Calculus II (part 1): Techniques of Integration (by Evan Dummit, 2012, v. 1.25)

Contents

5 Techniques of Integration

We discuss a number standard techniques for computing integrals: substitution methods, integration by parts, partial fractions, and improper integrals.

5.1 Basic Antiderivatives

- Here is a list of common indefinite integrals that should already be familiar:

$$\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1}+C,\ n\neq -1 \\
\int x^{-1}\,dx &= \ln(x)+C \\
\int e^x\,dx &= e^x+C \\
\int \sin(x)\,dx &= -\cos(x)+C \\
\int \cos(x)\,dx &= \sin(x)+C \\
\int \sec^2(x)\,dx &= \tan(x)+C \\
\int \sec(x)\tan(x)\,dx &= \sec(x)+C \\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x)+C \\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1}(x)+C \\
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}(x)+C
\end{aligned}$$

- Here are some other slightly more difficult antiderivatives that crop up occasionally:

$$\begin{aligned}
\int \ln(x)\,dx &= x\ln(x) - x + C \\
\int \tan(x)\,dx &= -\ln(\cos(x)) + C \\
\int \sec(x)\,dx &= \ln(\sec(x) + \tan(x)) + C \\
\int \csc(x)\,dx &= -\ln(\csc(x) + \cot(x)) + C \\
\int \cot(x)\,dx &= \ln(\sin(x)) + C
\end{aligned}$$

5.2 Substitution

- The general substitution formula states that $\boxed{\boxed{\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C}}$. It is just the Chain Rule, written in terms of integration via the Fundamental Theorem of Calculus. We generally don't use the formula written this way. To do a substitution, follow this procedure:
  - Step 1: Choose a substitution $u = g(x)$.
  - Step 2: Compute the differential $du = g'(x)\,dx$.
  - Step 3: Rewrite the original integral in terms of u:
    - 3a: Rewrite the integral to peel off what will become the new differential du.
    - 3b: Write the remaining portion of the integrand in terms of u.
    - 3c: Find the new limits of integration in terms of u, if the integral is a definite integral. [If the old limits are $x = a$ and $x = b$, the new ones will be $u = g(a)$ and $u = g(b)$. In a very concrete sense, these are "the same" points.]
    - 3d: Write down the new integral. If the integral is indefinite, substitute back in for the original variable.
- Substitution is best learned by doing examples:
- Example: Evaluate $\int_0^3 2x\,e^{x^2}\,dx$.
  - Step 1: The exponential has a 'complicated' argument x^2, so we try setting $u = x^2$.
  - Step 2: The differential is $du = 2x\,dx$.
  - Step 3a: We can rearrange the integral as $\int_0^3 e^{x^2} \cdot (2x\,dx)$.
  - Step 3b: The "remaining portion" of the integrand is e^{x^2}, which is just e^u.
  - Step 3c: We see that $x = 0$ corresponds to $u = 0^2$ and $x = 3$ corresponds to $u = 3^2$.
  - Step 3d: Putting it all together gives $\int_0^9 e^u\,du = e^u\big|_{u=0}^{9} = \boxed{e^9 - 1}$.
- Example: Evaluate $\int_1^e \dfrac{(\ln(x))^2}{x}\,dx$.
  - Step 1: It might not look like any function has a 'complicated' argument, but if we think carefully we can see that the numerator is what we get if we plug in $\ln(x)$ to the squaring function. So we try setting $u = \ln(x)$.
  - Step 2: The differential is $du = \dfrac{1}{x}\,dx$.
  - Step 3a: We can rearrange the integral as $\int_0^3 [\ln(x)]^2 \cdot \left(\dfrac{1}{x}\,dx\right)$.

- Step 3b: The "remaining portion" of the integrand is $[\ln(x)]^2$, which is just u^2.
- Step 3c: We see that $x = 1$ corresponds to $u = \ln(1) = 0$ and $x = e$ corresponds to $u = \ln(e) = 1$.
- Step 3d: Putting it all together gives $\int_0^1 u^2\,du = \frac{1}{3}u^3|_{u=0}^1 = \boxed{\frac{1}{3}}$.

- Example: Evaluate $\int_0^1 x\sqrt{3x^2+1}\,dx$.
  - Step 1: Here we see that the square root function has the 'complicated' argument $3x^2+1$ so we try $u = 3x^2+1$.
  - Step 2: The differential is $du = 6x\,dx$.
  - Step 3a: We can rearrange the integral as $\int_0^1 \sqrt{3x^2+1}\cdot\frac{1}{6}\cdot(6x\,dx)$. Note that we introduced a factor of $6\cdot\frac{1}{6}$, which is okay since it's just multiplication by 1.
  - Step 3b: The "remaining portion" of the integrand is $\sqrt{3x^2+1}\cdot\frac{1}{6}$, which is just $\frac{1}{6}u^{1/2}$.
  - Step 3c: We see that $x = 0$ corresponds to $u = 1$ and $x = 1$ corresponds to $u = 4$.
  - Step 3d: Putting it all together gives $\int_1^4 \frac{1}{6}u^{1/2}\,du = \frac{1}{6}\cdot\frac{2}{3}u^{3/2}|_{u=1}^4 = \boxed{\frac{7}{9}}$.

- Example: Evaluate $\int_2^3 \frac{2x}{x^2-1}\,dx$.
  - Step 1: Try the denominator: $u = x^2-1$.
  - Step 2: The differential is $du = 2x\,dx$.
  - Step 3a: We can rearrange the integral as $\int_2^3 \frac{1}{x^2-1}\cdot(2x\,dx)$.
  - Step 3b: The "remaining portion" of the integrand is $\frac{1}{x^2-1}$, which is just u^{-1}.
  - Step 3c: We see that $x = 2$ corresponds to $u = 3$ and $x = 3$ corresponds to $u = 8$.
  - Step 3d: Putting it all together gives $\int_3^8 u^{-1}\,du = \ln(u)|_{u=3}^8 = \boxed{\ln 8 - \ln 3}$.
  - Remark: It's possible to do this one without substitution, by using partial fractions to see that $\frac{2x}{x^2-1} = \frac{1}{x+1} + \frac{1}{x-1}$. This gives $I = [\ln(x+1) + \ln(x-1)]\,|_{x=2}^3 = \ln(4) + \ln(2) - \ln(3) = \ln(8) - \ln(3)$ as before.

5.3 Integration by Parts

- The integration by parts formula states $\boxed{\boxed{\int f'\cdot g\,dx = f\cdot g - \int f\cdot g'\,dx}}$. It is just the Product Rule, rearranged and rewritten in terms of integrals using the Fundamental Theorem of Calculus. To perform an integration by parts, all that is required is to pick a function $f(x)$ and a function $g(x)$ such that the product $f'(x)\cdot g(x)$ is equal to the original integrand. Examples will make everything clear.

- Example: Evaluate $\int_0^1 x\,e^x\,dx$.
  - The integrand $x\cdot e^x$ is an obvious product, and so we need to decide which of x and e^x should be f'. Since x gets more complicated if we take its antiderivative (since we'd get $\frac{1}{2}x^2$) we try $f' = e^x$ and $g = x$, to get $f = e^x$ and $g' = 1$. Plugging into the formula gives $\int_0^1 x\,e^x\,dx = x\,e^x|_{x=0}^1 - \int_0^1 1\cdot e^x\,dx = (x\,e^x - e^x)|_{x=0}^1 = \boxed{1}$.

- Example: Evaluate $\int_1^e [\ln(x)]^2\,dx$.

- We can write $[\ln(x)]^2 = \ln(x) \cdot \ln(x)$, but this doesn't help unless we remember the antiderivative of $\ln(x)$. Instead we write the integrand as $1 \cdot [\ln(x)]^2$, so as to take $f' = 1$ and $g = [\ln(x)]^2$. Then we get $f = x$ and $g' = 2 \cdot \ln(x) \cdot \dfrac{1}{x}$ by the Chain Rule. So plugging in will yields

$$\begin{aligned} \int_1^e [\ln(x)]^2 \, dx &= x \cdot \ln(x)^2 \big|_1^e - \int_1^e x \cdot 2\ln(x) \cdot \frac{1}{x} \, dx \\ &= e - \int_1^e 2\ln(x) \\ &= e - \left[2x \cdot \ln x \big|_1^e - \int_1^e 2x \cdot \frac{1}{x} \, dx \right] \quad \text{(IBP again)} \\ &= e - [2e - (2e-2)] \\ &= \boxed{e-2}. \end{aligned}$$

- Example: Evaluate $\int_0^1 (x^2 - 2x + 2)e^{3x} \, dx$.
  - We want g to be the thing which gets simpler when we differentiate. The polynomial $x^2 - 2x + 2$ gets much simpler if we differentiate it, while the exponential e^{3x} stays basically the same. So we should take $g = x^2 - 2x + 2$ and $f' = e^{3x}$, so that $g' = 2x - 2$ and $f = \frac{1}{3}e^{3x}$. Then integrating by parts yields an expression which we can't evaluate directly – we have to integrate by parts again:

$$\begin{aligned} \int_0^1 (x^2 - 2x + 2)e^{3x} \, dx &= (x^2 - 2x + 2) \cdot \frac{1}{3}e^{3x} \big|_0^1 - \int_0^1 (2x-2) \cdot \frac{1}{3}e^{3x} \, dx \quad \text{(IBP once)} \\ &= \left(\frac{2}{3}e^3 - \frac{2}{3} \right) - \int_0^1 (2x-2) \cdot \frac{1}{3}e^{3x} \, dx \\ &= \left(\frac{2}{3}e^3 - \frac{2}{3} \right) - \left[(2x-2) \cdot \frac{1}{9}e^{3x} \big|_0^1 - \int_0^1 2 \cdot \frac{1}{9}e^{3x} \, dx \right] \quad \text{(IBP again)} \\ &= \left(\frac{2}{3}e^3 - \frac{2}{3} \right) - \left[\frac{2}{9} - \left[\frac{2}{27}e^3 - \frac{2}{27} \right] \right] \\ &= \boxed{\frac{20}{27}e^3 - \frac{26}{27}}. \end{aligned}$$

- Example: Evaluate $\int x^3 \sin(x) \, dx$.
  - Here we just need to integrate by parts repeatedly. We get

$$\begin{aligned} \int x^3 \sin(x) \, dx &= -x^3 \cos(x) + \int 3x^2 \cos(x) \, dx \quad \text{(IBP once)} \\ &= -x^3 \cos(x) + \left[3x^2 \sin(x) - \int 6x \sin(x) \, dx \right] \quad \text{(IBP again)} \\ &= -x^3 \cos(x) + 3x^2 \sin(x) - \left[-6x \cos(x) + \int 6\cos(x) \, dx \right] \quad \text{(IBP again)} \\ &= \boxed{-x^3 \cos(x) + 3x^2 \sin(x) + 6x \cos(x) + 6\sin(x) + C} \end{aligned}$$

- Example: Find $\int x^3 \sin(x^2) \, dx$.
  - First we look for a substitution. We try the argument of the sine: $u = x^2$. The differential is $du = 2x \, dx$, and we can rearrange the integral as $\int x^2 \sin(x^2) \cdot \dfrac{1}{2} \cdot (2x \, dx) = \int u \sin(u) \cdot \dfrac{1}{2} \, du$.
  - Now we integrate by parts, to get $-\dfrac{1}{2}u \cos(u) + \int \dfrac{1}{2} \cos(u) \, du = \dfrac{1}{2} [-u \cos(u) + \sin(u)] + C$.
  - Finally substitute back for x to get $\boxed{\dfrac{1}{2} \left[-x^2 \cos(x^2) + \sin(x^2) \right] + C}$.

5.4 Trigonometric Substitution

- Some kinds of integrals require a more clever sort of substitution to evaluate, one that's sort of 'backwards' from the usual way we try to do substitutions: instead of $u = f(x)$ we try $x = f(u)$ for some appropriate (trigonometric) function f. The idea is to use one of the Pythagorean relations (e.g., $\sin^2(x) + \cos^2(x) = 1$) to simplify something more complicated. As always, examples make everything clear.

- Example: Evaluate $\int \sqrt{1-x^2}\, dx$.
  - Traditional substitution – along the lines of $u = 1 - x^2$ – doesn't work like we'd hope.
  - Instead we try $x = \sin(u)$; then $dx = \cos(u)\, du$.
  - So we get $\int \sqrt{1-\sin^2(u)} \cdot \cos(u)\, du = \int \sqrt{\cos^2(u)} \cdot \cos(u)\, du = \int \cos^2(u)\, du$.
  - Remembering $\cos^2(u) = \dfrac{1+\cos(2u)}{2}$ we can evaluate the integral to get $\dfrac{u}{2} + \dfrac{\sin(2u)}{4} + C$.
  - Finally substitute back for $u = \sin^{-1}(x)$ to obtain $\boxed{\dfrac{\sin^{-1}(x)}{2} + \dfrac{\sin(2\sin^{-1}(x))}{4} + C}$. If desired, we can simplify this to the equivalent form $\boxed{\dfrac{\sin^{-1}(x)}{2} + \dfrac{x\sqrt{1-x^2}}{2} + C}$.

- Example: Evaluate $\int \dfrac{1}{(1+x^2)^2}\, dx$.
  - This time we think of the arctangent antiderivative and try $x = \tan(u)$. Then $dx = \sec^2(u)\, du$.
  - We obtain $\int \dfrac{1}{(1+\tan^2(u))^2} \cdot \sec^2(u)\, du = \int \dfrac{1}{\sec^4(u)} \cdot \sec^2(u)\, du = \int \cos^2(u)\, du$.
  - Remembering the identity $\cos^2(u) = \dfrac{1+\cos(2u)}{2}$ we can evaluate the integral to get $\dfrac{u}{2} + \dfrac{\sin(2u)}{4} + C$.
  - Finally substitute back for $u = \tan^{-1}(x)$ to obtain $\boxed{\dfrac{\tan^{-1}(x)}{2} + \dfrac{\sin(2\tan^{-1}(u))}{4} + C}$. If desired, we can simplify this to the equivalent form $\boxed{\dfrac{\tan^{-1}(x)}{2} + \dfrac{x}{2(1+x^2)} + C}$.

- Example: Evaluate $\int \dfrac{2-x}{\sqrt{4-x^2}}\, dx$.
  - We'd like to do the $x = \sin(u)$ substitution again but it doesn't quite work, since then $\sqrt{4-x^2}$ isn't nice.
  - Instead we have to try $x = 2\sin(u)$, since then $\sqrt{4-x^2} = \sqrt{4-4\sin^2(u)} = 2\cos(u)$, and we're much happier.
  - Then we have $dx = 2\cos(u)\, du$, so we get $\int \dfrac{2-2\sin(u)}{2\cos(u)} \cdot 2\cos(u)\, du = \int [2-2\sin(u)]\, du = 2u + 2\cos(u) + C$.
  - Substituting back yields $\boxed{2\sin^{-1}(x) + 2\cos(\sin^{-1}(x)) + C}$, or equivalently, $\boxed{2\sin^{-1}(x) + 2\sqrt{1-x^2} + C}$.

5.5 Partial Fractions

- Generally, it is difficult to integrate rational functions without rearranging them in some way first. (Try finding the antiderivative of $\dfrac{2x}{x^3+x^2+x+1}$ directly.)

- There is a general technique for evaluating such integrals, called partial fraction decomposition (PFD): the idea is to break down rational functions into simpler parts which we know how to integrate. To find partial fraction decompositions, follow these steps:

- Step 1: Factor the denominator.
- Step 2: Find the form of the PFD:
  - For each term $(x+a)^n$ the PFD has terms $\dfrac{C_1}{x+a} + \dfrac{C_2}{(x+a)^2} + \cdots + \dfrac{C_n}{(x+a)^n}$.
  - For each non-factorable term $(x^2+ax+b)^n$, the PFD has terms $\dfrac{C_1 x + D_1}{x^2+ax+b} + \dfrac{C_2 x + D_2}{(x^2+ax+b)^2} + \cdots + \dfrac{C_n x + D_n}{(x^2+ax+b)^n}$.
- Step 3: Solve for the coefficients $C_1, C_2, \dots, C_n$.
  - The best way is to clear all denominators and then substitute 'intelligent' values for x (e.g., the roots of the linear factors).
  - Sometimes plugging in other values of x is necessary, in order to find the coefficients of the higher terms.
  - One can also employ other more clever methods, such as taking derivatives.
- Step 4: Evaluate the integral.
  - Terms of the form $\dfrac{C}{(x+a)^n}$ can be integrated directly using the Power Rule.
  - Terms of the form $\dfrac{Cx+D}{(x^2+ax+b)^n}$ should be separated further as $\dfrac{E(2x+a)}{(x^2+ax+b)^n} + \dfrac{F}{(x^2+ax+b)^n}$. The first term can then be integrated by substituting $u = x^2+ax+b$, and the second term can be integrated by completing the square as $(x+a/2)^2 + (b-a^2/4) = (x+c)^2 + d^2 = d^2\left[\left(\dfrac{x+c}{d}\right)^2 + 1\right]$, and then substituting $\tan(t) = \dfrac{x+c}{d}$.

- As ever, examples make the procedure clear.
- <u>Example</u>: Evaluate $\int \dfrac{1}{x^2+3x}\,dx$.
  - We factor and get $\dfrac{1}{x(x+3)}$ so we want to write $\dfrac{1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}$.
    - Clear denominators so that $1 = A(x+3) + B(x)$.
    - Set $x=0$ and $x=-3$ to see that $1 = 3A$ and $1 = -3B$.
  - Then $\int \dfrac{1}{x^2+3x}\,dx = \int \left[\dfrac{1/3}{x} - \dfrac{1/3}{x+3}\right] dx = \boxed{\dfrac{1}{3}\ln(x) - \dfrac{1}{3}\ln(x+3) + C}$.
- <u>Example</u>: Evaluate $\int \dfrac{1}{x^3+x^2}\,dx$.
  - We factor and get $\dfrac{1}{x^2(x+1)}$ so we want to write $\dfrac{1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{D}{x+1}$.
    - Clear denominators so that $1 = Ax(x+1) + B(x+1) + Dx^2$.
    - Set $x=0$ to get $1 = B$.
    - Set $x=-1$ to get $1 = D$.
    - Set $x=1$ to get $1 = 2A + 2B + D$ so that $A = -1$.
  - Then $\int \dfrac{1}{x^3+x^2}\,dx = \int \left[\dfrac{-1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x+1}\right] dx = \boxed{-\ln(x) - \dfrac{1}{x} + \ln(x+1) + C}$.
- <u>Example</u>: Evaluate $\int \dfrac{2x}{(x+1)(x^2+1)}\,dx$.
  - We want $\dfrac{2x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+D}{x^2+1}$.

* Clear denominators so that $2x = A(x^2+1) + (Bx+D)(x+1)$.
    * Set $x = -1$ to get $-2 = 2A$ so that $A = -1$.
    * So we want $x^2 + 2x + 1 = (Bx+D)(x+1)$ so we see that we need $B = D = 1$.
  - Then $\int \frac{2x}{(x+1)(x^2+1)}\,dx = \int \left[\frac{-1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\right] dx = \boxed{-\ln(x+1) + \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C}$.

5.6 The Weierstrass Substitution

- There is one additional substitution that deserves discussion.
  - It is rarely covered at any length in modern textbooks, but it should be, since it allows one to integrate any rational function in $\sin(x)$ and $\cos(x)$.
- The substitution is $\boxed{\boxed{t = \tan\left(\frac{x}{2}\right)}}$, which is called the Weierstrass substitution, or, occasionally, "the magic substitution", since it allows one to evaluate complicated trigonometric integrals.
  - With this definition, one can check that $\cos(x) = \frac{1-t^2}{1+t^2}$, $\sin(t) = \frac{2t}{1+t^2}$, and $dx = \frac{2}{1+t^2}\,dt$.
  - Then as we can see, the trigonometric integral $\displaystyle\int \frac{p(\sin(x),\cos(x))}{q(\sin(x),\cos(x))}\,dx$ becomes the integral $\displaystyle\int \frac{p\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)}{q\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)} \frac{2}{1+t^2}$ which is a rational function of t (though complicated).
  - The magic substitution can also be used to reduce any problem involving trigonometric identities in rational multiples of a single variable x to a finite computation with rational functions. (In practice, this is not as useful as it might seem, because the polynomial algebra is usually harder than using other techniques like the angle-addition formulas.)
- Example: Find $\int \frac{1}{2+\cos(x)}\,dx$.
  - We set $t = \tan\left(\frac{x}{2}\right)$, with $\cos(x) = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}\,dt$, to obtain

$$\begin{aligned}
\int \frac{1}{2+\cos(x)}\,dx &= \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt \\
&= \int \frac{2}{2(1+t^2) + (1-t^2)}\,dt \\
&= \int \frac{2}{3+t^2}\,dt = \int \frac{2}{3} \cdot \frac{1}{1+(t/\sqrt{3})^2}\,dt
\end{aligned}$$

  - In this new integral we set $u = t/\sqrt{3}$ with $du = dt/\sqrt{3}$ to obtain $\int \frac{2}{3} \cdot \frac{\sqrt{3}}{1+u^2}\,du = \frac{2}{\sqrt{3}}\tan^{-1}(u) + C$.
  - Substituting back for t and then x yields the answer as $\boxed{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{2}\right)\right) + C}$.

5.7 Improper Integration

- Sometimes we like to integrate to infinity, by which we mean, take the limit as a limit of integration becomes arbitrarily large.

- In other words, we write $\int_a^\infty f(x)\,dx$ as shorthand for $\lim_{q\to\infty} \int_a^q f(x)\,dx$.

- Other times, we like to integrate through a 'singularity' of a function – that is, through a point where a function is undefined because it blows up: for example, $x = 0$ for the function $f(x) = 1/x$.

  - Again, we will write $\int_0^1 \frac{1}{x}\,dx$ as shorthand for $\lim_{q\to 0^+} \int_q^1 \frac{1}{x}\,dx$.

- Integrals with either of these two kinds of 'bad behaviors' are called <u>improper integrals</u>. (Perhaps the terminology stems from the fact that trying to evaluate such integrals is not something done in polite company.)

- Typically we will be interested in asking (i) if the integral actually converges to a finite value, and (ii) if it does, what the value is.

- There are roughly two ways to solve problems like this:

  - <u>Method 1</u>: Find the indefinite integral, and then evaluate the limit. If the limit is hard, try rewriting the function to simplify the limit.
  - <u>Method 2</u>: Compare the integrand to another function whose integral is easier to evaluate. We are aided by the following theorem:
    * <u>Theorem (Comparison Test for Integrals)</u>: If $0 < f(x) < g(x)$ for all x in the interval $[a, b]$ where one or both of a and b can be infinite, then if $\int_a^b f(x)\,dx$ diverges, so does $\int_a^b g(x)\,dx$. If $\int_a^b g(x)\,dx$ converges, then so does $\int_a^b f(x)\,dx$.
    * The idea behind the theorem is to say that if $0 < f(x) < g(x)$ then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$, and so if the $f(x)$-integral goes to ∞ then so must the $g(x)$-integral, and conversely if the $g(x)$-integral stays finite then so must the $f(x)$-integral.

- <u>Example</u>: Evaluate $\int_0^\infty e^{-x}\,dx$.

  - We just evaluate to see $\int_0^\infty e^{-x}\,dx = \lim_{q\to\infty} (-e^{-x})\big|_{x=0}^q = \lim_{q\to\infty} \left[-e^{-q} + 1\right] = \boxed{1}$.

- <u>Example</u>: Evaluate $\int_1^\infty \frac{1}{x^2 + x}\,dx$.

  - We do partial fractions to see $\int \frac{1}{x^2 + x}\,dx = \ln(x) - \ln(x + 1) + C$.
  - Now to take the limit as $x \to \infty$ is not so easy, unless we notice that we can rewrite the indefinite integral as $\ln\left(\frac{x}{x+1}\right)$.
  - Then we have $\int_1^\infty \frac{1}{x^2 + x}\,dx = \lim_{q\to\infty} \left[\ln\left(\frac{x}{x+1}\right)\right]\big|_{x=1}^q = \lim_{q\to\infty} \left[\ln\left(\frac{x}{x+1}\right) - \ln(\frac{1}{2})\right] = \boxed{\ln(2)}$.
  - If the problem had only asked about convergence, we could have observed that $x^2 + x > x^2$ for positive x, so $\frac{1}{x^2 + x} < \frac{1}{x^2}$. Then the Comparison Test would have said that, because $\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\big|_{x=1}^\infty = 1$ converges, then so does $\int_1^\infty \frac{1}{x^2 + x}\,dx$. However, the Comparison Test doesn't help us in computing the actual value.

- <u>Example</u>: Determine whether the integral $\int_0^\infty \frac{1}{(x + 1)(x + 2)(x + 3)(x + 4)}\,dx$ converges.

  - Doing partial fractions is too much work (although it does give the correct answer, if the trick in the previous example is used repeatedly).
  - But it's easier to notice that $\frac{1}{x + 4} < \frac{1}{x + 3} < \frac{1}{x + 2} < \frac{1}{x + 1}$.

- Then $\int_0^q \frac{1}{(x+1)(x+2)(x+3)(x+4)}\,dx < \int_0^q \frac{1}{(x+1)^4}\,dx = -\frac{1}{4}(x+1)^{-3}\big|_{x=0}^{q}$. As $q \to \infty$ this remains finite (in fact, it is $\frac{1}{4}$).
- Therefore this integral $\boxed{\text{converges}}$.

• Example: Determine whether the integral $\int_0^1 \frac{1}{x}\,dx$ converges.

- We integrate to get $\int_q^1 \frac{1}{x}\,dx = -\ln(q)$.
- So as $q \to 0^+$ we see that this diverges to $-\infty$.
- Therefore this integral $\boxed{\text{diverges to } -\infty}$.

Well, you're at the end of my handout. Hope it was helpful.