Calculus II (part 3): Sequences and Series (by Evan Dummit, 2015, v. 2.00)

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7 Sequences and Series

Our first goal in this chapter is to introduce the notion of a convergent sequence and to discuss the closely related concept of a series. We give a definition for "infinite series", and then to discuss how to calculate the value of infinite sums in some cases.

We then turn our attention to the question of whether a given infinite series converges. We discuss the most commonly used convergence tests for positive series (the Integral Test, Comparison Tests, Ratio Test, and Root Test) and then discuss general series whose terms may be positive or negative (in particular, alternating series) and the ideas of absolute and conditional convergence.

We will not dwell extensively on the technical details involved in the justifications of all of the results: although we will give proofs of the main results when feasible, learning the details of the proofs is far less important than understanding how the results are used. Our goal is primarily to state the tests, illuminate the underlying ideas behind them, and explain how they are used. To this end, we close our discussion with an extensive array of examples illustrating the convergence tests.

7.1 Sequences and Convergence

- Definition: A sequence is a (finite or infinite) ordered list of numbers a_1, a_2, a_3, a_4, ... , a_n, ... We will usually be concerned with infinite sequences.
  - The sequence 1, 2, 3, 4, ... is the sequence of positive integers, whose nth term is defined as $a_n = n$.
  - The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is the sequence of reciprocals of positive integers, whose nth term is $a_n = \frac{1}{n}$.
  - Another sequence is $1, -1, 1, -1, 1, -1, \ldots$, whose nth term is $a_n = (-1)^{n+1}$.

- One of the most immediate questions for an infinite sequence is "how does this sequence behave as we go further and further out?" In other words, what happens to a_n as $n \to \infty$?
  - One possibility is that the terms could approach (closer and closer) to some fixed "limiting value" L. Notice, for example, that the terms in the sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... are getting closer and closer to 0.
  - Another possibility is that the terms could grow without bound, like the terms in the sequence 1, 2, 3, 4, ...
  - Yet another possibility is that the terms could just bounce around and not settle down to anything, like the terms in the sequence 1, -1, 1, -1, 1, -1, ...
  - Ultimately, we are asking about whether this sequence "converges to a limit". This is very much the same question we could ask about a function $f(x)$: namely, what happens to the value of $f(x)$ as $x \to \infty$?

- Definition: We say a sequence a_1, a_2, ... converges to the limit L if, for any $\epsilon > 0$ (no matter how small) there exists some positive integer N such that for every $n \geq N$, it is true that $|a_n - L| < \epsilon$. If there is no value of L such that the sequence a_n converges to L, then we say the sequence a_n diverges.
  - Intuitively, the definition says that the sequence converges to L if the terms of the sequence eventually get and stay arbitrarily close to L.
  - This definition is almost identical to the formal $\epsilon - \delta$ definition of the limit of a function. Like with that definition, we use it primarily as a starting point.
  - Example: The sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ... whose nth term is $a_n = \frac{1}{n}$ $\boxed{\text{converges to } 0}$.
    - In the formal definition, we can take $N = 2/\epsilon$: then for any $n \geq N$, we see that $|a_n - L| = \left|\frac{1}{n}\right| \leq \frac{1}{N} \leq \frac{\epsilon}{2} < \epsilon$, as desired.
  - Example: The sequence -1, 1, -1, 1, -1, ... whose nth term is $a_n = (-1)^n$ $\boxed{\text{diverges}}$, since it alternates forever between $+1$ and -1.
    - This series, in an imprecise way, really is trying to converge to 'two values' (namely, 1 and -1).
  - Remark: The convergence of a sequence is not affected if we remove a finite number of terms from the sequence. So, for example, the sequence $a_1, a_2, a_3, a_4, a_5, \cdots$ has exactly the same convergence properties as the sequence $a_5, a_6, a_7, \cdots$.

- As with limits of a function, the limit of a sequence obeys a number of simple rules (sometimes collectively called the "Limit Laws").

- Theorem (Limit Laws for Sequences): Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and A, B be real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then the following properties hold:
  - The addition rule: $\lim_{n\to\infty} [a_n + b_n] = A + B$.
  - The subtraction rule: $\lim_{n\to\infty} [a_n - b_n] = A - B$.
  - The multiplication rule: $\lim_{n\to\infty} [a_n \cdot b_n] = A \cdot B$.
    - Note that the multiplication rule yields as a special case (when b_n is identically equal to a constant c) the constant-multiplication rule: $\lim_{n\to\infty} [c \cdot a_n] = c \cdot A$, where c is any real number.
  - The division rule: $\lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{A}{B}$. provided that B is not zero.
  - The squeeze rule (also called the sandwich rule): If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ (meaning that both limits exist and are equal to L) then $\lim_{n\to\infty} b_n$ exists and is also L.
  - The proofs of these results follow in the same way as for limits of functions. We omit the details.

- We also have a few additional results concerning limits of sequences:

- The Monotone Convergence Theorem: If the sequence a_1, a_2, ... is monotone increasing and bounded above, then it converges.
  - A sequence is monotone increasing if $a_1 < a_2 < a_3 < \cdots$, and a sequence is bounded above if there exists some M with all $a_i \le M$.
  - This property is almost equivalent to part of the definition of the real numbers: it follows from what is called the least upper bound property.
  - By multiplying everything by -1, the theorem also says that if a sequence of real numbers is monotone decreasing and bounded below, then it has a limit.
  - Example: The sequence a_n with $a_n = 1 - \frac{1}{n^2}$ $\boxed{\text{converges}}$, because the terms are monotone increasing, and they are all bounded above by 1 because the square is always positive. (In fact, the limit of this sequence is 1.)
- The Continuous Function Theorem: If $f(x)$ is any continuous function and $\{a_n\}$ is any convergent sequence with $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} f(a_n) = f(A)$.
  - This theorem is intuitively very natural: for a function to be continuous, it must be the case that as we approach any point $x = A$, the value of the function must get closer and closer to $f(A)$. This is precisely the behavior captured by the convergent sequence.
  - Example: The sequence b_n with $b_n = 2^{1/n}$ $\boxed{\text{converges to 1}}$, because if we set $a_n = \frac{1}{n}$ and $f(x) = 2^x$, then $b_n = f(a_n)$. Since $\lim_{n\to\infty} a_n = 0$, the Theorem indicates that $\lim_{n\to\infty} f(a_n) = 2^0 = 1$.

- Just like with limits of functions, we can sometimes be more precise about the way in which a sequence diverges: if the terms grow very large and positive, for example, it is natural to want to say that the sequence diverges to $+\infty$.
  - We will say that a sequence a_1, a_2, ... diverges to $+\infty$ if for any $M > 0$ (no matter how large) there exists some N such that for every $n \ge N$, it is true that $a_n > M$.
  - We will also say that a sequence a_1, a_2, ... diverges to $-\infty$ if for any $M > 0$ (no matter how large) there exists some N such that for every $n \ge N$, it is true that $a_n < -M$.
  - These are just formalisms that give a more careful meaning to the idea of "the terms get and stay arbitrarily large and positive" or "the terms get and stay arbitrarily large and negative".

- Frequently, in computing limits of sequences, the terms of the sequence are the values $f(1)$, $f(2)$, $f(3)$, ... for some simple function $f(x)$. There is a very natural relation between the limit of the sequence and the limit of the function:

- Proposition: If $f(x)$ is any function such that $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $f(1)$, $f(2)$, $f(3)$, ... is equal to L. Also, if the limit of the function is ∞ (or $-\infty$), then so is the limit of the sequence.
  - The idea is simply that if the sequence $f(1)$, $f(2)$, $f(3)$, ... did not have limit L, then this would contradict the statement that $\lim_{x\to\infty} f(x) = L$, since both notions of convergence are capturing the idea that the values of $f(x)$ must get and stay close to L for large x.
  - Example: The sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ whose nth term is $a_n = \frac{1}{n}$ $\boxed{\text{converges to 0}}$: it is the sequence $f(1)$, $f(2)$, $f(3)$, ... with $f(x) = \frac{1}{x}$, and $\lim_{x\to\infty} \frac{1}{x} = 0$.
  - Example: The sequence 1, 2, 4, ... whose nth term is $a_n = 2^n$ $\boxed{\text{diverges to } +\infty}$: it is the sequence $f(1)$, $f(2)$, $f(3)$, ... with $f(x) = 2^x$, and $\lim_{x\to\infty} 2^x = \infty$.

- Frequently, it is useful to invoke L'Hôpital's Rule in combination with this result.
  - Recall that L'Hôpital's Rule says that if f and g are differentiable functions, and $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$, assuming the second limit exists.

- The result above says that we can use L'Hôpital's Rule to compute limits of sequences, provided they are a quotient of the necessary form.
- Example: The sequence $\frac{1}{2}, \frac{3}{3}, \frac{5}{4}, \frac{7}{5}, \frac{9}{6}, \ldots$ whose nth term is $a_n = \frac{2n-1}{n+1}$ $\boxed{\text{converges to } 2}$: it is the sequence $f(1), f(2), f(3), \ldots$ with $f(x) = \frac{2n-1}{n+1}$. We then compute $\lim_{n\to\infty} \frac{2n-1}{n+1} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{2}{1} = 2$, where we used l'Hôpital's rule in the equality labeled "l'H".

7.2 Infinite Series

- If $a_1, a_2, \ldots$ is a sequence, recall the summation notation $\sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$: it means "add up all terms of the form a_n, where n runs from 1 to k".
  - We would like to give meaning to an "infinite sum": something like $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_k + \cdots$.
  - As written, this doesn't really make sense: how do we add infinitely many things? The answer is: we add the first k of them, and then consider what happens as $k \to \infty$.
- Definition: If $a_1, a_2, a_3, \ldots$ is an infinite sequence, we define the value (or sum) of the infinite series $\sum_{n=1}^{\infty} a_n$ to be the limit $\lim_{k\to\infty} S_k$, where $S_k = \sum_{n=1}^{k} a_n$ is the kth partial sum of the series (namely, the sum of the first n terms in the series). We say the series converges to L if the limit $\lim_{k\to\infty} S_k$ exists and equals L, and we say it diverges otherwise. (We also include the possibility that it could diverge to ∞ or to $-\infty$.)
  - Important Note: Notice that a series is different from a sequence: a series is a sum, while a sequence is a list of numbers.
  - Example: The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\boxed{\text{converges to } 1}$: the sequence of partial sums is $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots$. We see that the kth partial sum is $S_k = 1 - \frac{1}{2^k}$, and since $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{2^k}\right) = 1$, the series converges to 1.
  - Example: The series $\sum_{n=1}^{\infty} (-1)^n$ $\boxed{\text{diverges}}$: the sequence of partial sums is $-1, 0, -1, 0, -1, 0, \ldots$, which alternates forever and does not converge.
  - Example: The series $\sum_{n=1}^{\infty} 2^n$ $\boxed{\text{diverges to } +\infty}$: the sequence of partial sums is $2, 6, 14, 30, 62, \ldots$. We see that the kth partial sum is $S_k = 2^{k+1} - 2$, which diverges to ∞. (Alternatively, if we didn't see the pattern, clearly the kth term is bigger than k.)
- In order for a series to converge, the terms of the series must eventually be small. Explicitly:
- Test ("Divergence" Test): The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$. If the limit exists and is positive, then the series diverges to $+\infty$; if it is negative, the series diverges to $-\infty$.
  - The "limit does not equal zero" part includes the case where the limit doesn't exist.
  - Intuitively, if the terms aren't eventually very small, then the partial sums will bounce around too much to converge to a limit L. This is the essence of the proof of the test.
  - Note that this test is only a test for *divergence*, not a test for *convergence*.
  - Important Warning: The converse of the theorem is FALSE! Even if the terms a_n tend to 0, the series doesn't have to converge. (We will give an example in a moment.)

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converges.
  - We see that $\lim_{n\to\infty} \frac{n}{n+1} \overset{\text{l'Hô}}{=} \lim_{n\to\infty} \frac{1}{1} = 1$, where we used l'Hôpital's rule in the middle.
  - Since this is not zero, this series $\boxed{\text{diverges}}$. (In fact, it diverges to $+\infty$.)

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^4+1}}$ converges.
  - We see that $\lim_{n\to\infty} \frac{(-1)^n n^2}{\sqrt{n^4+1}} = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{1+1/n^4}} = \lim_{n\to\infty} (-1)^n$, and this last limit does not exist.
  - Thus, this series $\boxed{\text{diverges}}$.

- Example: Determine whether the series $\frac{1}{2} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ times}} + \cdots + \underbrace{\frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ times}} + \cdots$ converges.
  - Notice that the terms in this series do approach zero; nonetheless, we claim that this series diverges!
  - We just group all of the equal terms together: when we add all of the 2^n copies of $\frac{1}{2^n}$ that show up, we get 1.
  - Thus, the partial sum including all terms whose denominator is $\frac{1}{2^n}$ or smaller is equal to $\underbrace{1+1+\cdots+1}_{n \text{ times}} = n$, since there are n such "blocks" of terms.
  - Since this goes to $+\infty$, we conclude that the series $\boxed{\text{diverges to } +\infty}$.

- There are two simple types of series for which we can write down easy summation formulas: geometric series and telescoping series.

7.2.1 Geometric Series

- Definition: A geometric sequence is of the form a, ar, ar^2, ... , for some initial value a and some common ratio r. We are interested in summing the corresponding geometric series $a + ar + ar^2 + ar^3 + \cdots$.
  - Consider the partial sum $S_k = a + ar + ar^2 + ar^3 + \cdots + ar^k$. Then $rS_k = ar + ar^2 + ar^3 + \cdots + ar^{k+1}$, and so $S_k - rS_k = a - ar^{k+1}$.
  - Therefore, $\boxed{S_k = a \cdot \frac{1-r^{k+1}}{1-r}}$ is the sum of the finite geometric series $\sum_{n=0}^{k} ar^n$, provided $r \neq 1$.

- Proposition: If $-1 < r < 1$, then the infinite geometric series $\sum_{n=0}^{\infty} ar^n$ converges to the value $\frac{a}{1-r}$. If $r \geq 1$, then the series diverges to $+\infty$ if $a > 0$ and to $-\infty$ if $a < 0$. Finally, if $r \leq -1$ and $a \neq 0$, then the series diverges (in an oscillatory way).
  - Proof: Write the summation formula as $S_k = \frac{a}{1-r} - \frac{a}{1-r} r^{k+1}$ for $r \neq 1$.
    * If $-1 < r < 1$, then the limit as $n \to \infty$ of the second term is zero, so the series converges to $\frac{a}{1-r}$.
    * If $r = 1$, then the series is just $a + a + a + \cdots$, which diverges to $+\infty$ if $a > 0$ and to $-\infty$ if $a < 0$.
    * If $r > 1$, then the limit of the second term is $+\infty$ if $a > 0$ and $-\infty$ if $a < 0$.
    * If $r = -1$, then the series is just $a - a + a - a + \cdots$, which diverges for $a \neq 0$.
    * If $r < -1$, then the second term oscillates between positive and negative, and grows larger (in magnitude) as n grows, so the series diverges.

- Example: We sum the series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-1/2} = \boxed{2}$.

- Example: We sum the series $\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{9/10}{1-1/10} = \boxed{1}$.

  ◦ In other words, the infinite repeating decimal 0.9999999... is actually equal to 1!

- Example: The series $\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots \boxed{\text{diverges to } +\infty}$.

- Example: The series $\sum_{n=0}^{\infty} (-2)^n = 1 - 2 + 4 - 8 + 16 - 32 + - \cdots \boxed{\text{diverges}}$.

7.2.2 Telescoping Series

- A telescoping series is of the form $[f(1) - f(2)] + [f(2) - f(3)] + [f(3) - f(4)] + \cdots = \sum_{n=1}^{\infty} [f(n) - f(n+1)]$, for some function $f(x)$.

  ◦ We can see just by removing the parentheses and cancelling that $\sum_{n=1}^{k} [f(n) - f(n+1)] = f(1) - f(k+1)$.

  ◦ Thus, to compute the sum of the infinite series $\sum_{n=1}^{\infty} [f(n) - f(n+1)]$, we simply take the limit as $k \to \infty$ of the partial sum $f(1) - f(k+1)$. (If the sum diverges, then so does the series.)

- Example: Find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

  ◦ We use partial fraction decomposition to see that $\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$.

  ◦ Then we can write $\sum_{n=1}^{k} \frac{1}{n^2+n} = \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1}\right] = 1 - \frac{1}{k+1}$.

  ◦ As $k \to \infty$, the sum therefore $\boxed{\text{converges to } 1}$.

- Example: Find the sum of the infinite series $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$.

  ◦ We use logarithms properties to write $\ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$.

  ◦ Then we can write $\sum_{n=1}^{k} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{k} [\ln(n+1) - \ln(n)] = \ln(k+1) - \ln(1) = \ln(k+1)$.

  ◦ As $k \to \infty$, the sum therefore $\boxed{\text{diverges to } \infty}$.

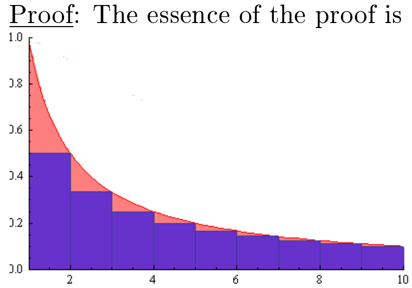
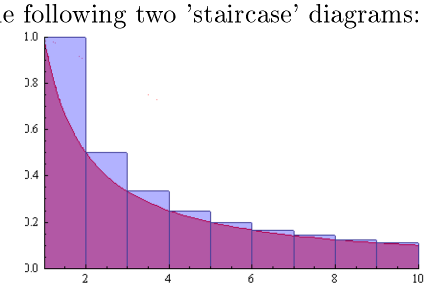
7.3 Positive Series: Integral Test, Comparison Tests, Ratio and Root Tests

- In general, it is difficult (and frequently, impossible) to give "closed-form", simple expressions for the sums of infinite series.

- We are therefore willing to settle for determining whether or not a given series converges to a finite value. If we can see that it does converge, then usually with only a little more effort we can (in principle, most of the time) compute the value to as much accuracy as we want using a computer.
- There is a fairly extensive array of "series convergence tests", giving various criteria for when a series will converge or diverge. In this section, we will discuss the most commonly used tests for series whose terms are all positive. (In the next section, we will treat sequences with negative terms.)
- The fundamental idea behind each of these series tests is to compare the given series to something else that is easier to understand: either an integral, or a similar (but simpler) series.
- Note also that if all the terms of a series are positive, either the series diverges to $+\infty$, or it is bounded above. In the latter case, the Monotone Convergence Theorem implies that the series converges to a finite limit. Thus, we need only determine whether the series converges or diverges.

7.3.1 Integral Test

- Test (Integral Test): If $f(x)$ is a decreasing, positive function, then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the integral $\int_1^{\infty} f(x)\,dx$ converges.
  - Almost always we will use this theorem to say something about the sum, by computing (or at the least, checking convergence of) the integral. The theorem is useful because sums are hard to evaluate exactly, but integrals are often easier.
  - Proof: The essence of the proof is contained in the following two 'staircase' diagrams:

    - In the picture on the left, the red region is the area under the curve $y = f(x)$ on the interval $[1, k+1]$, while the blue region is composed of k rectangles each of width 1, having height $f(n+1)$ on the interval $[n, n+1]$ for each integer $1 \le n \le k$. It is easy to see that the area of the blue region is equal to the sum $\sum_{n=2}^{k+1} f(n+1)$, while the area of the red region is equal to the integral $\int_1^{k+1} f(x)\,dx$. Since $f(x)$ is decreasing, the blue region is contained in the red region, so $\sum_{n=1}^{k+1} f(n+1) \le \int_1^{k+1} f(x)\,dx$.
    - In the picture on the right, the red region is the area under the curve $y = f(x)$ on the interval $[1, k+1]$, while the blue region is composed of k rectangles each of width 1, having height $f(n)$ on the interval $[n, n+1]$. This time, the area of the blue region is equal to the sum $\sum_{n=1}^{k} f(n)$, while the area of the red region is equal to the integral $\int_1^{k+1} f(x)\,dx$. Since $f(x)$ is decreasing, the blue region contains the red region, so $\int_1^{k+1} f(x)\,dx \le \sum_{n=1}^{k} f(n)$.
    - Combining the two inequalities gives $\sum_{n=2}^{k+1} f(n+1) \le \int_1^{k+1} f(x)\,dx \le \sum_{n=1}^{k} f(n)$. Taking the limit as

$k \to \infty$ shows that $\left[\sum_{n=1}^{\infty} f(n)\right] - f(1) \le \left[\int_1^{\infty} f(x)\,dx\right] \le \left[\sum_{n=1}^{\infty} f(n)\right]$. Thus, we see that the sum being finite forces the integral to be finite, and vice versa.

- We also remark that we do not need to start the summation at 1, since the convergence of a series is not affected if we remove a finite number of terms. We may replace 1 with any integer and the result will still hold.

- Our proof of the Integral Test also gives us a technique for bounding the value of a series. Explicitly:

- Corollary: If $f(x)$ is a decreasing positive function and $S = \sum_{n=1}^{\infty} f(n)$ is finite, then for $S_k = \sum_{n=1}^{k} f(n)$ we have the inequality $S_k + \int_{k+1}^{\infty} f(x)\,dx \le S \le S_k + \int_k^{\infty} f(x)\,dx$.
  - We can rearrange this to see that $|S - S_k| \le \int_k^{\infty} f(x)\,dx$.
  - Hence, if we want to estimate the value of S within an error of ϵ, we simply need to find the smallest k such that $\int_k^{\infty} f(x)\,dx \le \epsilon$: then S_k will be within ϵ of S.

- Example (p-series): For each positive real number p, determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
  - We apply the Integral Test: we thus need to determine the convergence of the integral is $\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$.
  - If $p < 1$ then the integral is $\left(\frac{x^{1-p}}{1-p}\right)\Big|_{x=1}^{\infty}$, which $\boxed{\text{diverges to } \infty}$, because x^{1-p} tends to ∞ as x does, since $1 - p > 0$.
  - If $p = 1$ then the integral is $(\ln(x))\,\Big|_{x=1}^{\infty}$, which $\boxed{\text{diverges to } \infty}$, because $\ln(x)$ tends to ∞ as $x \to \infty$.
  - If $p > 1$ then the integral is $\left(\frac{x^{1-p}}{1-p}\right)\Big|_{x=1}^{\infty} = \frac{1}{1-p}$, since $1-p < 0$. The integral converges, and therefore the sum $\boxed{\text{converges}}$.
  - Remark: The sum $\sum_{n=1}^{\infty} \frac{1}{n}$, the p-series with $p = 1$, is called the harmonic series. It is the simplest example of a non-convergent series whose terms nonetheless shrink to zero.

- Example: Give an estimate for the value of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ that is accurate to (at least) two decimal places.
  - We know this series converges by the previous example. The corollary tells us to find the value of k such that $\int_k^{\infty} \frac{1}{x^3}\,dx \le 0.01$.
  - We compute $\int_k^{\infty} \frac{1}{x^3}\,dx = -\frac{1}{2}x^{-2}\,\Big|_{x=k}^{\infty} = \frac{1}{2}k^{-2}$. Thus we want to pick k such that $\frac{1}{2}k^{-2} \le 0.01$.
  - If we choose $k = 8$, then $\frac{1}{2}k^{-2} = \frac{1}{128} < 0.01$.
  - Hence the partial sum $S_8 = \sum_{n=1}^{8} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{8^3} \approx \boxed{1.195}$ is guaranteed to be within 0.01 of the full infinite sum.

- Example: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\,\ln(n)}$ is convergent or divergent.

- Applying the Integral Test indicates we should determine the convergence of the integral $\int_2^\infty \frac{1}{x \cdot (\ln(x))}\,dx$.
- Upon making the substitution $u = \ln(x)$, with $du = \frac{1}{x}\,dx$, we obtain $\int_2^\infty \frac{1}{x \cdot (\ln(x))}\,dx = \int_{\ln(2)}^\infty \frac{1}{u}\,du = \ln(u)\Big|_{u=\ln(2)}^\infty = \infty$, since the natural logarithm goes to ∞.
- Since the integral diverges to ∞, by the Integral Test we conclude that the sum $\boxed{\text{diverges to } \infty}$.

• Example: Determine whether the series $\sum_{n=1}^\infty \frac{n}{e^n}$ is convergent or divergent.

- Applying the Integral Test indicates we should determine the convergence of the integral $\int_1^\infty \frac{x}{e^x}dx = \int_1^\infty x\,e^{-x}\,dx$.
- Now we integrate by parts: recall that the integration by parts formula says $\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx$.
- We take $f'(x) = e^{-x}$ with $f(x) = -e^{-x}$, and $g(x) = x$ with $g'(x) = 1$.
- This yields $\int x\,e^{-x}\,dx = -x\,e^{-x} - \int(-e^{-x}) \cdot 1\,dx = -x\,e^{-x} + \int e^{-x}\,dx = -x\,e^{-x} - e^{-x} + C$.
- Hence $\int_1^\infty x\,e^{-x}\,dx = \left[-x\,e^{-x} - e^{-x}\right]\Big|_{x=1}^\infty = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}$, where we computed $\lim_{x\to\infty} -xe^{-x} = \lim_{x\to-\infty} -\frac{x}{e^x} = 0$ via L'Hôpital's Rule.
- Since the integral converges, by the Integral Test we conclude that the sum $\boxed{\text{converges}}$.

7.3.2 Comparison Test

• Test (Comparison Test): Given two sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers such that $a_n < b_n$ for all n, if $\sum_{n=1}^\infty b_n$ converges then so does $\sum_{n=1}^\infty a_n$. Also, if $\sum_{n=1}^\infty a_n$ diverges then so does $\sum_{n=1}^\infty b_n$.

- This just says that if a (positive) series converges, then any other series with smaller (positive) terms also converges.
- Similarly, if a (positive) series diverges, then any other series with bigger terms also diverges.
- The proof of the test is merely these observations applied to the sequence of partial sums of the two series, along with an appeal to the Monotone Convergence Theorem.

• Example: Determine whether the series $\sum_{n=2}^\infty \frac{n}{n^2-1}$ is convergent or divergent.

- We observe that $\frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$.
- Since the sum $\sum_{n=2}^\infty \frac{1}{n}$ diverges by the Integral Test, the Comparison Test says that $\sum_{n=2}^\infty \frac{n}{n^2-1}$ $\boxed{\text{diverges}}$ as well.

• Example: Determine whether the series $\sum_{n=1}^\infty \frac{|\sin(n)|}{3n^2+5}$ is convergent or divergent.

- We observe that $|\sin(n)| \le 1$ for any n, and $3n^2 + 5 > 3n^2$.
- Hence we have $\frac{|\sin(n)|}{3n^2+5} \le \frac{1}{3n^2+5} \le \frac{1}{3n^2}$.
- Since the sum $\sum_{n=1}^\infty \frac{1}{3n^2}$ converges by the Integral Test, the Comparison Test says that $\sum_{n=1}^\infty \frac{|\sin(n)|}{3n^2+5}$ $\boxed{\text{converges}}$.

7.3.3 Limit Comparison Test

- Test (Limit Comparison Test): Given sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers, if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is some positive constant c, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
  - This just says that if the terms in two series are (fairly close) to being a positive constant times the other, then the two series either both converge or both diverge. The proof of the test is essentially just this idea, done carefully.
  - In general, to use the Limit Comparison Test, the idea is to find a simpler series b_n such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and positive, such that b_n is easier to analyze.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{2n-1}{n^3+5}$ is convergent or divergent.
  - For large n, the numerator will be dominated by the $2n$ term and the denominator will be dominated by the n^3 term. Thus, we will try comparing the given series to the series with $b_n = \frac{2n}{n^3} = \frac{2}{n^2}$.
  - For $a_n = \frac{2n-1}{n^3+5}$ and $b_n = \frac{2}{n^2}$, we have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n-1)/(n^3+5)}{2/n^2} = \lim_{n\to\infty} \frac{2n^3-n^2}{2n^3+10} = 1$, by standard limit properties (or L'Hôpital's Rule applied three times).
  - So, because we know that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by the Integral Test, the Limit Comparison Test says that $\sum_{n=1}^{\infty} a_n$ also $\boxed{\text{converges}}$.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n-\sin(n)}{n^2+\ln(n)}$ is convergent or divergent.
  - We will deal with the numerator and denominator separately by making two comparisons.
  - First, the numerator will be dominated by the n term, because sine is between -1 and 1, so we begin by comparing the given series to the series with $b_n = \frac{n}{n^2+\ln(n)}$.
  - For $a_n = \frac{n-\sin(n)}{n^2+\ln(n)}$ and $b_n = \frac{n}{n^2+\ln(n)}$, we have

    $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n-\sin(n))/(n^2+\ln(n))}{n/(n^2+\ln(n))} = \lim_{n\to\infty} \frac{n-\sin(n)}{n} = \lim_{n\to\infty} \left(1 - \frac{\sin(n)}{n}\right) = 1,$$

    where in the last step we used the squeeze theorem (since sine is bounded as $n \to \infty$).
  - So we are reduced to determining whether $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+\ln(n)}$ converges.
  - We do the same procedure for this series: the denominator will be dominated by the n^2 term, so we will compare to the series with $c_n = \frac{n}{n^2} = \frac{1}{n}$.
  - For $b_n = \frac{n}{n^2+\ln(n)}$ and $c_n = \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n/(n^2+\ln(n))}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+\ln(n)} = 1$ by a few applications of L'Hôpital's Rule.
  - Finally, since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Integral Test, we conclude that $\sum_{n=1}^{\infty} a_n$ also $\boxed{\text{diverges}}$.

7.3.4 Ratio Test

- Test (Ratio Test): If the sequence $\{a_n\}$ of positive real numbers has the property that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equals some constant ρ, then the sum $\sum_{n=1}^{\infty} a_n$ converges if $\rho < 1$, and diverges if $\rho > 1$. If $\rho = 1$ then the test is inconclusive, while if $\rho = \infty$ then the series diverges.
    - The idea behind the proof of the test is to compare the sequence $\{a_n\}$ to a geometric series with common ratio ρ.
    - Proof: First suppose that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ is less than 1.
        * Let $\epsilon = \frac{1-\rho}{2}$. By definition, there exists some N such that $\frac{a_{n+1}}{a_n} \le 1-\epsilon$ for every $n \ge N$. (This follows because $1-\epsilon$ is equal to $\rho+\epsilon$, whereas the fact that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ is equal to ρ means that there cannot be infinitely many terms with $\frac{a_{n+1}}{a_n} > 1-\epsilon$.)
        * Thus we have $a_{N+1} \le (1-\epsilon)a_N$, $a_{N+2} \le (1-\epsilon)a_{N+1} \le (1-\epsilon)^2 a_N$, and, by repeating this argument, $a_{N+k} \le (1-\epsilon)^k a_N$.
        * Therefore, we have the upper bound $\sum_{n=N}^{\infty} a_n \le \sum_{n=N}^{\infty} (1-\epsilon)^{n-N} a_N = a_N \cdot \frac{1}{\epsilon}$, since this last sequence is a geometric series with common ratio $1-\epsilon$. We conclude that $\sum_{n=1}^{\infty} a_n$ is bounded above, so by the Monotone Convergence Theorem it converges.
        * If the value of ρ is bigger than 1, then we can use essentially the same argument to compare the series to a geometric series with common ratio larger than 1, to see that the series diverges. (We will omit the details.)

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ is convergent or divergent.
    - We see that for $a_n = \frac{e^n}{n!}$ we have $\frac{a_{n+1}}{a_n} = \frac{e^{n+1}/(n+1)!}{e^n/n!} = \frac{e^{n+1}\cdot n!}{e^n\cdot(n+1)!} = \frac{e}{n+1}$, so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$.
    - Therefore the series $\boxed{\text{converges}}$ by the Ratio Test.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ is convergent or divergent.
    - We see that for $a_n = \frac{(2n)!}{(n!)^2}$ we have $\frac{a_{n+1}}{a_n} = \frac{(2n+2)!/\left[(n+1)!\right]^2}{(2n)!/[n!]^2} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n^2+6n+2}{n^2+2n+1}$.
    - Now we see that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4$, either by basic limits or two applications of L'Hôpital's Rule.
    - Therefore the series $\boxed{\text{diverges}}$ by the Ratio Test.

7.3.5 Root Test

- Test (Root Test): If the sequence $\{a_n\}$ of positive real numbers has the property that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and equals some constant ρ, then the sum $\sum_{n=1}^{\infty} a_n$ converges if $\rho < 1$, and diverges if $\rho > 1$. If $\rho = 1$ the test is inconclusive, while if $\rho = \infty$ then the series diverges.
    - This test is like the Ratio Test but can work better for certain types of series. The idea of the proof is the same, though: it compares the sequence $\{a_n\}$ to a geometric series with common ratio ρ.

- Proof: First suppose that the limit $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$ is less than 1.
  - Let $\epsilon = \frac{1-\rho}{2}$. By definition, there exists some N such that $\sqrt[n]{a_n} \le 1-\epsilon$ for every $n \ge N$. (This follows because $1-\epsilon$ is equal to $\rho+\epsilon$, whereas the fact that the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$ is equal to ρ means that there cannot be infinitely many terms with $\sqrt[n]{a_n} > 1-\epsilon$.)
  - Hence, for any $n \ge N$, we have $a_n \le (1-\epsilon)^n$.
  - Therefore, we have the upper bound $\sum_{n=N}^{\infty} a_n \le \sum_{n=N}^{\infty} (1-\epsilon)^n = \frac{(1-\epsilon)^N}{\epsilon}$, since this last sequence is a geometric series with common ratio $1-\epsilon$. We conclude that $\sum_{n=1}^{\infty} a_n$ is bounded above, so by the Monotone Convergence Theorem it converges.
  - If the value of ρ is bigger than 1, then we can use essentially the same argument to compare the series to a geometric series with common ratio larger than 1, to see that the series diverges. (We will omit the details.)

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ is convergent or divergent.
  - For $a_n = \frac{e^n}{n^n}$, we see that $\sqrt[n]{a_n} = \frac{e}{n}$, and so $\lim_{n\to\infty} \sqrt[n]{a_n} = 0$.
  - Therefore the series $\boxed{\text{converges}}$ by the Root Test.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n^n}$ is convergent or divergent.
  - For $a_n = \frac{2^{n^2}}{n^n}$, we see that $\sqrt[n]{a_n} = \frac{2^n}{n}$. Then $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2^n}{n} = \infty$ by basic limit properties or an application of L'Hôpital's Rule.
  - Therefore the series $\boxed{\text{diverges}}$ by the Root Test.

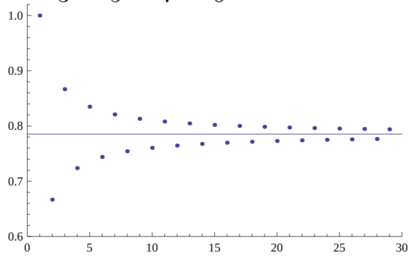
7.4 General Series: Alternating Series Test, Absolute and Conditional Convergence

- Up until now we have dealt primarily with series whose terms are positive, but we will now broaden our analysis to series which have both positive and negative terms.
- The most common of these are alternating series, which are of the form $\sum_{n=0}^{\infty} (-1)^n u_n = u_0 - u_1 + u_2 - u_3 + - \cdots$, with each $u_n > 0$.
- We will first analyze alternating series, and then broaden our discussion to general series.

7.4.1 Alternating Series Test

- We have a special convergence test for alternating series:
- Test (Alternating Series Test): Suppose $\sum_{n=0}^{\infty} (-1)^n u_n$ is an alternating series with $u_n > u_{n+1} > 0$ for all n, and $\lim_{n\to\infty} u_n = 0$. Then the series $\sum_{n=0}^{\infty} (-1)^n u_n$ converges. Furthermore, if S is the value of the infinite series and S_k is the kth partial sum, then $|S - S_k| \le u_{k+1}$ for every k.
  - Note: If $\lim_{n\to\infty} u_n \ne 0$, then the alternating series diverges by our earlier results.

- The idea behind the proof of this test is that the partial sums alternate above and below the limit of the sum. Since the partial sums get closer and closer to each other, eventually they must converge in on a single limiting value. Here is an illustration of this phenomenon for the alternating series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$:



- Proof of Test: Let S_k denote the kth partial sum.
  - First, observe that $S_{2k+1} + (u_{2k+2} - u_{2k+3}) = S_{2k+3}$. Since the term in the parentheses is positive, by the assumption that $u_n > u_{n+1}$ for all n, we conclude that $S_{2k+1} < S_{2k+3}$ for every k. Thus, $S_1 < S_3 < S_5 < S_7 < \cdots$, so the odd-numbered partial sums form an increasing sequence.
  - In a similar way, we observe that $S_{2k} - (u_{2k+1} - u_{2k+1}) = S_{2k+2}$. Again, since the term in the parentheses is positive, we conclude that $S_{2k} > S_{2k+2}$ for every k. Thus, $S_2 > S_4 > S_6 > S_7 < \cdots$, so the even-numbered partial sums form a decreasing sequence.
  - Since $S_{2k} - u_{2k+1} = S_{2k+1}$, we obtain the chain of inequalities $S_1 < S_3 < S_5 < \cdots < S_6 < S_4 < S_2$.
  - In particular, the odd-numbered partial sums form an increasing sequence that is bounded above by S_2. Hence by the Monotone Convergence Theorem, the odd-numbered partial sums converge to a limit L. Then we also have $\lim_{k\to\infty} S_{2k} = \lim_{k\to\infty} S_{2k-1} + \lim_{k\to\infty} u_{2k} = L + 0 = L$, so the even-numbered partial sums converge to the same limit.
  - For the error estimate, we simply observe that S always lies between S_k and S_{k+1} for any k, and therefore $|S - S_k| \le |S_{k+1} - S_k| = u_{k+1}$.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent or divergent.
  - Here $u_n = \frac{1}{n}$ and we can see that the criteria $u_n > u_{n+1} > 0$ and $\lim_{n\to\infty} u_n = 0$ for the Alternating Series Test are both satisfied.
  - Therefore, by the Alternating Series Test, this series $\boxed{\text{converges}}$.
  - Note: This series is called the alternating harmonic series; compare it to the (regular) harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Observe in particular that the regular harmonic series does not converge, but the alternating harmonic series does converge.

- Example: Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ is convergent and estimate its value within an error of 0.01.
  - Here $u_n = \frac{1}{n^2+1}$ and we can see that the criteria $u_n > u_{n+1} > 0$ and $\lim_{n\to\infty} u_n = 0$ for the Alternating Series Test are both satisfied. Therefore, by the Alternating Series Test, this series $\boxed{\text{converges}}$.
  - For the estimation of the value, we know that $|S - S_k| \le u_{k+1}$, so we want to choose k so that $u_{k+1} \le \frac{1}{100}$ (since this will give enough accuracy). Since $u_{10} = \frac{1}{101}$, we can take $k = 9$.
  - The desired estimate is $\sum_{n=1}^{9} \frac{(-1)^{n+1}}{n^2+1} \approx \boxed{0.3694}$: we are then guaranteed that the infinite sum is within 0.01 of this value.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^2+1}$ is convergent or divergent.
  - Here $u_n = \frac{n}{n^2+1}$ and we can check that $u_n > u_{n+1} > 0$ by observing that $u_n = \frac{1}{n+1/n}$, and noting that $(n+1) + \frac{1}{n+1} > n+1 > n + \frac{1}{n}$.
  - Since clearly $\lim_{n\to\infty} u_n = 0$, the Alternating Series Test applies and says that the series $\boxed{\text{converges}}$.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{n^2+1}$ is convergent or divergent.
  - Here $u_n = \frac{n^2}{n^2+1}$. We try applying the Alternating Series Test, but we cannot, because $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$, by basic limit properties (or L'Hôpital's Rule).
  - From this limit we see that the terms of the series do not tend to zero: thus, the series actually $\boxed{\text{diverges}}$.

- Example: Estimate the value of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to four decimal places.
  - It is easy to apply the Alternating Series Test to see that this series converges: we have $u_n = \frac{1}{n!}$, and the terms are clearly positive and decrease to zero as $n \to \infty$.
  - For the estimation of the value, we know that $|S - S_k| \le u_{k+1}$, so we want to choose k so that $u_{k+1} \le 10^{-4}$ (since this will give enough accuracy). Since $8! = 40320$ we see that $u_8 < 10^{-4}$, so we can take $k = 7$.
  - We obtain the estimate $\sum_{n=0}^{7} \frac{(-1)^n}{n!} \approx 0.367857$, so to four decimal places the sum is $\boxed{0.3679}$.

7.4.2 Absolute and Conditional Convergence

- Sometimes alternating series will converge because the terms u_n decrease in size so rapidly that the series would have converged even if we summed the series without the alternating signs: namely, if the series $\sum_{n=0}^{\infty} |(-1)^n u_n| = u_0 + u_1 + u_2 + u_3 + \cdots$ is convergent. This idea is captured in the following theorem:

- Test (Absolute Convergence Test): If a_1, a_2, ... is a sequence of real numbers and the series $\sum_{n=0}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=0}^{\infty} a_n$.
  - The idea of the proof is to use the Comparison Test on $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} |a_n|$. In order to do this, however, we need to modify the first series slightly to make its terms nonnegative.
  - Proof: Suppose $\sum_{n=0}^{\infty} |a_n|$ converges and let $b_n = a_n + |a_n|$.
    * Notice that $0 \le a_n + |a_n| \le 2|a_n|$ for each n, because $|a_n|$ is either equal to a_n or to $-a_n$.
    * By hypothesis, the sequence $\sum_{n=0}^{\infty} |a_n|$ converges hence so does the sequence $\sum_{n=0}^{\infty} 2|a_n|$.
    * Now because $0 \le b_n \le 2|a_n|$, applying the Comparison Test shows that $\sum_{n=0}^{\infty} b_n$ converges.

- * Finally, since $a_n = b_n - |a_n|$, we see that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n - \sum_{n=0}^{\infty} |a_n|$ is a difference of two convergent series, hence is convergent.

- Definition: A series which still converges when we take the absolute values of all the terms is said to converge absolutely. A series which itself converges but whose absolute-value series does not converge is said to converge conditionally.
  - The theorem above says that every absolutely convergent series converges.
  - In general, absolutely convergent series are much better behaved than conditionally convergent series.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent, conditionally convergent, or divergent.
  - We analyze the series of absolute values first.
  - Notice that this series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and the Integral Test says that this series converges.
  - Therefore, the original series $\boxed{\text{converges absolutely}}$.

- Example: Determine whether the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is absolutely convergent, conditionally convergent, or divergent.
  - We analyze the series of absolute values first.
  - Notice that this series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by the Integral Test.
  - For the original series, as we saw above, the Alternating Series Test implies that the series converges.
  - Therefore, the series $\boxed{\text{converges conditionally}}$.

7.4.3 Rearrangements of Infinite Series

- Another topic we can investigate is what happens if we sum the terms of a convergent series in a different order.
  - In other words: what happens if we rearrange the terms in the series? (For example, one rearrangement of $\sum_{n=1}^{\infty} a_n$ is given by $a_1 + a_3 + a_{15} + a_{36} + a_5 + a_7 + a_{80} + \cdots$.)
  - Based on the behavior of finite sums, it may seem that rearranging the terms cannot possibly make a difference, since (for example) $a + b + c + d$ is equal to $d + c + a + b$.
  - However, this turns out not to be the case: it is (quite unexpectedly) possible to change the value of an infinite order by rearranging its terms!
  - For an explicit example, let

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

    be the alternating harmonic series.
  - Now divide each of the terms by 2 and "pad" the series by including zeroes: we get

$$\frac{1}{2}S = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + \cdots.$$

- Now add these two series together term-by-term. By the limit laws, the summed series is

$$\frac{3}{2}S = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \cdots$$

  which, after removing the zero terms, yields

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots.$$

- It is straightforward to verify that every term of the alternating harmonic series occurs exactly once. (The sequence now has two positive terms followed by a negative term, rather than alternating.) But notice that we have changed the sum by doing this rearrangement: it is now $\frac{3}{2}$ of the original value!

• Theorem (Riemann Rearrangement Theorem): Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of the series has the same sum. If the series $\sum_{n=1}^{\infty} a_n$ converges conditionally, then there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ of the series that has any desired real number value, $+\infty$, or $-\infty$.

  - The non-intuitive behavior displayed by conditionally convergent series underscores the fact that infinite summation can be *extremely* (!) delicate.
  - In general, it is of central importance to be scrupulously careful when dealing with infinite series, because even something as seemingly innocuous as rearranging the terms can completely change the behavior of the series.
  - Outline Proof (first part): Consider the kth partial sum $B_k = \sum_{n=1}^{k} b_n$.
    * Choose k large enough so that each of $a_1, a_2, \cdots, a_N$ appear in the sum $\sum_{n=1}^{k} b_n$.
    * Then the difference between $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{k} b_n$ is at most $\sum_{n=N+1}^{\infty} |a_n|$, since the second sum contains $\sum_{n=1}^{N} a_n$.
    * Taking N larger and larger shows that the partial sums $\sum_{n=1}^{k} b_n$ approach the value $\sum_{n=1}^{\infty} a_n$, as claimed.
  - Outline Proof (second part): First, we note that the sum of all of the negative terms in a conditionally convergent series must be $-\infty$.
    * This follows because if the sum of all the negative terms were a finite integer $-N$, then the series $\sum_{n=1}^{\infty} |a_n|$ would be equal to $2N + \sum_{n=1}^{\infty} a_n$, which converges. (This contradicts the assumption that the series is not absolutely convergent.)
    * Similarly, the sum of all the positive terms in the series must be $+\infty$.
    * Now, in order to get a rearrangement $\sum_{n=1}^{\infty} b_n$ whose sum is $r \geq 0$, we sum positive terms until the sum exceeds r, then negative terms until the sum drops below r, then more positive terms until the sum exceeds r, and so on. The nature of the summation will make the sum hone in on the value r, since each partial sum is a distance at most a_t from r (where t is a parameter that increases as we take partial sums farther out in the series), and the terms a_i shrink to zero.

* For a negative sum, we simply sum negative terms first. To get a sum of $+\infty$, we add positive terms until the sum exceeds 2, then negative terms until the sum drops below 1, then positive terms until the sum exceeds 4, then negative terms until the sum drops below 3, and so on and so forth. And for $-\infty$, we simply interchange positive and negative.

7.5 Further Examples of Series Convergence Tests

- In this section, we give a number of examples of series and apply the various series tests to determine absolute / conditional convergence of the corresponding series. We have included these examples in a separate section to give additional practice for determining which tests to use on different types of infinite series.

- Here is a general list of steps to follow when trying to determine the convergence of a given series $\sum_{n=1}^{\infty} a_n$:
  - First, determine $\lim_{n\to\infty} |a_n|$: if this limit fails to exist, or exists but is nonzero, the series diverges.
  - Next, examine the terms in the series. There are different strategies depending on the form of the terms of the series:
  - Suppose first that the terms of the series are positive:
    * If the series is a geometric series of the form $\sum_{n=1}^{\infty} ar^n$, it can be summed directly. Also consider the possibility that the series may be a telescoping series of the form $\sum_{n=1}^{\infty} [f(n) - f(n+1)]$ for some nice function $f(x)$, or a sum of several telescoping or geometric series.
    * If $a_n = q(n)$ where $q(n)$ is a rational function (or more generally, an algebraic function, possibly involving radicals), use the Limit Comparison Test to compare the series to an appropriate p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.
    * More generally, if the terms of a_n can be easily compared to a simpler function, use the Limit Comparison Test to convert the problem to one about analyzing a simpler series.
    * If $a_n = f(n)$ where $f(x)$ is a nice function (whose integral $\int_1^{\infty} f(x)\,dx$ is easily evaluated), use the Integral Test.
    * If $a_n = (b_n)^n$ where b_n is simple, then try using the Root Test. Particularly worth noting is the fact that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ (which may be derived from L'Hôpital's Rule).
    * If a_n involves factorials, exponentials, or other kinds of products, try using the Ratio Test. (Note that using the Ratio Test on quotients of polynomials will never work, because the quotient $q(n+1)/q(n)$ of successive terms always tends to 1 for any rational function $q(x)$.)
  - Now suppose that the series has negative terms:
    * First analyze the absolute value series $\sum_{n=1}^{\infty} |a_n|$ using the tests above: if this series converges, then so does the original series.
    * If the series is an alternating series of the form $\sum_{n=0}^{\infty} (-1)^n u_n$, try using the Alternating Series Test.
  - In some cases, it may be necessary to make algebraic manipulations to simplify the terms of the series before applying any of the tests.

- Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{3n+1}$ is absolutely convergent, conditionally convergent, or divergent.
  - We compute $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3n}{3n+1} = 1$. Since this is nonzero, we conclude that the series is $\boxed{\text{divergent}}$.

- Example: Determine whether the series $\sum_{n=0}^{\infty} \frac{(-3)^n n^3}{n!}$ is absolutely convergent, conditionally convergent, or divergent.
  - We examine $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n n^3}{n!}$. This limit will be zero because factorials grow faster than exponentials and polynomial functions.
  - Now we analyze the absolute value series $\sum_{n=0}^{\infty} \frac{3^n n^3}{n!}$, with $b_n = \frac{3^n n^3}{n!}$.
  - Since the terms $b_n = \frac{3^n n^3}{n!}$ involve factorials and exponentials, we try using the Ratio Test.
  - We compute $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}(n+1)^3/(n+1)!}{3^n n^3/n!} = \frac{3^{n+1}(n+1)^3 \cdot n!}{3^n n^3 \cdot (n+1)!} = \frac{3(n+1)^2}{n^3}$.
  - As $n \to \infty$, we see that $\frac{b_{n+1}}{b_n} \to 0$, since the denominator has a higher degree than the numerator.
  - Hence by the Ratio Test, the absolute value series converges. We conclude that the original series is $\boxed{\text{absolutely convergent}}$.

- Example: Determine whether the series $\sum_{n=0}^{\infty} \frac{n!}{2^{n^2}}$ is absolutely convergent, conditionally convergent, or divergent.
  - Since the terms of this series are positive, we only need to determine convergence.
  - It is not immediate whether $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{2^{n^2}}$ converges to zero or not.
  - Since the terms $a_n = \frac{n!}{2^{n^2}}$ involve factorials and exponentials, we try using the Ratio Test.
  - We compute $\frac{a_{n+1}}{a_n} = \frac{(n+1)!/2^{(n+1)^2}}{n!/2^{n^2}} = \frac{(n+1)! \cdot 2^{n^2}}{n! \cdot 2^{n^2+2n+1}} = \frac{n+1}{2^{2n+1}}$.
  - As $n \to \infty$, we see that $\frac{a_{n+1}}{a_n} \to 0$, since the exponential in the denominator will dominate the numerator.
  - Hence by the Ratio Test, the series is $\boxed{\text{(absolutely) convergent}}$.

- Example: Determine whether the series $\sum_{n=3}^{\infty} \frac{1}{n^{1+1/n}}$ is convergent or divergent.
  - It is easy to see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{1+1/n}}$ converges to zero, because the denominator is bigger than $n^1 = n$.
  - The function $f(x) = \frac{1}{x^{1+1/x}}$ is not easy to integrate. It is also not easy to take the nth root of the terms, nor is the ratio $\frac{a_{n+1}}{a_n}$ particularly nice.
  - We do notice that the terms are rather similar to the terms of the harmonic series $\sum_{n=3}^{\infty} \frac{1}{n}$. We will try using the Limit Comparison Test with $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^{1+1/n}}$.
  - We see that $\frac{a_n}{b_n} = \frac{1/n}{1/n^{1+1/n}} = \frac{n^{1+1/n}}{n} = \sqrt[n]{n}$.
  - We know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$, so the Limit Comparison Test says that our series is $\boxed{\text{divergent}}$, because we know that the harmonic series is divergent. More specifically, it $\boxed{\text{diverges to } +\infty}$.

- <u>Example</u>: Determine whether the series $\sum_{n=0}^{\infty}(-1)^n \dfrac{\sqrt[3]{n^2+1}}{\sqrt{n^3+4}}$ is absolutely convergent, conditionally convergent, or divergent.
    - We examine $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\dfrac{\sqrt[3]{n^2+1}}{\sqrt{n^3+4}}$. This limit will be zero because the denominator grows like $n^{3/2}$ while the numerator grows like $n^{2/3}$.
    - Since the terms $a_n = \dfrac{\sqrt[3]{n^2+1}}{\sqrt{n^3+4}}$ involve algebraic functions, we try using the Limit Comparison Test.
    - By basic limit properties, $\lim_{n\to\infty}\dfrac{\sqrt[3]{n^2+1}}{\sqrt[3]{n^2}} = 1$, and $\lim_{n\to\infty}\dfrac{\sqrt{n^3+4}}{\sqrt{n^3}} = 1$, so the terms a_n grow at essentially the same rate as $\dfrac{\sqrt[3]{n^2}}{\sqrt{n^3}} = \dfrac{n^{2/3}}{n^{3/2}} = n^{-5/6}$. Explicitly, by the limit laws, we deduce that $\lim_{n\to\infty}\dfrac{a_n}{n^{-5/6}} = 1$.
    - But we know that $\sum_{n=1}^{\infty} n^{-5/6}$ diverges to ∞ since it is a p-series. Hence the original series is not absolutely convergent by the Limit Comparison Test.
    - Now we examine the original series, which is an alternating series.
    - We apply the Alternating Series Test with $u_n = \dfrac{\sqrt[3]{n^2+1}}{\sqrt{n^3+4}}$. It is a straightforward check that $f(x) = \dfrac{\sqrt[3]{n^2+1}}{\sqrt{n^3+4}}$ is a decreasing function for large enough n, and as we saw above, $\lim_{n\to\infty} u_n = 0$. Hence by the Alternating Series Test, the original series converges.
    - We conclude that the original series is $\boxed{\text{conditionally convergent}}$.

- <u>Example</u>: Determine whether the series $\sum_{n=3}^{\infty}\dfrac{1}{n\sqrt{\ln(n)}}$ is convergent or divergent.
    - Clearly, $\lim_{n\to\infty}|a_n| = 0$.
    - The nth root of a_n is not especially nice, nor is $\dfrac{a_{n+1}}{a_n}$, so the Root and Ratio Tests are unlikely to be useful. The only natural series for comparison is the harmonic series, but the limit of the ratio $\dfrac{a_n}{1/n} = \dfrac{1}{\sqrt{\ln(n)}}$ is zero as $n \to \infty$.
    - We try the Integral Test: we compute $\displaystyle\int_3^{\infty}\frac{1}{x\sqrt{\ln(x)}}\,dx$.
    - We substitute $u = \ln(x)$ with $du = \dfrac{1}{x}\,dx$ to obtain $\int_{\ln(3)}^{\infty}\dfrac{1}{\sqrt{u}}\,du = 2u^{1/2}\Big|_{u=\ln(3)}^{\infty} = \infty$.
    - Hence, by the Integral Test, we conclude that the original series is $\boxed{\text{divergent}}$. More specifically, it $\boxed{\text{diverges to } +\infty}$.

- <u>Example</u>: Determine whether the series $\sum_{n=0}^{\infty}(-1)^n\left[\sqrt{n+1}-\sqrt{n}\right]$ is absolutely convergent, conditionally convergent, or divergent.
    - For this series, we first rewrite the terms using $\sqrt{a}-\sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$: this gives $a_n = \dfrac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$.
    - Again, we start by analyzing the absolute value series with $|a_n| = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. By the Comparison (or Limit Comparison) Test, the series $\sum_{n=0}^{\infty}|a_n|$ diverges, because we can compare it to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = \dfrac{2}{\sqrt{n}}$, which is a divergent p-series.

- For the original series, we apply the Alternating Series Test. Each of the hypotheses is easy to verify (we skip the details), and so the test implies that the series converges. We conclude that the original series is $\boxed{\text{conditionally convergent}}$.

- Example: Determine whether the series $\sum_{n=0}^{\infty}(-1)^n\left[\ln(2n^2+3)-2\ln(n)\right]$ is absolutely convergent, conditionally convergent, or divergent.
  - For this series, we first combine the logarithms into a single term by writing $\ln(2n^2+3)-2\ln(n) = \ln(2n^2+3)-\ln(n^2) = \ln\left(\frac{2n^2+3}{n^2}\right) = \ln\left(2+\frac{3}{n^2}\right)$.
  - Now we compute $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\ln\left(2+\frac{3}{n^2}\right) = \ln(2)$, since the term inside the logarithm tends to 2 and $f(x)=\ln(x)$ is continuous.
  - Since the limit is not zero, we conclude that the original series is $\boxed{\text{divergent}}$.

- Example: Determine whether the series $\sum_{n=0}^{\infty}\frac{1-\cos(3n)}{n\sqrt{n}+2}$ is absolutely convergent, conditionally convergent, or divergent.
  - We compare the absolute value series to a simpler one. First, we observe that $\left|\frac{1-\cos(3n)}{n\sqrt{n}+2}\right| = \frac{|1-\cos(3n)|}{n\sqrt{n}+2} \le \frac{2}{n\sqrt{n}+2}$, because cosine is always between -1 and 1.
  - Thus, by the Comparison Test, if the second series converges, then so does the first one.
  - Now we use the Limit Comparison Test to compare $\sum_{n=0}^{\infty}\frac{2}{n\sqrt{n}+2}$ with $\sum_{n=0}^{\infty}\frac{2}{n\sqrt{n}}$: since the limit of the ratio is $\lim_{n\to\infty}\frac{2/(n\sqrt{n})}{2/(n\sqrt{n}+2)} = \lim_{n\to\infty}\frac{n\sqrt{n}+2}{n\sqrt{n}} = 1$, and the second series is a convergent p-series, we see that $\sum_{n=0}^{\infty}\frac{2}{n\sqrt{n}+2}$ converges.
  - Thus, by the Comparison Test, the original series is $\boxed{\text{absolutely convergent}}$.

- Example: Determine whether the series $\sum_{n=0}^{\infty}\frac{3^n+4^n}{5^n}$ is convergent or divergent.
  - Here, we observe that $3^n \le 4^n$, so $a_n \le \frac{2\cdot 4^n}{5^n}$.
  - Then we apply the Root Test to see that $\lim_{n\to\infty}\sqrt[n]{a_n} \le \lim_{n\to\infty}\frac{\sqrt[n]{2}\cdot 4}{5} = \frac{4}{5}$.
  - Since this limit is less than 1, the series $\boxed{\text{converges}}$.
  - In fact, if we rewrite the series as $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n$, we see that it is the sum of two geometric series, and that the exact value is $\frac{1}{1-3/5}+\frac{1}{1-4/5} = \frac{15}{2}$.

Well, you're at the end of my handout. Hope it was helpful.