x - 2$.
    - Step 1: Replacing terms in $Q(x)$ gives $D_0 + D_1 x$.
    - Step 2: The homogeneous solution is $A + Be^{-x}$.
    - Step 3: There is an overlap (the solution $D_0$) so we multiply the corresponding trial solution terms by $x$, to get $D_0 x + D_1 x^2$. Now there is no overlap, so $\boxed{D_0 x + D_1 x^2}$ is the trial solution.
  - Example: $y'' - y = e^x$.
    - Step 1: Replacing terms in $Q(x)$ gives $D_0 e^x$.
    - Step 2: The homogeneous solution is $Ae^x + Be^{-x}$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by $x$, to get $D_0 x e^x$. Now there is no overlap, so $\boxed{D_0 x e^x}$ is the trial solution.
  - Example: $y'' - 2y' + y = 3e^x$.
    - Step 1: Replacing terms in $Q(x)$ gives $D_0 e^x$.
    - Step 2: The homogeneous solution is $Ae^x + Bxe^x$.
    - Step 3: There is an overlap (the solution $D_0 e^x$) so we multiply the trial solution term by $x^2$, to get rid of the overlap, giving us the trial solution $\boxed{D_0 x^2 e^x}$.
  - Example: $y'' - 2y' + y = x^3 e^x$.
    - Step 1: We fill in the lower-degree terms to get $D_0 e^x + D_1 x e^x + D_2 x^2 e^x + D_3 x^3 e^x$.
    - Step 2: The homogeneous solution is $A_0 e^x + A_1 x e^x$.
    - Step 3: There is an overlap (namely $D_0 e^x + D_1 x e^x$) so we multiply the trial solution terms by $x^2$ to get $\boxed{D_0 x^2 e^x + D_1 x^3 e^x + D_2 x^4 e^x + D_3 x^5 e^x}$ as the trial solution.
  - Example: $y'' + y = \sin(x)$.
    - Step 1: We fill in the missing cosine term to get $D_0 \cos(x) + E_0 \sin(x)$.
    - Step 2: The homogeneous solution is $A\cos(x) + B\sin(x)$.
    - Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms by $x$ to get $D_0 x\cos(x) + E_0 x \sin(x)$. There is now no overlap so $\boxed{D_0 x\cos(x) + E_0 x\sin(x)}$ is the trial solution.
  - Example: $y'' + y = x\sin(x)$.
    - Step 1: We fill in the missing cosine term and then all the lower-degree terms to get $D_0 \cos(x) + E_0 \sin(x) + D_1 x\cos(x) + E_1 x\sin(x)$.
    - Step 2: The general homogeneous solution is $A\cos(x) + B\sin(x)$.
    - Step 3: There is an overlap (all of $D_0 \cos(x) + E_0 \sin(x)$) so we multiply the trial solution terms in that group by $x$ to get $\boxed{D_0 x\cos(x) + E_0 x\sin(x) + D_1 x^2 \cos(x) + E_1 x^2 \sin(x)}$, which is the trial solution since now there is no overlap.

- Here is a series of examples finding the general trial solution and then solving for the coefficients:

- Example: Find a function $y$ such that $y'' + y' + y = x$.
  - The procedure produces our trial solution as $y = D_0 + D_1 x$, because there is no overlap with the solutions to the homogeneous equation.
  - We plug in and get $0 + (D_1) + (D_1 x + D_0) = x$, so that $D_1 = 1$ and $D_0 = -1$.
  - So our solution is $\boxed{y = x - 1}$.
- Example: Find a function $y$ such that $y'' - y = 2e^x$.
  - The procedure gives the trial solution as $y = D_0 x e^x$, since $D_0 e^x$ overlaps with the solution to the homogeneous equation.
  - If $y = D_0 x e^x$ then $y'' = D_0(x+2)e^x$ so plugging in yields $y'' - y = [D_0(x+2)e^x] - [D_1 x\, e^x] = 2e^x$. Solving yields $D_0 = 1$, so our solution is $\boxed{y = xe^x}$.
- Example: Find a function $y$ such that $y'' - 2y' + y = x + \sin(x)$.
  - The procedure gives the trial solution as $y = (D_0 + D_1 x) + (D_2 \cos(x) + D_3 \sin(x))$, by filling in the missing constant term and cosine term, and because there is no overlap with the solutions to the homogeneous equation.
  - Then we have $y'' = -D_2 \cos(x) - D_3 \sin(x)$ and $y' = D_1 - D_2 \sin(x) + D_3 \cos(x)$ so plugging in yields

    $$y'' - 2y' + y = [-D_2 \cos(x) - D_3 \sin(x)] - 2\,[D_1 - D_2 \sin(x) + D_3 \cos(x)] + [D_0 + D_1 x + D_2 \cos(x) + D_3 \sin(x)]$$

    and setting this equal to $x + \sin(x)$ then requires $D_0 - 2D_1 = 0$, $D_1 = 1$, $D_2 + 2D_3 - D_2 = 1$, $D_3 - 2D_2 - D_3 = 0$, so our solution is $\boxed{y = x + 2 + \frac{1}{2}\cos(x)}$.
- Example: Find all functions $y$ such that $y'' + y = \sin(x)$.
  - The solutions to the homogeneous system $y'' + y = 0$ are $y = C_1 \cos(x) + C_2 \sin(x)$.
  - Then the procedure gives the trial solution for the non-homogeneous equation as $y = D_0 x \cos(x) + D_1 x \sin(x)$, by filling in the missing cosine term and then multiplying both by $x$ due to the overlap with the solutions to the homogeneous equation.
  - We can compute (eventually) that $y'' = -D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)$.
  - Plugging in yields $y'' + y = (-D_0 x \cos(x) - 2D_0 \sin(x) - D_1 x \sin(x) + 2D_1 \cos(x)) + (D_0 x \sin(x) + D_1 x \cos(x))$, and so setting this equal to $\sin(x)$, we obtain $D_0 = 0$ and $D_1 = -\frac{1}{2}$.
  - Therefore the set of solutions is $\boxed{y = -\frac{1}{2} x \cos(x) + C_1 \cos(x) + C_2 \sin(x)}$, for constants $C_1$ and $C_2$.

### 9.10.2 Variation of Parameters

- This method requires more thought, but less computation, than the method of undetermined coefficients. However, it will work for a general function $Q(x)$. The derivation is not terribly enlightening, so we will just give the steps to follow to solve $ay'' + by' + cy = Q(x)$.
  - Step 1: Solve the corresponding homogeneous equation $ay'' + by' + cy = 0$ and find two (linearly independent) solutions $y_1$ and $y_2$. Also calculate $y_1'$ and $y_2'$.
  - Step 2: Look for functions $v_1$ and $v_2$ making $y_p = v_1 \cdot y_1 + v_2 \cdot y_2$ a solution to the original equation: do this by requiring $v_1'$ and $v_2'$ to satisfy the two equations

    $$\begin{aligned} v_1' \cdot y_1 + v_2' \cdot y_2 &= 0 \\ v_1' \cdot y_1' + v_2' \cdot y_2' &= Q(x)/a \end{aligned}$$

    Solve the relations for $v_1'$ and $v_2'$ in any way you would normally solve a system of two linear equations in two variables. (Cramer's Rule will work, or you can just multiply the first equation by $y_1'$, the second by $y_1$, and subtract.) Or, even easier, plug in to these explicit formulas:
    - $\boxed{v_1' = \begin{vmatrix} 0 & y_2 \\ Q(x)/a & y_2' \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{-y_2 \cdot Q(x)/a}{y_1 y_2' - y_1' y_2}}$ and

* $\boxed{v_2' = \begin{vmatrix} y_1 & 0 \\ y_1' & Q(x)/a \end{vmatrix} \Big/ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \dfrac{+y_1 \cdot Q(x)/a}{y_1 y_2' - y_1' y_2}}$.

  ◦ Step 3: Integrate to find $v_1$ and $v_2$. (Ignore constants of integration.)

  ◦ Step 4: Write down the particular solution to the nonhomogeneous equation, $\boxed{y_p = v_1 \cdot y_1 + v_2 \cdot y_2}$.

  ◦ Step 5: If asked, the general solution to the nonhomogeneous equation is $\boxed{y = y_p + C_1 y_1 + C_2 y_2}$. If there are any initial conditions, plug them in to solve for the constants $C_1$ and $C_2$.

- Example: Find all functions $y$ for which $y'' + y = \sec(x)$.

  ◦ Step 1: The homogeneous equation is $y'' + y = 0$ which has two solutions of $y_1 = \cos(x)$ and $y_2 = \sin(x)$. Observe $y_1' = -\sin(x)$ and $y_2' = \cos(x)$.

  ◦ Step 2: We have $Q(x)/a = \sec(x)$. Also we have $y_1 y_2' - y_1' y_2 = \cos(x) \cdot \cos(x) - (-\sin(x)) \cdot \sin(x) = 1$. Thus plugging in to the formulas gives $v_1' = -\sin(x) \cdot \sec(x) = -\tan(x)$ and $v_2' = \cos(x) \cdot \sec(x) = 1$.

  ◦ Step 3: Integrating yields $v_1 = \ln(\cos(x))$ and $v_2 = x$.

  ◦ Step 4: We obtain the particular solution of $y_p = \ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)$.

  ◦ Step 5: The general solution is, therefore, given by $\boxed{y = [\ln(\cos(x)) \cdot \cos(x) + x \cdot \sin(x)] + C_1 \sin(x) + C_2 \cos(x)}$.

- Example: Find a function $y$ for which $y'' - y = e^x$.

  ◦ We could use undetermined coefficients to solve this – we would end up with $\frac{1}{2} x\, e^x$ – but let's use variation of parameters instead.

  ◦ Step 1: The homogeneous equation is $y'' - y = 0$ which has two solutions of $y_1 = e^{-x}$ and $y_2 = e^x$; then $y_1' = -e^{-x}$ and $y_2' = e^x$.

  ◦ Step 2: We have $G(x)/a = e^x$. Also we have $y_1 y_2' - y_1' y_2 = e^{-x} \cdot (e^x) - (-e^{-x}) e^x = 2$. Thus plugging in to the formulas gives $v_1' = -e^x \cdot e^x/2 = -e^{2x}/2$ and $v_2' = e^{-x} \cdot e^x/2 = 1/2$.

  ◦ Step 3: Integrating yields $v_1 = -e^{2x}/4$ and $v_2 = x/2$.

  ◦ Step 4: We obtain the particular solution of $y_p = e^{-x}(-e^{2x}/4) + e^x (x/2) = \boxed{-\frac{1}{4} e^x + \frac{1}{2} x\, e^x}$.

## 9.11 Second-Order: Applications to Newtonian Mechanics

- One of the applications we somewhat care about is the use of second-order differential equations to solve certain physics problems. Most of the examples involve springs, because springs are easy to talk about.

  ◦ Note: Second-order linear equations also arise often in basic circuit problems in physics and electrical engineering. All of the discussion of the behaviors of the solutions to these second-order equations also carries over to that setup.

### 9.11.1 Spring Problems and Damping

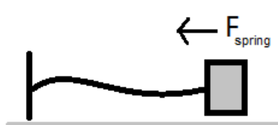
- Basic setup: An object is attached to one end of a spring whose other end is fixed. The mass is displaced some amount from the equilibrium position, and the problem is to find the object's position as a function of time.

  ◦ Various modifications to this basic setup include any or all of (i) the object slides across a surface thus adding a force (friction) depending on the object's velocity or position, (ii) the object hangs vertically thus adding a constant gravitational force, (iii) a motor or other device imparts some additional nonconstant force (varying with time) to the object.

- In order to solve problems like this one, follow these steps:

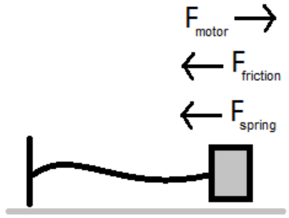
- Step 1: Draw a diagram and label the quantity or quantities of interest (typically, it is the position of a moving object) and identify and label all forces acting on those quantities.
- Step 2: Find the values of the forces involved, and then use Newton's Second Law ($F = ma$) to write down a differential equation modeling the problem. Also use any information given to write down initial conditions.
  - In the above, $F$ is the net force on the object – i.e., the sum of each of the individual forces acting on the mass (with the proper sign) – while $m$ is the mass of the object, and $a$ is the object's acceleration.
  - Remember that acceleration is the second derivative of position with respect to time – thus, if $y(t)$ is the object's position, acceleration is $y''(t)$.
  - You may need to do additional work to solve for unknown constants – e.g., for a spring constant, if it is not explicitly given to you – before you can fully set up the problem.
- Step 3: Solve the differential equation and find its general solution.
- Step 4: Plug in any initial conditions to find the specific solution.
- Step 5: Check that the answer obtained makes sense in the physical context of the problem.
  - In other words, if you have an object attached to a fixed spring sliding on a frictionless surface, you should expect the position to be sinusoidal, something like $C_1 \sin(\omega t) + C_2 \cos(\omega t) + D$ for some constants $C_1, C_2, \omega, D$.
  - If you have an object on a spring sliding on a surface imparting friction, you should expect the position to tend to some equilibrium value as $t$ grows to $\infty$, since the object should be 'slowing down' as time goes on.

- Basic Example: An object, mass $m$, is attached to a spring of spring constant $k$ whose other end is fixed. The object is displaced a distance $d$ from the equilibrium position of the spring, and is let go with velocity $v_0$ at time $t = 0$. If the object is restricted to sliding horizontally on a frictionless surface, find the position of the object as a function of time.

  - Step 1: 

    Take $y(t)$ to be the displacement of the object from the equilibrium position. The only force acting on the object is from the spring, $F_{spring}$.
  - Step 2: We know that $F_{spring} = -k \cdot y$ from Hooke's Law (aka, the only thing we know about springs). Therefore we have the differential equation $-k \cdot y = m \cdot y''$. We are also given the initial conditions $y(0) = d$ and $y'(0) = v_0$.
  - Step 3: We can rewrite the differential equation as $m \cdot y'' + k \cdot y = 0$, or as $y'' + \frac{k}{m} \cdot y = 0$. The characteristic equation is then $r^2 + \frac{k}{m} = 0$ with roots $r = \pm\sqrt{\frac{k}{m}}i$. Hence the general solution is $\boxed{y = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$, where $\omega = \sqrt{\frac{k}{m}}$.
  - Step 4: The initial conditions give $d = y(0) = C_1$ and $v_0 = y'(0) = \omega C_2$ hence $C_1 = d$ and $C_2 = v_0/\omega$. Hence the solution we want is $\boxed{y = d \cdot \cos(\omega t) + \frac{v_0}{\omega} \cdot \sin(\omega t)}$.
  - Step 5: The solution we have obtained makes sense in the context of this problem, since on a frictionless surface we should expect that the object's motion would be purely oscillatory – it should just bounce back and forth along the spring forever since there is nothing to slow its motion. We can even see that the form of the solution agrees with our intuition: the fact that the frequency $\omega = \sqrt{\frac{k}{m}}$ increases with bigger spring constant but decreases with bigger mass makes sense – a stronger spring with larger $k$ should pull back harder on the object and cause it to oscillate more quickly, while a heavier object should resist the spring's force and oscillate more slowly.

- Most General Example: An object, mass $m$, is attached to a spring of spring constant $k$ whose other end is fixed. The object is displaced a distance $d$ from the equilibrium position of the spring, and is let go with velocity $v_0$ at time $t = 0$. A motor attached to the object imparts a force along its direction of motion given by $R(t)$. If the object is restricted to sliding horizontally on a surface which imparts a frictional force of $\mu$ times the velocity of the object (opposite to the object's motion), set up a differential equation modeling the problem.

  - Here is the diagram: 

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  - As before we take $y(t)$ to be the displacement of the object from the equilibrium position. The forces acting on the object are from the spring, $F_{spring}$, from friction, $F_{friction}$, and from the motor, $F_{motor}$.
  - We know that $F_{spring} = -k \cdot y$ from Hooke's Law (aka, the only thing we know about springs). We are also given that $F_{fric} = -\mu \cdot y'$, since the force acts opposite to the direction of motion and velocity is given by $y'$. And we are just given $F_{motor} = R(t)$.
  - Plugging in gives us the differential equation $-k \cdot y - \mu \cdot y' + R(t) = m \cdot y''$, which in standard form is $\boxed{m \cdot y'' + \mu \cdot y' + k \cdot y = R(t)}$. We are also given the initial conditions $y(0) = d$ and $y'(0) = v_0$.

- Some Terminology: If we were to solve the differential equation $m \cdot y'' + \mu \cdot y' + k \cdot y = 0$ (here we assume that there is no outside force acting on the object, other than the spring and friction), we would observe a few different kinds of behavior depending on the parameters $m$, $\mu$, and $k$.

  - Overdamped Case: If $\mu^2 - 4mk > 0$ and $R(t) = 0$, we would end up with general solutions of the form $C_1 e^{-r_1 t} + C_2 e^{-r_2 t}$, which when graphed is just a sum of two exponentially-decaying functions. Physically, as we can see from the condition $\mu^2 - 4mk > 0$, this means we have 'too much' friction, since we can just see from the form of the solution function that the position of the object will just slide back towards its equilibrium at $y = 0$ without oscillating at all. This is the "overdamped" case. ["Overdamped" because there is 'too much' damping.]
  - Critically Damped Case: If $\mu^2 - 4mk = 0$ and $R(t) = 0$, we would end up with general solutions of the form $(C_1 + C_2 t)e^{-rt}$, which when graphed is a slightly-slower-decaying exponential function that still does not oscillate, but could possibly cross the position $y = 0$ once, depending on the values of $C_1$ and $C_2$. This is the "critically damped" case. ["Critically" because we give the name 'critical' to values where some kind of behavior transitions from one thing to another.]
  - Underdamped Case: If $\mu^2 - 4mk < 0$ and $R(t) = 0$, we end up with general solutions of the form $e^{-\alpha t} \cdot [C_1 \cos(\omega t) + C_2 \sin(\omega t)]$, where $\alpha = -\dfrac{\mu}{2m}$ and $\omega^2 = \dfrac{4mk - \mu^2}{4m^2}$ . When graphed this is a sine curve times an exponentially-decaying function. Physically, this means that there is some friction (the exponential), but 'not enough' friction to eliminate the oscillations entirely – the position of the object will still tend toward $y = 0$, but the sine and cosine terms will ensure that it continues oscillating. This is the "underdamped" case. ["Underdamped" because there's not enough damping.]
  - Undamped Case: If there is no friction (i.e., $\mu = 0$), we saw earlier that the solutions are of the form $y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$ where $\omega^2 = k/m$. Since there is no friction, it is not a surprise that this is referred to as the "undamped" case.

### 9.11.2 Resonance and Forcing

- Suppose an object of mass $m$ (sliding on a frictionless surface) is oscillating on a spring with frequency $\omega$. Examine what happens to the object's motion if an external force $F(t) = A \cos(\omega t)$ is applied which oscillates at the same frequency $\omega$.

- From the solution to the "Basic Example" above, we know that $\omega = \sqrt{k/m}$ so we must have $k = m \cdot \omega^2$.
- Then if $y(t)$ is the position of the object once we add in this new force $R(t) = A\,\cos(\omega t + \theta)$, Newton's Second Law now gives $-k \cdot y + R(t) = m \cdot y''$, or $m \cdot y'' + k \cdot y = R(t)$.
- If we divide through by $m$ and put in $k = m \cdot \omega^2$ we get $y'' + \omega^2 y = \dfrac{A}{m}\cos(\omega t)$.
- Now we use the method of undetermined coefficients to find a solution to this differential equation.
- We would like to try something of the form $y = D_1 \cos(\omega t) + D_2 \sin(\omega t)$, but this will not work because functions of that form are already solutions to the homogeneous equation $y'' + \omega^2 y = 0$.
- Instead the method instructs that the appropriate solution will be of the form $y = D_1 t \cdot \cos(\omega t) + D_2 t \cdot \sin(\omega t)$. We can use a trigonometric formula (the sum-to-product formula) to rewrite this as $y = D\,t \cdot \cos(\omega t + \phi)$, where $\phi$ is a "phase shift". (We can solve for the coefficients in terms of $A, m, \omega$ but it will not be so useful.)
- We can see from this formula that as $t$ grows, so does the "amplitude" $D \cdot t$: in other words, as time goes on, the object will continue oscillating with frequency $\omega$ around its equilibrium point, but the swings back and forth will get larger and larger.
- You can observe this phenomenon for yourself if you sit in a rocking chair, or swing an object back and forth – you will quickly find that the most effective way to rock the chair or swing the object is to push back and forth at the same frequency that the object is already moving at.

- We may work out the same computation with an external force $F(t) = A\,\cos(\omega_1 t)$ oscillating at a frequency $\omega_1 \neq \omega$.
  - In this case (using the same argument as above) we have $y'' + \omega^2 y = \dfrac{A}{m}\cos(\omega_1 t)$.
  - The trial solution (again by undetermined coefficients) is $\boxed{y(t) = B\cos(\omega_1 t)}$, where $B = \dfrac{A/m}{\omega^2 - \omega_1^2}$.
  - Thus the overall solution is $B\cos(\omega_1 t)$, plus a solution to the homogeneous system.
  - Now as we can see, if $\omega_1$ and $\omega$ are far apart (i.e., the driving force is oscillating at a very different frequency from the frequency of the original system) then $B$ will be small, and so the overall change $B\cos(\omega_1 t)$ that the driving force adds will be relatively small.
  - However, if $\omega_1$ and $\omega$ are very close to one another (i.e., the driving force is oscillating at a frequency close to that of the original system) then $B$ will be large, and so the driving force will cause the system to oscillate with a much bigger amplitude.
  - As $\omega_1$ approaches $\omega$, the amplitude $B$ will go to $\infty$, which agrees with the behavior seen in the previous example (where we took $\omega_1 = \omega$).

- Important Remark: Understanding how resonance arises (and how to minimize it!) is a very, very important application of differential equations to structural engineering.
  - A poor understanding of resonance is something which, at several times in the not-too-distant past, has caused bridges to fall down, airplanes to crash, and buildings to fall over.
  - We can see from the two examples that resonance arises when an external force acts on a system at (or very close to) the same frequency that the system is already oscillating at.
  - Of course, resonance is not always bad. The general principle, of applying an external driving force at (one of) a system's "natural resonance frequencies", is the underlying physical idea behind the construction of many types of musical instruments.

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Well, you're at the end of my handout. Hope it was helpful.