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4 Introduction to Integration

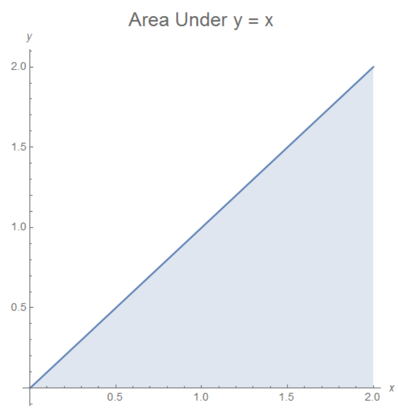
In this chapter, we discuss integration, which is motivated by the problem of calculating the area underneath the graph of a function. We motivate the definition of the definite integral using Riemann sums to calculate areas, and prove the Fundamental Theorem of Calculus, which describes the close relationship between derivatives and integrals. We then introduce indefinite integrals of basic functions and discuss substitution techniques for evaluating definite and indefinite integrals, and close with a discussion of some basic applications of integration to computing areas, arclengths, volumes, and moments and masses.

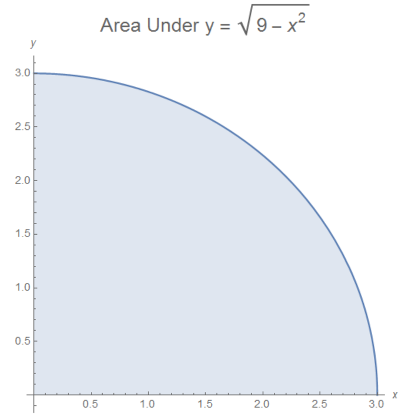
4.1 Definite Integrals and Riemann Sums

- We originally motivated our development of the derivative by asking how to determine the instantaneous rate of change of a function.
- We now pose a new question with a similar flavor: given a continuous, positive function $f(x)$ on an interval $[a, b]$, what is the area of the region that lies under the graph of $y = f(x)$ and above the x-axis, between $x = a$ and $x = b$?

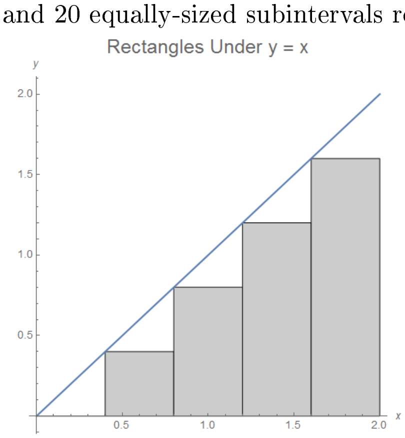
4.1.1 Riemann Sums

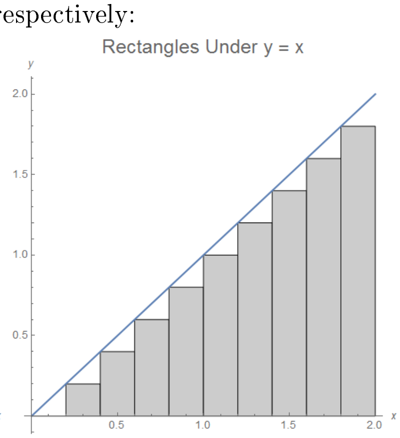
- In some cases we can find the area under a curve using basic geometry.
- Example: Find the area under the graph of $f(x) = x$ between $x = 0$ and $x = 2$.
    - Since $f(x) = x$ is linear and passes through the origin, the area forms a right triangle, with base and height both equal to 2. The area is therefore $\frac{1}{2} \cdot 2 \cdot 2 = \boxed{2}$.
    - Below on the left is a graph of the area in question:

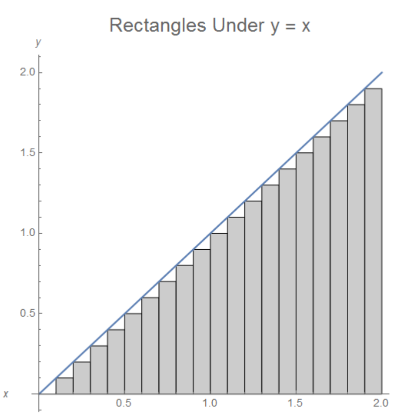




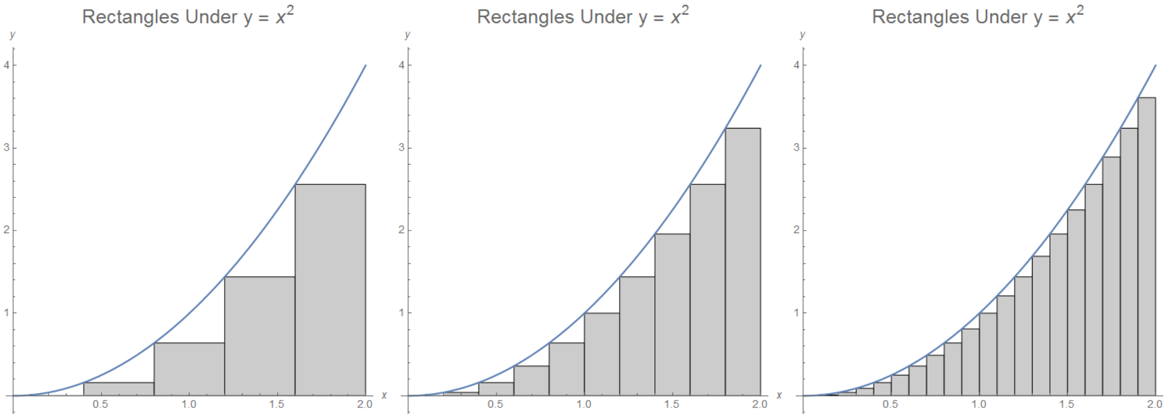
- Example: Find the area under the graph of $g(x) = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$.
    - If we write $y = g(x) = \sqrt{9 - x^2}$ we can see that $x^2 + y^2 = 9$, and so the graph of $y = g(x)$ is the upper half of a circle of radius 3 centered at the origin, as can be easily seen by the graph above on the right.
    - Aided by the picture, we can see that the region is the interior of a quarter-circle of radius 3, so since the area of the circle is 9π, the desired region has area $\boxed{\frac{9\pi}{4}}$.
- However, to evaluate areas more complicated than those which have formulas from basic geometry, we will need a more general method.
    - Here is one possible approach (which was, in fact, essentially first used by Archimedes): divide the interval $[a, b]$ into pieces, and then in each interval draw a rectangle with base on the x-axis with one vertex on the graph of $y = f(x)$. Then add up the areas of all of the small rectangles: this will give an approximation to the area under the graph.
    - As we use more and smaller rectangles, the collective area of the rectangles will approximate the total area under the graph more and more closely.
    - Here are some illustrations of this idea for the function $f(x) = x$ on $[0, 2]$, dividing the interval into 4 and 20 equally-sized subintervals respectively:







- The same procedure will work for any continuous function, such as $f(x) = x^2$:



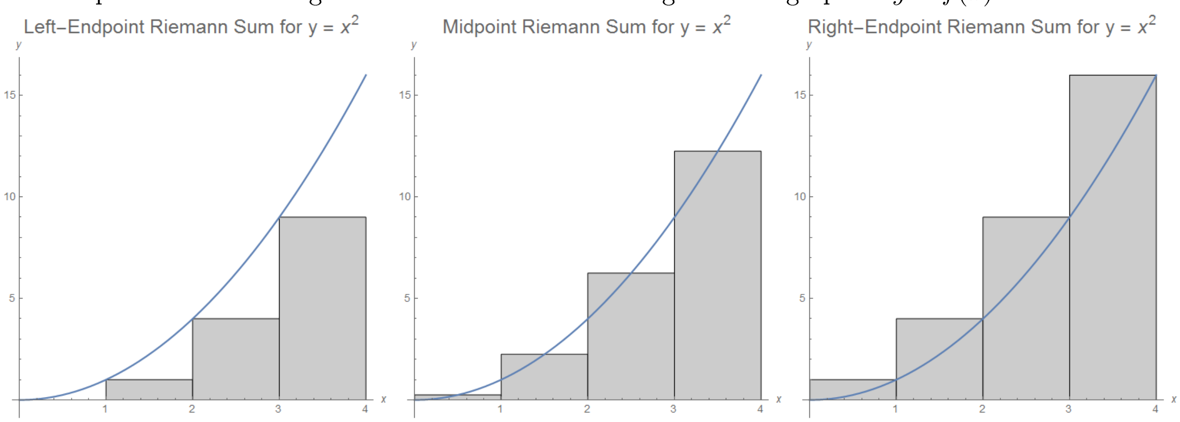
- To formalize these ideas we first need to define some terminology:
- Definition: For an interval $[a, b]$, a partition of $[a, b]$ into n subintervals is a list of x-coordinates $x_0, x_1, x_2, \cdots, x_{n-1}, x_n$ with $x_0 = a$, $x_n = b$, and $x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$. This list of x-coordinates divides $[a, b]$ into the subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. A tagged partition is a partition of $[a, b]$ together with a point x_i^* is a point in the ith interval $[x_{i-1}, x_i]$ for each $1 \le i \le n$.
  - The only partitions we will be interested in are partitions of $[a, b]$ into n equally-sized subintervals. In that case, for $0 \le i \le n$ we have $x_i = a + i \cdot \left[\dfrac{b-a}{n}\right]$.
  - However, in some applications (and also when working in a more formal context) it is useful to use partitions where the intervals have different sizes.
  - Example: The partition of $[0, 8]$ having 4 equal subintervals is $[0, 2]$, $[2, 4]$, $[4, 6]$, $[6, 8]$. If we wish to give a tagged partition, we simply select a point to go along with each interval.
  - In general, we say the norm of the partition P is the width of the largest subinterval.
- Now we can give the formal definition of a Riemann sum, which represents the sum of the areas of the rectangles we described above:
- Definition: Suppose that $f(x)$ is a continuous function and P^* is a tagged partition of the interval $[a, b]$ into n subintervals. If x_i^* is the tagged point in the ith interval $[x_{i-1}, x_i]$, we define the associated Riemann sum of $f(x)$ on $[a, b]$ corresponding to P^* to be $\text{RS}_{P^*}(f) = \displaystyle\sum_{i=1}^{n} f(x_i^*) \cdot [x_i - x_{i-1}]$.
  - Recall that if $g(x)$ is a function, then the notation $\displaystyle\sum_{k=1}^{n} g(k)$ means $g(1) + g(2) + g(3) + \cdots + g(n)$. Thus $\displaystyle\sum_{i=1}^{n} f(x_i^*) \cdot [x_i - x_{i-1}]$ is the sum $f(x_0^*) \cdot [x_1 - x_0] + f(x_1^*) \cdot [x_2 - x_1] + \cdots + f(x_{n-1}^*) \cdot [x_n - x_{n-1}]$.
  - Although this definition is somewhat complicated, it is simply a formalization of what we discussed above: on each interval $[x_{i-1}, x_i]$ in the partition, we draw a rectangle above the interval $[x_{i-1}, x_i]$ whose height is $f(x_i^*)$, so that it lies on the graph of $y = f(x)$. The area of this rectangle is the length of its base $x_i - x_{i-1}$ times its height $f(x_i^*)$. We then add up the areas of all of these rectangles, which is the sum given above.
  - We will primarily be interested in three case: the first case is where $x_i^* = x_{i-1}$ is the left endpoint of its interval which we call the left-endpoint Riemann sum, the second case is where $x_i^* = x_i$ is the right endpoint of its interval which we call the right-endpoint Riemann sum, and the third case is where $x_i^* = (x_{i-1} + x_i)/2$ is the midpoint of its interval which we call the midpoint Riemann sum.
- In the case where P is the partition with n equally-sized subintervals, we can write the Riemann sum as $\displaystyle\sum_{i=1}^{n} f(x_i^*) \, \Delta x$, where $\Delta x = \dfrac{b-a}{n}$ is the common width of the subintervals.

- We can give explicit formulas for the left-endpoint, midpoint, and right-endpoint Riemann sums as well.
- Specifically, we have $\text{RS}_{\text{left}}(f) = \sum_{i=1}^{n} f[a+(i-1)\Delta x]\,\Delta x$, $\text{RS}_{\text{mid}}(f) = \sum_{i=1}^{n} f[a+(i-\frac{1}{2})\Delta x]\,\Delta x$, and $\text{RS}_{\text{right}}(f) = \sum_{i=1}^{n} f[a+i\Delta x]\,\Delta x$.

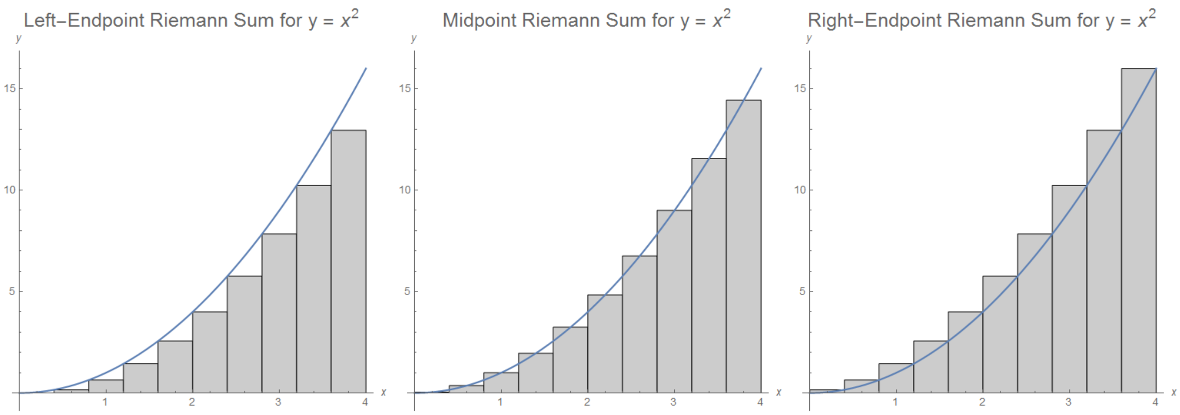
• Here are some examples of Riemann sum computations:

• Example: Find the left-endpoint, midpoint, and right-endpoint Riemann sums for $f(x) = x^2$ on the interval $[0,4]$ with (i) 4 equal subintervals, and (ii) 10 equal subintervals.

- First, we have $a = 0$ and $b = 4$. If there are 4 subintervals, then $\Delta x = \frac{b-a}{4} = 1$, and the subintervals themselves are $[0,1]$, $[1,2]$, $[2,3]$, and $[3,4]$.
- The left-endpoint Riemann sum is then $f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 = 0^2\cdot 1 + 1^2\cdot 1 + 2^2\cdot 1 + 3^2\cdot 1 = \boxed{14}$.
- The midpoint Riemann sum is then $f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 = 0.5^2\cdot 1 + 1.5^2\cdot 1 + 2.5^2\cdot 1 + 3.5^2\cdot 1 = \boxed{21}$.
- The right-endpoint Riemann sum is then $f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 = 1^2\cdot 1 + 2^2\cdot 1 + 3^2\cdot 1 + 4^2\cdot 1 = \boxed{30}$.
- Here are plots of the rectangles for these Riemann sums against the graph of $y = f(x)$:



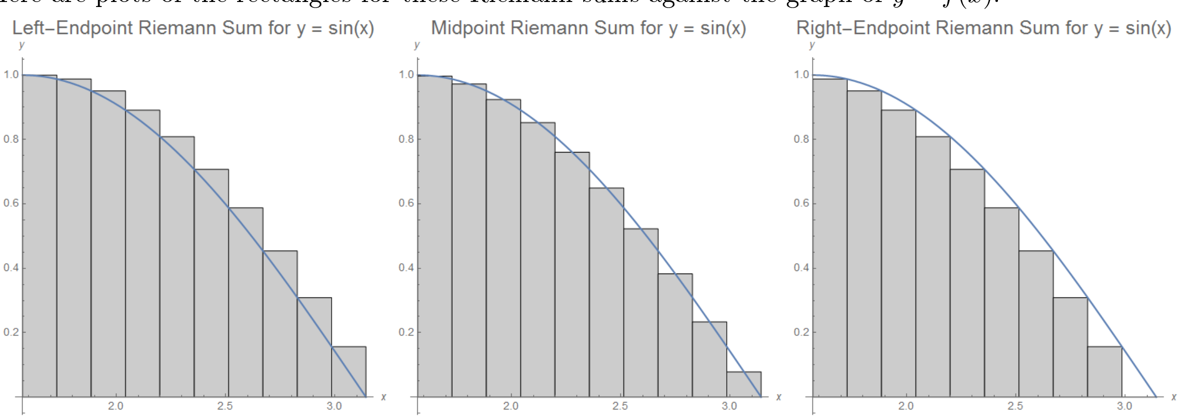
- In a similar way we can compute the Riemann sums with 10 subintervals: in this case $\Delta x = \frac{b-a}{10} = 0.4$, and the subintervals are $[0,0.4]$, $[0.4,0.8]$, $[0.8,1.2]$, ... , $[3.6,4]$.
- The left-endpoint Riemann sum is $\sum_{i=1}^{10} f[0.4\cdot(i-1)]\cdot 0.4 = 0^2\cdot 0.4 + 0.4^2\cdot 0.4 + 0.8^2\cdot 0.4 + \cdots + 3.6^2\cdot 0.4 = \boxed{18.24}$.
- The midpoint Riemann sum is $\sum_{i=1}^{10} f[0.4\cdot(i-1/2)]\cdot 0.4 = 0.2^2\cdot 0.4 + 0.6^2\cdot 0.4 + 1^2\cdot 0.4 + \cdots + 3.8^2\cdot 0.4 = \boxed{21.28}$.
- The right-endpoint Riemann sum is $\sum_{i=1}^{10} f[0.4\cdot i]\cdot 0.4 = 0.4^2\cdot 0.4 + 0.8^2\cdot 0.4 + 1.2^2\cdot 0.4 + \cdots + 4.0^2\cdot 0.4 = \boxed{24.64}$.
- Here are plots of the rectangles for these Riemann sums against the graph of $y = f(x)$:



- We can see from the values that the Riemann sums with 10 rectangles give much better approximations of the actual area under the curve than the Riemann sums with 4 rectangles do.
- We also observe that because the function $f(x) = x^2$ is increasing, all of the left-endpoint rectangles lie below the graph, and thus the left-endpoint Riemann sum is less than the total area under the curve. Likewise, all of the right-endpoint rectangles lie above the graph, and thus the right-endpoint Riemann sum is greater than the total area under the curve.

• <u>Example</u>: Find the left-endpoint, midpoint, and right-endpoint Riemann sums for $f(x) = \sin(x)$ on the interval $[\frac{\pi}{2}, \pi]$ with 10 equal subintervals.

- First, we have $a = \frac{\pi}{2}$ and $b = \pi$. If there are 10 subintervals, then $\Delta x = \dfrac{b-a}{10} = \dfrac{\pi}{20}$, and the subintervals themselves are $[\frac{\pi}{2}, \frac{11\pi}{20}]$, $[\frac{11\pi}{20}, \frac{12\pi}{20}]$, $[\frac{12\pi}{20}, \frac{13\pi}{20}]$, ... , and $[\frac{19\pi}{20}, \pi]$.
- The left-endpoint Riemann sum is $\displaystyle\sum_{i=1}^{10} f[\tfrac{\pi}{2} + \tfrac{\pi}{2}(i-1)] \cdot \tfrac{\pi}{20} = \sin(\tfrac{\pi}{2}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{11\pi}{20}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{12\pi}{20}) \cdot \tfrac{\pi}{20} + \cdots + \sin(\tfrac{19\pi}{20}) \cdot \tfrac{\pi}{20} \approx \boxed{1.076}$.
- The midpoint Riemann sum is $\displaystyle\sum_{i=1}^{10} f[\tfrac{\pi}{2} + \tfrac{\pi}{2}(i-\tfrac{1}{2})] \cdot \tfrac{\pi}{20} = \sin(\tfrac{21\pi}{40}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{23\pi}{40}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{25\pi}{40}) \cdot \tfrac{\pi}{20} + \cdots + \sin(\tfrac{39\pi}{40}) \cdot \tfrac{\pi}{20} \approx \boxed{1.001}$.
- The right-endpoint Riemann sum is $\displaystyle\sum_{i=1}^{10} f[\tfrac{\pi}{2} + \tfrac{\pi}{2}(i)] \cdot \tfrac{\pi}{20} = \sin(\tfrac{11\pi}{20}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{12\pi}{20}) \cdot \tfrac{\pi}{20} + \sin(\tfrac{13\pi}{20}) \cdot \tfrac{\pi}{20} + \cdots + \sin(\pi) \cdot \tfrac{\pi}{20} \approx \boxed{0.919}$.
- Here are plots of the rectangles for these Riemann sums against the graph of $y = f(x)$:



- Based on the values we have computed, it seems like the area under the graph of $y = \sin(x)$ above the x-axis on the interval for $\pi/2 \le x \le \pi$ is equal to 1. In fact, this is true, but in order to establish this fact formally, we must first develop some more results about Riemann sums.

- For suffiently simple functions, we can evaluate certain Riemann sums exactly, for an arbitrary number of equal subintervals.
- Example: Compute the left-endpoint and right-endpoint Riemann sums for $f(x) = x^2$ on the interval $[0, 1]$ with n equal subintervals. By using the behavior as $n \to \infty$ and the fact that f is increasing, show that the region under $y = x^2$ above the x-axis on $[0, 1]$ has area $1/3$.
  - For this interval we have $a = 0$ and $b = 1$, and also $\Delta x = \dfrac{b-a}{n} = \dfrac{1}{n}$. The intervals are $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n}]$, $[\frac{2}{n}, \frac{3}{n}]$, ... , $[\frac{n-1}{n}, 1]$.
  - Then $\text{RS}_{\text{left}}(f) = 0^2 \cdot \dfrac{1}{n} + (\dfrac{1}{n})^2 \cdot \dfrac{1}{n} + (\dfrac{2}{n})^2 \cdot \dfrac{1}{n} + \cdots + (\dfrac{n-1}{n})^2 \cdot \dfrac{1}{n} = \dfrac{0^2 + 1^2 + 2^2 + \cdots + (n-1)^2}{n^3}$.
  - Also, $\text{RS}_{\text{right}}(f) = (\dfrac{1}{n})^2 \cdot \dfrac{1}{n} + (\dfrac{2}{n})^2 \cdot \dfrac{1}{n} + (\dfrac{3}{n})^2 \cdot \dfrac{1}{n} + \cdots + 1^2 \cdot \dfrac{1}{n} = \dfrac{1^2 + 2^2 + \cdots + n^2}{n^3}$.
  - By using the summation formula $1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$, we can evaluate both sums.
  - Using the summation formula, we see that $\text{RS}_{\text{left}}(f) = \dfrac{(n-1)n(2n-1)/6}{n^3} = \dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2}$, while $\text{RS}_{\text{right}}(f) = \dfrac{n(n+1)(2n+1)/6}{n^3} = \dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}$.
  - Now since f is increasing, the left-endpoint Riemann sum is less than the total area (since all of its rectangles lie under the graph) while the right-endpoint Riemann sum is greater than the total area (since all of its rectangles lie above the graph).
  - Hence, if A is the desired area, we see that $\dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2} < A < \dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}$.
  - If we let $n \to \infty$, then since $\lim\limits_{n\to\infty} \left[\dfrac{1}{3} - \dfrac{1}{2n} + \dfrac{1}{6n^2}\right] = \dfrac{1}{3} = \lim\limits_{n\to\infty} \left[\dfrac{1}{3} + \dfrac{1}{2n} + \dfrac{1}{6n^2}\right]$, we must have $A = \dfrac{1}{3}$.
- In principle, we could employ a similar procedure to compute the area under the graph of other continuous functions (at least on intervals where the function is increasing, or where it is decreasing).
  - However, even for a comparatively simple function like $f(x) = x^2$, these explicit calculations are already quite lengthy.
  - Instead, we will take a slightly different approach: we will instead define the integral of a continuous function to be the limit of its Riemann sums over partitions with smaller and smaller rectangles.
  - By definition, the integral will give the value of the area under the graph of $y = f(x)$, at least when $f(x)$ is positive.
  - We will then relate integrals to derivatives, and thereby obtain methods for calculating areas.

4.1.2 The Definite Integral

- We now give a precise definition for the integral of a continuous function f on an interval $[a, b]$, which is a formalization of the area under the graph:
- Definition: The function $f(x)$ is Riemann-integrable on the interval $[a, b]$ if there exists a value L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ) such that for every tagged partition P^* all of whose subintervals have width less than δ, it is true that $|RS_{P^*}(f) - L| < \epsilon$.
  - Essentially, what this definition means is: the function $f(x)$ is integrable if L is the "limiting value" of the Riemann sums of f as the size of the subintervals in the partition becomes small.
  - Like the formal definition of the limit of a function, it takes a great deal of time and effort to become comfortable with this definition[1].

[1] In fact, most modern analytic treatments of integration typically use a slightly different formulation of integrability: instead of using Riemann sums, it is more technically convenient to use what are called "upper" and "lower" sums, which leads to what is called the Darboux integral, rather than the Riemann integral. However, the Darboux integral can be shown to be the same as the Riemann integral (in that the class of functions that can be integrated is the same, and the resulting integrals always have the same value). In treatments of elementary calculus, most authors nevertheless use Riemann sums, since they have an older history.

- Definition: If $f(x)$ is Riemann-integrable on $[a,b]$, we define the definite integral of f on $[a,b]$, denoted $\int_a^b f(x)\,dx$, to be the limiting value L of the Riemann sums for f.
  - Example: Our analysis of $f(x) = x^2$ on $[0,1]$ shows that x^2 is integrable on this interval, and that $\int_0^1 x^2\,dx = \dfrac{1}{3}$.
  - Notation: All of the parts of the definite integral notation are needed when writing an integral. The dx part labels the variable of integration (and behaves exactly as a differential), and $f(x)$ indicates the function being integrated. The values a and b are called the limits of integration, and specify the range $[a,b]$ on which the function is to be integrated.
  - Observe the similarity between the notation $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ for a Riemann sum, and the notation $\int_a^b f(x)\,dx$ for the definite integral. This similarity is deliberate: the idea of Leibniz (who developed the notation) is that "in the limit" of $\Delta x \to 0$, the term Δx becomes the differential dx, and the sum becomes an integral.
- A fundamental result is that every continuous function is Riemann-integrable:
- Theorem (Continuous Functions are Integrable): If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is Riemann-integrable on $[a,b]$.
  - The proof is quite technical (so we will omit the lengthy details), but we can outline the basic idea: first, one shows piecewise-constant functions are integrable. Next, one shows that on sufficiently small intervals, a continuous function can be approximated closely by a piecewise-constant function. By taking a sufficiently fine partition, it then follows that the corresponding Riemann sums for the two functions must also be close together. Finally, by taking an appropriate limit, one may establish that continuous functions are integrable.
- We will mention that there exist discontinuous functions that are also integrable, and also discontinuous functions that are not integrable.
  - An example of a discontinuous function that is integrable on the interval $[0,1]$ is the "step function" $f(x) = \begin{cases} 0 & \text{for } 0 \le x \le 1/2 \\ 1 & \text{for } 1/2 < x \le 1 \end{cases}$. It is not hard to show that for this function $f(x)$, its Riemann integral on $[0,1]$ is $1/2$.
  - An example of a discontinuous function that is not integrable on the interval $[0,1]$ is the function $g(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.
  - For any partition of $[0,1]$, no matter how small the intervals, if we choose all of the tagging points x_i^* to be rational numbers then the corresponding Riemann sum for g is 1, while if we choose all of the tagging points x_i^* to be irrational numbers then the corresponding Riemann sum for g is 0. This means that the Riemann sums do not converge to a limit, and so g is not integrable.
  - Because there exist non-integrable discontinuous functions, we will focus from this point only on continuous functions.
- Note that our geometric motivation for integration involved finding the area under the graph of a function $y = f(x)$, where we implicitly assumed that $f(x) \ge 0$. However, the definition via Riemann sums does not require that $f(x)$ be nonnegative: it makes perfectly good sense for negative-valued functions as well.
  - If we follow the definition through and evaluate Riemann sums for $-f(x)$ where $f(x)$ is positive, we obtain -1 times the result for $+f(x)$.
  - So we can interpret the definite integral of a negative function as giving a negative area: that is, if we interpret the area as being negative if $f(x) < 0$, the definite integral makes sense for all functions.

- As with limits and derivatives, it is much easier to work with integrals after we have proven some basic results on manipulating them. Here are some properties of definite integrals which are more or less immediate consequences of the Riemann sum definition:

- Proposition (Properties of Definite Integrals): Let $a < b < c$ be arbitrary constants, let C be an arbitrary constant, and let $f(x)$ and $g(x)$ be continuous functions. Then the following properties hold:

  1. Integral of constant: $\int_a^b C\,dx = C \cdot (b-a)$.
  2. Integral of constant multiple: $\int_a^b C \cdot f(x)\,dx = C \cdot \int_a^b f(x)\,dx$.
  3. Integral of sum: $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [f(x)+g(x)]\,dx$.
  4. Integral of difference: $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x)-g(x)]\,dx$.
  5. Nonnegativity: If $f(x) \geq 0$, then $\int_a^b f(x)\,dx \geq 0$.
  6. Integral of inequality: If $f(x) \leq g(x)$ for all x in $[a,b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
  7. Union of intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
  8. Backwards interval: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$.

     - Proof: Properties (1)-(4) and (7) follow from algebraic manipulations of Riemann sums: for example, (3) follows by observing that the Riemann sum for $f+g$ is the sum of a Riemann sum for f with a Riemann sum for g.
     - Property (5) follows by observing that any Riemann sum for a nonnegative function is also nonnegative.
     - Property (6) follows by applying property (5) to the nonnegative function $g(x) - f(x) \geq 0$, and then using property (4).
     - The statements in property (8) are actually notational conventions (rather than actual facts to be proven). They are chosen so that property (7) is true for any choice of a, b, c, regardless of order.

- By using these properties in tandem with some of the results we have already found using Riemann sums or geometry, we can evaluate a small number of integrals.

  - Using geometry, we can see that for $a > 0$, we have $\int_0^a x\,dx = \frac{1}{2}a^2$, since the corresponding area is a right triangle with base and height both equal to a.
  - We also showed that $\int_0^1 x^2\,dx = \frac{1}{3}$.
  - So, for example, we can find $\int_0^1 (x^2+2x+3)\,dx = \int_0^1 x^2\,dx + 2\int_0^1 x\,dx + \int_0^1 3\,dx = \frac{1}{3} + 2 \cdot \frac{1}{2} + 3 = \boxed{\frac{13}{3}}$ using the various properties listed above.

- For integrals that we cannot evaluate, we can in some cases give upper and lower bounds using the properties of inequalities.

  - Example: Because $0 \leq \sin(x) \leq 1$ for $0 \leq x \leq \pi$, the integral $\int_0^\pi \sqrt{\sin(x)}\,dx$ is between $\int_0^\pi 0\,dx = 0$ and $\int_0^\pi 1\,dx = \pi$.

4.2 The Fundamental Theorem of Calculus

- We would now like to extend our ability to evaluate integrals directly, without resorting to cumbersome Riemann sum calculations.

  - To do this, we will establish a fundamental relation between differentiation and integration, namely that they are essentially inverse to one another.
  - More explicitly, we will show that integrating the derivative of a continuous function, or differentiating the integral of a continuous function, will (essentially) give back the original function.

4.2.1 Statement and Proof of the Fundamental Theorem of Calculus

- Our starting point is a simple inequality that follows from the observation that the integral of a nonnegative function is always nonnegative:
- Theorem (Min-Max Inequality): If $f(x)$ is a continuous function on $[a,b]$, then $(b-a)\cdot\min_{[a,b]}(f) \le \int_a^b f(t)dt \le (b-a)\cdot\max_{[a,b]}(f)$.
  - The notation $\min_{[a,b]}(f)$ refers to the absolute minimum of f on the interval $[a,b]$, while $\max_{[a,b]}(f)$ refers to the absolute maximum. These values are guaranteed to exist by the Extreme Value Theorem, since f is continuous on a closed interval.
  - Proof: By definition of the minimum and maximum values, for any x in the interval $[a,b]$, it is true that $\min(f)_{[a,b]} \le f(x) \le \max(f)_{[a,b]}$.
  - Now we apply the "integration of an inequality" integral property (6) twice to see that $\int_a^b \min_{[a,b]}(f)dx \le \int_a^b f(x)dx \le \int_a^b \max_{[a,b]}(f)dx$.
  - Then since the first and last integrals are integrals of constants, evaluating them yields $(b-a)\cdot\min_{[a,b]}(f) \le \int_a^b f(t)dt \le (b-a)\cdot\max_{[a,b]}(f)$, as claimed.
- Using this inequality, we can establish a version of the Mean Value Theorem for integrals:
- Theorem (Mean Value Theorem for Integrals): If $f(x)$ is a continuous function on $[a,b]$, then there exists some c in (a,b) for which $f(c) = \dfrac{1}{b-a}\int_a^b f(t)dt$.
  - The value $\dfrac{1}{b-a}\int_a^b f(t)dt$ is called the average value (or mean value) of f on the interval $[a,b]$. Intuitively, the Mean Value Theorem says that there is a point in the interval where the function is equal to its average value.
  - Proof: Dividing through by $(b-a)$ everywhere in the Min-Max inequality gives $\min_{[a,b]}(f) \le \dfrac{1}{b-a}\int_a^b f(t)dt \le \max_{[a,b]}(f)$.
  - Then since f is continuous, it attains its minimum and maximum values, and then by the Intermediate Value Theorem it takes every value in between.
  - But the inequalities above say that the average value $\dfrac{1}{b-a}\int_a^b f(t)dt$ is between the minimum and maximum, and therefore is one of the values attained.
- We can now establish both parts of the Fundamental Theorem of Calculus:
- Theorem (Fundamental Theorem of Calculus, Part 1): For any continuous function f on $[a,b]$, the function $F(x) = \int_a^x f(t)dt$ is continuous, differentiable, and has the property that $F'(x) = f(x)$ on $[a,b]$.
  - In other words, this result says that the function $F(x) = \int_a^x f(t)dt$ is an antiderivative of f on the interval $[a,b]$.
  - Note that the integration variable is t and not x: this is necessary because the limits of integration cannot contain the variable of integration (an expression like $\int_a^x f(x)\,dx$ does not make sense: when interpreted literally, it would say to integrate the function $f(x)$ from $x=a$ to $x=x$). Since we cannot use x for the variable of integration, we replace it with a different variable t instead.
  - Proof: To show that $F'(x) = f(x)$, we look at the difference quotient

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] = \lim_{h\to 0}\frac{1}{h}\left[\int_x^{x+h} f(t)dt\right].$$

  - The quantity inside the limit is the mean value of f on the interval $[x,x+h]$.
  - Applying the Mean Value Theorem for integrals shows that there exists a value c_h in $(x,x+h)$ for which $\dfrac{1}{h}\left[\int_x^{x+h} f(t)dt\right] = f(c_h)$.

- Then $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c_h)$, and this last limit is just $f(x)$ because f is continuous and the points c_h approach x as $h \to 0$, since c_h is in the interval $(x, x+h)$.
- Therefore $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ exists and is equal to $f(x)$, so F is differentiable and $F'(x) = f(x)$. Finally, since F is differentiable, it is continuous.

- From our results on antiderivatives, we know that any two antiderivatives of a function defined on an interval must differ by a constant.
  - Therefore, if we are able to find an antiderivative of the function $f(x)$ somehow, it must differ by a constant from the function $F(x) = \int_a^x f(t)dt$.
  - This key insight allows us to evaluate definite integrals, and is the second part of the Fundamental Theorem of Calculus:

- Theorem (Fundamental Theorem of Calculus, Part 2): If F is any antiderivative of the continuous function f on the interval $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$.
  - Proof: By the first part of the Fundamental Theorem of Calculus, we know that $G(x) = \int_a^x f(t)dt$ is an antiderivative of f, since $G'(x) = f(x)$.
  - But we have also shown that any two antiderivatives of a function on an interval differ by a constant: therefore, $G(x) = F(x) + C$ for some constant C.
  - We also can see easily that $G(a) = \int_a^a f(t)\,dt = 0$ and that $G(b) = \int_a^b f(t)dt$.
  - Hence $\int_a^b f(t)dt = G(b) - G(a) = [F(b) + C] - [F(a) + C] = F(b) - F(a)$, as desired.

4.2.2 Evaluating Definite Integrals

- The content of the second part of the Fundamental Theorem of Calculus is that we can evaluate definite integrals using antiderivatives, instead of using the complicated process of computing limits of Riemann sums.
  - Specifically, if $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)dx = F(x)\,\big|_{x=a}^{b} = F(b) - F(a)$.
  - The notation $F(x)\,\big|_{x=a}^{b}$ means to evaluate the function f "from $x = a$ to b", and is simply shorthand for $F(b) - F(a)$.
  - Thus, in order to compute a definite integral $\int_a^b f(t)dt$, we need only find an antiderivative of f, and then evaluate it at the endpoints a and b and subtract the results.

- Example: Evaluate $\int_0^1 x^2\,dx$ using the Fundamental Theorem of Calculus, and interpret the result as an area.
  - As we can easily see, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.
  - Therefore, by the Fundamental Theorem of Calculus, we have $\int_0^1 x^2\,dx = \int_0^1 f(x)\,dx = F(x)\,\big|_{x=0}^{1} = F(1) - F(0) = \boxed{\frac{1}{3}}$.
  - This evaluation $\int_0^1 x^2\,dx = \frac{1}{3}$ corresponds to the area of the region underneath the graph of $y = x^2$ above the x-axis for $0 \le x \le 1$.
  - Remark: Note how much simpler this calculation was, in comparison to the very lengthy arguments using Riemann sums we needed earlier to compute the area of this region!

- Example: Evaluate $\int_1^{16} \sqrt{x}\,dx$ using the Fundamental Theorem of Calculus, and interpret the result as an area.
  - We wish to find an antiderivative of $f(x) = \sqrt{x} = x^{1/2}$. Since the derivative of $x^{3/2}$ is $\frac{3}{2}x^{1/2}$, we see the derivative of $\frac{2}{3}x^{3/2}$ is $x^{1/2}$, and so we may take $F(x) = \frac{2}{3}x^{3/2}$.

- Then by the Fundamental Theorem of Calculus, we have $\int_1^{16} \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\,\Big|_{x=1}^{16} = \frac{2}{3}16^{3/2} - \frac{2}{3}1^{3/2} = \boxed{42}$.
- This evaluation $\int_1^{16} \sqrt{x}\, dx = 42$ corresponds to the area of the region underneath the graph of $y = \sqrt{x}$ above the x-axis for $1 \le x \le 16$.

- Example: Evaluate $\int_1^e \frac{1}{x}\, dx$ using the Fundamental Theorem of Calculus.
  - Observe that an antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln(x)$.
  - Then by the Fundamental Theorem of Calculus, we have $\int_1^e \frac{1}{x}\, dx = \ln(x)\,\Big|_{x=1}^{e} = \ln(e) - \ln(1) = \boxed{1}$.

- Example: Evaluate $\int_1^2 2^x\, dx$.
  - Since the derivative of 2^x is $2^x \ln(2)$, we see that an antiderivative of $2^x \ln(2)$ is 2^x.
  - Since we want the antiderivative of $f(x) = 2^x$ itself, we can just divide by $\ln(2)$ to see that an antiderivative is $F(x) = \frac{2^x}{\ln(2)}$.
  - Then by the Fundamental Theorem of Calculus, we have $\int_1^2 2^x\, dx = \frac{2^x}{\ln(2)}\,\Big|_{x=1}^{2} = \frac{4}{\ln(2)} - \frac{2}{\ln(2)} = \boxed{\frac{2}{\ln(2)}}$.

- Example: Evaluate $\int_0^{\pi/4} [4\sin(x) - 2\cos(x)]\, dx$.
  - Since an antiderivative of $\sin(x)$ is $-\cos(x)$, and an antiderivative of $\cos(x)$ is $\sin(x)$, we can see that an antiderivative of $f(x) = 4\sin(x) - 2\cos(x)$ is $F(x) = -4\cos(x) - 2\sin(x)$.
  - Then by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\int_0^{\pi/4} [4\sin(x) - 2\cos(x)]\, dx &= [-4\cos(x) - 2\sin(x)]\,\Big|_{x=0}^{\pi/4} \\
&= [-4\cdot\frac{\sqrt{2}}{2} - 2\cdot\frac{\sqrt{2}}{2}] - [-4\cdot 1 - 2\cdot 0] \\
&= \boxed{4 - 3\sqrt{2}}.
\end{aligned}$$

4.2.3 Indefinite Integrals

- In evaluating integrals via the Fundamental Theorem of Calculus, we need to compute general antiderivatives. We refer to such antiderivatives as "indefinite integrals", since they essentially tell us the value of the integral of a function on an unspecified interval:

- Definition: The indefinite integral of $f(x)$ with respect to x, denoted $\int f(x)\, dx$, is the set of all antiderivatives of $f(x)$.
  - By our results on antiderivatives, if f is defined on the interval I and $F(x)$ is one antiderivative of f on I, then any other antiderivative of f is of the form $F(x) + C$ for some arbitrary constant C.
  - We traditionally write $+C$ at the end of an indefinite integral to ensure that the arbitrary constant is not lost.
  - Some examples are $\int x\, dx = \frac{1}{2}x^2 + C$, $\int x^2\, dx = \frac{1}{3}x^3 + C$, and $\int e^x\, dx = e^x + C$.
  - Extremely Important Note: When writing an indefinite integral, the $+C$ must always be included!

- Computing indefinite integrals is in general very difficult: unlike with derivatives, there is no straightforward procedure for computing antiderivatives of arbitrary functions.

- In fact, it is known that there exist elementary functions which have no elementary antiderivative (a function is "elementary" if it can be written in terms of polynomials, radicals, exponentials, logarithms, and trigonometric and inverse trigonometric functions), meaning that there is no "nice" formula for the antiderivative in terms of familiar functions.
- Some examples of simple functions with no elementary antiderivative are e^{x^2}, $\sqrt{1-x^4}$, $\sin(x^2)$, $\dfrac{1}{\ln(x)}$, $\ln(\ln x)$, and $\dfrac{\sin(x)}{x}$.

- From the derivatives we have calculated, we can write a list of simple indefinite integrals:

$$\begin{aligned}
\int x^n\,dx &= \frac{x^{n+1}}{n+1}+C, \quad n \neq -1 \\
\int \frac{1}{x}\,dx &= \ln(x)+C \\
\int a^x\,dx &= \frac{a^x}{\ln(a)}+C, \quad a \neq 1 \\
\int \sin(x)\,dx &= -\cos(x)+C \\
\int \cos(x)\,dx &= \sin(x)+C \\
\int \sec^2(x)\,dx &= \tan(x)+C \\
\int \sec(x)\tan(x)\,dx &= \sec(x)+C \\
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x)+C \\
\int \frac{1}{1+x^2}\,dx &= \tan^{-1}(x)+C \\
\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}(x)+C
\end{aligned}$$

- Remark: For the indefinite integral of $\frac{1}{x}$, there are many sources which write $\int \frac{1}{x}\,dx = \ln|x|+C$, with absolute values. This has the advantage of being defined for negative values of x (which $\ln(x)$ is not), but the downside is that this formula may appear to give finite values for definite integrals that are actually undefined. If one takes the viewpoint of defining logarithms of negative numbers as having non-real values, then the two formulas $\int \frac{1}{x}\,dx = \ln(x)+C$ and $\int \frac{1}{x}\,dx = \ln|x|+C$ actually are equivalent: the minus sign gets absorbed into the constant of integration when x is negative. (Many computer algebra systems declare that $\int \frac{1}{x}\,dx = \ln(x)+C$ for this reason.) We will take the convention of avoiding the absolute values, but by using $\ln(-x)+C$ for definite integrals where x is negative.

- Here are some other antiderivatives of basic functions that may be verified by differentiation:

$$\begin{aligned}
\int \ln(x)\,dx &= x\ln(x)-x+C \\
\int \tan(x)\,dx &= -\ln(\cos(x))+C \\
\int \sec(x)\,dx &= \ln(\sec(x)+\tan(x))+C \\
\int \csc(x)\,dx &= -\ln(\csc(x)+\cot(x))+C \\
\int \cot(x)\,dx &= \ln(\sin(x))+C
\end{aligned}$$

4.2.4 Evaluating Definite and Indefinite Integrals

- By using these basic antiderivatives along with our rules for combining them, we can evaluate a moderately wide array of definite and indefinite integrals:
- Example: Find $\int \left(\sin(x) + x^2\right)\,dx$ and $\int_0^\pi \left(\sin(x) + x^2\right)\,dx$.
  - From the basic integrals we see $\int \left(\sin(x) + x^2\right)\,dx = \boxed{-\cos(x) + \dfrac{1}{3}x^3 + C}$.
  - Then $\int_0^\pi \left(\sin(x) + x^2\right)\,dx = \left(-\cos(x) + \dfrac{1}{3}x^3\right)\Big|_{x=0}^{\pi} = \boxed{2 + \dfrac{1}{3}\pi^3}$.
- Example: Find $\displaystyle\int \frac{x^2 + 2x\sqrt{x} + \sqrt[3]{x}}{x^3}\,dx$.
  - We can distribute the fraction in the integrand, and then integrate each term separately.
  - This yields $\displaystyle\int \frac{x^2 + 2x\sqrt{x} + \sqrt[3]{x}}{x^3}\,dx = \int \left[\frac{1}{x} + x^{-1/2} + x^{-8/3}\right]\,dx = \boxed{\ln(x) + 2x^{1/2} - \frac{3}{5}x^{-5/3} + C}$.
- Example: Find $\displaystyle\int \left[4\cos(x) + 2\sec^2(x)\right]\,dx$ and $\displaystyle\int_0^{\pi/4} \left[4\cos(x) + 2\sec^2(x)\right]\,dx$.
  - From the basic integrals we see $\displaystyle\int \left[4\cos(x) + 2\sec^2(x)\right]\,dx = \boxed{4\sin(x) + 2\tan(x) + C}$.
  - Then $\displaystyle\int_0^{\pi/4} \left[4\cos(x) + 2\sec^2(x)\right]\,dx = \left[4\sin(x) + 2\tan(x)\right]\Big|_{x=0}^{\pi/4} = \boxed{2\sqrt{2} + 2}$.
- Example: Find $\displaystyle\int_0^2 (2^x + 3^x + 4^x)\,dx$.
  - From the basic integrals we see $\displaystyle\int (2^x + 3^x + 4^x)\,dx = \frac{2^x}{\ln(2)} + \frac{3^x}{\ln(3)} + \frac{4^x}{\ln(4)} + C$.
  - Then $\displaystyle\int_0^2 (2^x + 3^x + 4^x)\,dx = \left[\frac{2^x}{\ln(2)} + \frac{3^x}{\ln(3)} + \frac{4^x}{\ln(4)}\right]\Big|_{x=0}^{2} = \boxed{\frac{3}{\ln(2)} + \frac{8}{\ln(3)} + \frac{15}{\ln(4)}}$.
- Example: Find $\displaystyle\int \frac{1 - \cos^2(x) + \sin(2x)}{\sin(x)}\,dx$.
  - We can use trigonometric identities to simplify the integrand, and then integrate each term separately.
  - This yields

$$\begin{aligned}\int \frac{1 - \cos^2(x) + \sin(2x)}{\sin(x)}\,dx &= \int \frac{\sin^2(x) + 2\sin(x)\cos(x)}{\sin(x)}\,dx \\ &= \int \left[\sin(x) + 2\cos(x)\right]dx = \boxed{-\cos(x) + 2\sin(x) + C}.\end{aligned}$$

- Example: Find $\int_{\pi/4}^{\pi/3} \tan(x)\,dx$.
  - From the basic integrals we have $\int \tan(x)\,dx = -\ln(\cos(x)) + C$.
  - Then $\int_{\pi/4}^{\pi/3} \tan(x)\,dx = \left(-\ln(\cos(x))\right)\Big|_{x=\pi/4}^{\pi/3} = \left(-\ln(\cos(\pi/3))\right) - \left(-\ln(\cos(\pi/4))\right) = \boxed{\ln(\sqrt{2}) = \dfrac{1}{2}\ln(2)}$.
- Example: Find $\int_2^8 dx$.
  - Note here that $dx = 1 \cdot dx$, so $\int_2^8 dx$ is really just shorthand for $\int_2^8 1\,dx = x\,\Big|_{x=2}^{8} = \boxed{6}$.

- Example: Find $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$.
  - Since $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$, we see $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(\frac{\sqrt{3}}{2}) - \sin^{-1}(\frac{1}{2}) = \boxed{\frac{\pi}{6}}$.

4.2.5 Differentiating Integrals

- By the first part of the Fundamental Theorem of Calculus, we can compute derivatives of integrals. In particular, by using integration, we can construct new functions.
- Example: The function $\operatorname{erf}(x)$ is defined via $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$. Find the derivative of $\operatorname{erf}(x)$.
  - To find the derivative, we can just use the Fundamental Theorem of Calculus: we have $\operatorname{erf}'(x) = \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt\right] = \boxed{\frac{2}{\sqrt{\pi}} e^{-x^2}}$ directly from the first part of the Fundamental Theorem.
  - Remark: This function is called the "error function" and shows up very often in statistics due to its close connection to the "normal distribution". It can be proven (though it is hard!) that there is no way to write the error function $\operatorname{erf}(x)$ in terms of the elementary functions (i.e., as a sum, product, or composition of any number of polynomials, exponentials, logarithms, and trigonometric or inverse trigonometric functions of x). Thus, the description above as an integral is, in some sense, the "simplest way" of describing the error function. So we have actually constructed a new function that we could not have described without using integration.
  - From our calculation of the derivative, we can see that this function is always increasing (since $\operatorname{erf}'(x) > 0$ for all x). From the second derivative $\operatorname{erf}''(x) = -\frac{4x}{\sqrt{\pi}} e^{-x^2}$ we can see that erf is concave up for negative x and concave down for positive x.
- Example: If $g(x) = \int_x^{x^2} \frac{\sin(t)}{t}\,dt$, find $g'(x)$.
  - The idea here is to rearrange $g(x)$ into simpler pieces to which we can apply the Fundamental Theorem of Calculus.
  - Specifically, the Fundamental Theorem tells us how to find the derivative of the function $h(x)$ defined by $h(x) = \int_0^x \frac{\sin(t)}{t}\,dt$: we have $h'(x) = \frac{\sin(x)}{x}$.
  - Now, by integration properties, we can write $g(x) = \int_0^{x^2} \frac{\sin(t)}{t}\,dt - \int_0^x \frac{\sin(t)}{t}\,dt = h(x^2) - h(x)$.
  - Now we can compute $g'(x)$ using these observations along with the Chain Rule. We obtain $g'(x) = \frac{d}{dx}\left[h(x^2) - h(x)\right] = 2x\,h'(x^2) - h'(x) = 2x \cdot \frac{\sin(x^2)}{x^2} - \frac{\sin(x)}{x} = \boxed{\frac{2\sin(x^2) - \sin(x)}{x}}$.
- Example: If $J(x) = \int_{-x}^{x^2} \ln(1+e^t)\,dt$, find $J'(x)$.
  - By the Fundamental Theorem of Calculus, for $F(x) = \int_0^x \ln(1+e^t)\,dt$, we have $F'(x) = \ln(1+e^x)$.
  - By integration properties, we have $J(x) = \int_0^{x^2} \ln(1+e^t)\,dt - \int_0^{-x} \ln(1+e^t)\,dt = F(x^2) - F(-x)$.
  - Then, by the Chain Rule, we get $J'(x) = F'(x^2) \cdot 2x - F(-x) \cdot (-1) = \boxed{2x \cdot \ln(1+e^{x^2}) + \ln(1+e^{-x})}$.

4.3 Substitution

- So far, we have discussed how to evaluate definite integrals using the Fundamental Theorem of Calculus.
  - However, evaluating integrals in this way requires being able to find an antiderivative of the function being integrated, which can often be difficult.
  - All of the differentiation formulas can be recast as (indefinite) integral formulas: we now examine how to rearrange the Chain Rule to compute integrals.

4.3.1 Overview of Substitution

- Since $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$, the Fundamental Theorem of Calculus says that $\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$.
  - Therefore, if we can write the integrand in the form $f'(g(x)) \cdot g'(x)$, we can evaluate the corresponding (in)definite integral: this is the heart of substitution.
- Example: Evaluate $\int \cos(e^x) \cdot e^x\,dx$.
  - Notice that if we take $f(x) = \sin(x)$ and $g(x) = e^x$, then $f'(x) = \cos(x)$ and $g'(x) = e^x$.
  - Thus, $\cos(e^x) \cdot e^x = f'(g(x)) \cdot g'(x)$.
  - So by the Chain Rule and the Fundamental Theorem of Calculus, we see that $\int \cos(e^x) \cdot e^x\,dx = \boxed{\sin(e^x) + C}$.
- Example: Evaluate $\int_0^1 (x^2+1)^4 \cdot 2x\,dx$.
  - For this integral, if we take $g(x) = x^2 + 1$, then $g'(x) = 2x$.
  - Then the integrand $(x^2+1)^4 \cdot 2x$ is equal to $f'(g(x)) \cdot g'(x)$, where $f'(x) = x^4$.
  - Hence if we take $f(x) = x^5/5$, we see that $f'(g(x)) \cdot g'(x) = (x^2+1)^4 \cdot 2x$.
  - So by the Chain Rule and the Fundamental Theorem of Calculus, we see that $\int (x^2+1)^4 \cdot 2x\,dx = \frac{1}{5}(x^2+1)^5 + C$.
  - Then the definite integral is $\int_0^1 (x^2+1)^4 \cdot 2x\,dx = \frac{1}{5}(x^2+1)^5 \big|_{x=0}^{1} = \boxed{\frac{31}{5}}$.
- Example: Evaluate $\int (x^3+5)^6 \cdot x^2\,dx$.
  - For this integral, if we take $g(x) = x^3 + 5$, then $g'(x) = 3x^2$.
  - Then the integrand $(x^3+5)^2 \cdot x^2$ is equal to $f'(g(x)) \cdot g'(x)$, where $f'(x) = \frac{1}{3}x^6$.
  - Hence if we take $f(x) = \frac{1}{21}x^7$, we see that $f'(g(x)) \cdot g'(x) = (x^3+5)^6 \cdot x^2$.
  - So by the Chain Rule and the Fundamental Theorem of Calculus, we see that $\int (x^3+5)^6 \cdot x^2\,dx = \boxed{\frac{1}{21}(x^3+5)^7 + C}$.
- Example: Evaluate $\int \frac{1}{e^x + e^{-x}}\,dx$.
  - For this integral, first observe that $\frac{1}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{(e^x)^2 + 1}$.
  - Therefore, if we write $g(x) = e^x$ and $f'(x) = \tan^{-1}(x)$, we have $f'(g(x)) \cdot g'(x) = \frac{1}{(e^x)^2+1} \cdot e^x$.

- So by the Chain Rule and the Fundamental Theorem of Calculus, we have $\int \frac{1}{e^x + e^{-x}}\, dx = \boxed{\tan^{-1}(e^x) + C}$.

- In many cases, it is not so obvious how exactly to write an integrand in the form $f'(g(x)) \cdot g'(x)$.
  - As the last two examples above show, it can be necessary to manipulate the integrand before it becomes clear how to find the functions $f(x)$ and $g(x)$.
  - We may simplify some of this procedure by using "substitution": writing $u = g(x)$, and then rewriting the integrand in terms of u.
  - The differential $du = g'(x)\, dx$ will then automatically keep track of the term $g'(x)$, and once we rewrite the integral, we may evaluate it as an integral in u, and then substitute back at the end to obtain a function of x.

- <u>Example</u>: Evaluate $\int x^2 e^{x^3}\, dx$.
  - If we substitute $u = x^3$, then we have $du = 3x^2\, dx$.
  - Then $\int x^2 e^{x^3}\, dx = \int e^{x^3} \cdot x^2\, dx = \int e^u \cdot \frac{1}{3} du = \frac{1}{3} e^u + C = \boxed{\frac{1}{3} e^{x^3} + C}$.

- <u>Example</u>: Evaluate $\int \frac{1}{x(1 + \ln x)}\, dx$.
  - If we substitute $u = 1 + \ln x$, then we have $du = \frac{1}{x}\, dx$.
  - Then $\int \frac{1}{x(1 + \ln x)}\, dx = \int \frac{1}{1 + \ln x} \cdot \frac{1}{x}\, dx = \int \frac{1}{u} du = \ln u + C = \boxed{\ln(1 + \ln x) + C}$.

- <u>Example</u>: Evaluate $\int_0^1 \frac{x^5}{(x^6 + 5)^2}\, dx$.
  - We evaluate the indefinite integral using substitution.
  - If we substitute $u = x^6 + 5$, then we have $du = 6x^5\, dx$.
  - Thus, $\int \frac{x^5}{(x^6 + 5)^2}\, dx = \int \frac{1/6}{(x^6 + 5)^2} \cdot 6x^5 dx = \int \frac{1/6}{u^2}\, du = -\frac{1}{6} u^{-1} + C = -\frac{1}{x^6 + 5} + C$.
  - Then the definite integral is $\int_0^1 \frac{x^5}{(x^6 + 5)^2}\, dx = -\frac{1}{x^6 + 5} \Big|_{x=0}^{1} = -\frac{1}{6} - (-\frac{1}{5}) = \boxed{\frac{1}{30}}$.

- We may also evaluate definite integrals directly using substitution.
  - However, when we do this, we must also transform the limits of integration: specifically, if the original limits are $x = a$ and $x = b$, then the new ones will be $u = g(a)$ and $u = g(b)$.
  - To emphasize the change of variables, we sometimes will write the variable explicitly in the limits of integration.
  - For example, to evaluate $\int_0^1 \frac{x^5}{(x^6 + 5)^2}\, dx$ in the example above with $u = x^6 + 5$, we see that $x = 0$ corresponds to $u = 5$ and $x = 1$ corresponds to $u = 6$.
  - Then $\int_{x=0}^{x=1} \frac{1/6}{(x^6 + 5)^2} \cdot 6x^5 dx = \int_{u=5}^{u=6} \frac{1}{6} u^{-2}\, du = -\frac{1}{6} u^{-1} \Big|_{u=5}^{6} = -\frac{1}{6} - (-\frac{1}{5}) = \boxed{\frac{1}{30}}$.

- It is not always obvious what substitution will help in evaluating an integral (or even whether a substitution will work at all).
  - In general, when trying to decide what substitution to make, one should try to assess whether the integrand is a composite function in any sensible way.

- In particular, if there is an obvious function composition $f(g(x))$ in the integrand, a good initial guess for is to take $u = g(x)$ to be the "inner" function in the composition. If there is a fraction in the integrand, substituting for a term in the denominator is often good.
- When testing whether $u = g(x)$ will be suitable, it is a good sign if the term $g'(x)$ is present in some form in the integrand. If it is not, then it must be possible to incorporate it into the substitution somehow: if doing so would introduce unwieldy terms, the function $g(x)$ may not be the best choice.

4.3.2 Examples of Substitution

- To evaluate an integral using substitution, follow this procedure:
  - Step 1: Choose a substitution $u = g(x)$.
  - Step 2: Compute the differential $du = g'(x)\,dx$.
  - Step 3: Rewrite the original integral in terms of u.
  - The most straightforward approach is to rewrite the integral to peel off the terms that will become the new differential du (including any necessary constant factors), and then write the remaining portion of the integrand in terms of u.
  - If the integral is a definite integral, it is also necessary to transform the limits of integration to the new variable: if the old limits are $x = a$ and $x = b$, then the new ones will be $u = g(a)$ and $u = g(b)$.
  - Step 4: Evaluate the new integral. If the integral is indefinite, substitute back in for the original variable x.
- Here are some more examples of substitutions:
- Example: Evaluate $\int_0^3 2x\, e^{x^2}\, dx$.
  - The exponential has a "complicated" argument x^2, so we try setting $u = x^2$.
  - Since the differential is $du = 2x\,dx$, we rearrange the integral as $\int_0^3 e^{x^2} \cdot (2x\,dx)$.
  - Then the "remaining portion" of the integrand is e^{x^2}, which is just e^u.
  - Also, we see that $x = 0$ corresponds to $u = 0^2$ and $x = 3$ corresponds to $u = 3^2$.
  - Putting all of this together gives $\int_0^3 2x\, e^{x^2}\, dx = \int_0^3 e^{x^2} \cdot (2x\,dx) = \int_0^9 e^u\, du = e^u \,\big|_{u=0}^{9} = \boxed{e^9 - 1}$.
- Example: Evaluate $\int_1^e \dfrac{(\ln(x))^2}{x}\, dx$.
  - It might not look like any function has a complicated argument, but notice that the numerator is what we get if we plug in $\ln(x)$ to the squaring function. So we try setting $u = \ln(x)$.
  - Since the differential is $du = \dfrac{1}{x}\, dx$, we rearrange the integral as $\int_0^e [\ln(x)]^2 \cdot \dfrac{1}{x}\, dx$.
  - Then the "remaining portion" of the integrand is $[\ln(x)]^2$, which is just u^2.
  - Also, we see that $x = 1$ corresponds to $u = \ln(1) = 0$ and $x = e$ corresponds to $u = \ln(e) = 1$.
  - Putting all of this together gives $\int_1^e \dfrac{(\ln(x))^2}{x}\, dx = \int_0^e [\ln(x)]^2 \cdot \dfrac{1}{x}\, dx = \int_0^1 u^2\, du = \dfrac{1}{3}u^3 \,\big|_{u=0}^{1} = \boxed{\dfrac{1}{3}}$.
- Example: Evaluate $\int x\sqrt{3x^2+1}\, dx$.
  - Since the square root function has the "complicated" argument $3x^2 + 1$, we try $u = 3x^2 + 1$.
  - The differential is $du = 6x\,dx$, and we can rearrange the integral as $\int \sqrt{3x^2+1} \cdot \dfrac{1}{6} \cdot 6x\,dx$. (Note that we introduced a factor $\dfrac{1}{6} \cdot 6$ in order to put the factor of 6 with the differential.)
  - We obtain $\int x\sqrt{3x^2+1}\, dx = \int \dfrac{1}{6} u^{1/2}\, du = \dfrac{1}{6} \cdot \dfrac{2}{3} u^{3/2} + C = \boxed{\dfrac{1}{9}(3x^2+1)^{3/2} + C}$.

- Example: Evaluate $\int_0^{\pi/16} \sec^2(4x)\,dx$.
  - Since secant has the "complicated" argument $4x$, we try $u = 4x$.
  - Then $du = 4\,dx$, and $x = 0$ corresponds to $u = 0$ and $x = \pi/16$ corresponds to $u = \pi/4$.
  - Hence $\int_0^{\pi/16} \sec^2(4x)\,dx = \int_0^{\pi/16} \sec^2(4x) \cdot \frac{1}{4} \cdot 4\,dx = \int_0^{\pi/4} \sec^2 u \cdot \frac{1}{4}\,du = \frac{1}{4}\tan(u)\Big|_{u=0}^{\pi/4} = \boxed{\frac{1}{4}}$.
- In some cases, it is not clear what substitution to try, and in other cases, the most obvious substitution does not work. In such cases we can try transforming or manipulating the integrand to see if other natural options will become visible.
- Example: Evaluate $\int_0^2 \sqrt{x^7 + x^4}\,dx$.
  - A natural first choice is to try setting $u = x^7 + x^4$, with $du = (7x^6 + 4x^3)\,dx$.
  - However, if we use this choice, there is no corresponding term that we can introduce that will take care of the differential, since there is no term $7x^6 + 4x^3$ in the integrand, and there is no nice way to write it in terms of u.
  - Searching for alternatives, we can try simplifying the square root by observing that $\sqrt{x^7 + x^4} = x^2\sqrt{x^3 + 1}$ for $x \geq 0$.
  - When we write the integrand in this way, it suggests a natural choice $u = x^3 + 1$ with $du = 3x^2\,dx$. This choice is much better because there is also a corresponding term x^2 in the integrand to go with the differential.
  - With $u = x^3 + 1$, we see $x = 0$ corresponds to $u = 1$ and $x = 2$ corresponds to $u = 9$.
  - Then $\int_0^2 \sqrt{x^7 + x^4}\,dx = \int_0^2 \sqrt{x^3 + 1} \cdot \frac{1}{3} \cdot 3x^2\,dx = \int_1^9 u^{1/2} \cdot \frac{1}{3}\,du = \frac{2}{9}u^{3/2}\,\Big|_{u=2}^{9} = \boxed{\frac{52}{9}}$.
- Example: Evaluate $\int \frac{1}{x^2 + a^2}\,dx$, where a is a positive constant.
  - Notice that this integrand looks similar to $\frac{1}{x^2 + 1}$, which is the derivative of $\tan^{-1}(x)$.
  - We can factor out an a^2 from the denominator to see that $\frac{1}{x^2 + a^2} = \frac{1}{(x/a)^2 + 1} \cdot \frac{1}{a^2}$.
  - Now we substitute $u = x/a$, with $du = dx/a$: this yields

$$\int \frac{1}{x^2 + a^2}\,dx = \int \frac{1}{(x/a)^2 + 1} \cdot \frac{1}{a} \cdot \frac{dx}{a} = \int \frac{1}{u^2 + 1} \cdot \frac{1}{a}\,du = \frac{1}{a}\tan^{-1}(u) + C = \boxed{\frac{1}{a}\tan^{-1}(\frac{x}{a}) + C}.$$

- Example: Evaluate $\int \sin^2 x\,dx$.
  - In this case, a natural first guess would be to try $u = \sin x$. But then $du = \cos x\,dx$, and this is an issue because there is no $\cos x$ term in the integrand.
  - Instead, a better approach is to use the double-angle identity $\sin^2 x = \frac{1 - \cos 2x}{2}$: then we can make a substitution $u = 2x$, with $du = 2\,dx$, to integrate the $\cos 2x$ term.
  - This yields $\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx = \int \frac{1 - \cos u}{2} \cdot \frac{1}{2}\,du = \frac{u}{4} - \frac{\sin u}{4} + C = \boxed{\frac{x}{2} - \frac{\sin 2x}{4} + C}$.
  - Remark: In a similar way, we can evaluate $\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$. Alternatively, we could derive this formula from the one above by integrating the identity $\sin^2 x + \cos^2 x = 1$.

4.3.3 Other Substitution Methods

- It is also possible to perform substitution "in reverse": instead of using a substitution of the form $u = g(x)$, we can instead write $x = g(u)$. (Both of these are allowable, since we are simply interchanging the roles of the variables.)
  - When we perform a substitution as $x = g(u)$, with $dx = g'(u)\,du$, it is much easier to write down the new u-integral because we do not need to manipulate the integrand; rather, we can simply plug in for x and for dx.
- Example: Evaluate $\int \frac{x}{\sqrt[3]{x+5}}\,dx$.
  - We try a substitution $u = \sqrt[3]{x+5}$. Instead of computing du as a function of x, which would then require us to do substantial manipulation of the function to isolate the du terms, we instead observe that $u = \sqrt[3]{x+5}$ is the same as $x = u^3 - 5$.
  - Then we have $dx = 3u^2\,du$, and can now plug in immediately to see that

$$\int \frac{x}{\sqrt[3]{x+5}}\,dx \quad = \quad \int \frac{u^3-5}{u}\cdot 3u^2 du = \int (3u^4 - 15u)\,du \frac{3}{4}u^4 - \frac{15}{2}u^2 + C = \boxed{\frac{3}{4}(x+5)^{4/3} - \frac{15}{2}(x+5)^{2/3} + C}.$$

- Example: Evaluate $\int \frac{1}{x+2\sqrt{x}+1}\,dx$.
  - Notice that the denominator factors as $(\sqrt{x}+1)^2$, so we will try the substitution $u = \sqrt{x}+1$. Again, we can save some effort by writing this as $x = (u-1)^2$ so that $dx = 2(u-1)\,du$.
  - We plug in to obtain

$$\int \frac{1}{(\sqrt{x}+1)^2}\,dx = \int \frac{1}{u^2}\cdot 2(u-1)du = \int (2u^{-2} - 2u^{-3})\,du = -2u^{-1} + u^{-2} + C = \boxed{-2(\sqrt{x}+1)^{-1} + (\sqrt{x}+1)^{-2} + C}.$$

- Example: Evaluate $\int \sqrt{1-x^2}\,dx$.
  - The obvious substitution $u = 1 - x^2$ does not work for this integral, since there is no corresponding $du = 2x\,dx$ term.
  - Instead, we can try to look for a "reverse" substitution in which the expression $\sqrt{1-x^2}$ will simplify.
  - Some thought will eventually suggest that the function $x = \sin(u)$ might work, since then $\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$.
  - So we try the substitution $x = \sin(u)$, so that $dx = \cos(u)\,du$.
  - This yields $\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(u)}\cdot\cos(u)\,du = \int \cos^2(u)\,du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$, using the calculation of the integral of $\cos^2 u$ we did earlier.
  - Substituting back $u = \sin^{-1}(x)$ yields $\int \sqrt{1-x^2}\,dx = \boxed{\frac{\sin^{-1}(x)}{2} + \frac{\sin(2\sin^{-1}(x))}{4} + C}$.
  - Remark: If desired, we could simplify this to the equivalent form $\frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C$ by noting that $\sin(2\sin^{-1}(x)) = x\sqrt{1-x^2}$.
- Example: Evaluate $\int \frac{1}{(1+x^2)^{3/2}}\,dx$.
  - As above, the obvious substitution $u = 1 + x^2$ does not work for this integral. If we search for "reverse" substitutions in which the expression $(1+x^2)^{3/2}$ will simplify, some thought will eventually suggest that the function $x = \tan(u)$ might work, since then $\sqrt{1+x^2} = \sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$, and so the denominator will simply be $\sec^3 x$.

- If we substitute $x = \tan(u)$, with $dx = \sec^2(u)\,du$, we obtain

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{\sec^3(u)} \cdot \sec^2(u)\,du = \int \cos(u)\,du = \sin(u) + C = \boxed{\sin(\tan^{-1} u) + C}.$$

- If desired, we could simplify this to the equivalent form $\dfrac{x}{\sqrt{1+x^2}} + C$.

- Remark: The approach given in the two examples above is sometimes referred to as trigonometric substitution.
  - In general, integrals with terms $\sqrt{a^2 - x^2}$ can often be evaluated by substituting $x = a\sin(u)$ since then $\sqrt{a^2 - x^2} = a\cos(u)$.
  - Likewise, integrals with terms $\sqrt{x^2 + a^2}$ can often be evaluated by substituting $x = a\tan(u)$ since then $\sqrt{x^2 + a^2} = a\sec(u)$.
  - Also, integrals with terms $\sqrt{x^2 - a^2}$ can often be evaluated by substituting $x = a\sec(u)$ since then $\sqrt{x^2 - a^2} = a\tan(u)$.

- In some cases, when we perform an initial substitution, the new integral still requires additional substitution and manipulation to evaluate. In such cases we continue simplifying until we obtain an integral we can evaluate:

- Example: Evaluate $\int \dfrac{\sin(\sqrt{x})}{\sqrt{x}\cos(\sqrt{x})}\,dx$.
  - Since the sine and cosine both have argument $\sqrt{x}$, we first try substituting $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$.
  - This yields $\int \dfrac{\sin(\sqrt{x})}{\sqrt{x\cos(\sqrt{x})}}\,dx = \int \dfrac{\sin(u)}{\sqrt{u^2\cos(u)}} \cdot 2u\,du = \int \dfrac{\sin(u)}{u\sqrt{\cos(u)}} \cdot 2u\,du = \int \dfrac{\sin(u)}{2\sqrt{\cos(u)}}\,du$.
  - Then, since the denominator has the term $\sqrt{\cos(u)}$ in it, we try setting $w = \cos(u)$ with $dw = -\sin(u)\,du$.
  - We obtain $\int \dfrac{\sin(u)}{2\sqrt{\cos(u)}}\,du = \int \dfrac{-dw}{2\sqrt{w}} = \int -\dfrac{1}{2}w^{-1/2}\,dw = -w^{1/2}+C = -\sqrt{\cos(u)}+C = \boxed{-\sqrt{\cos(\sqrt{x})} + C}$.

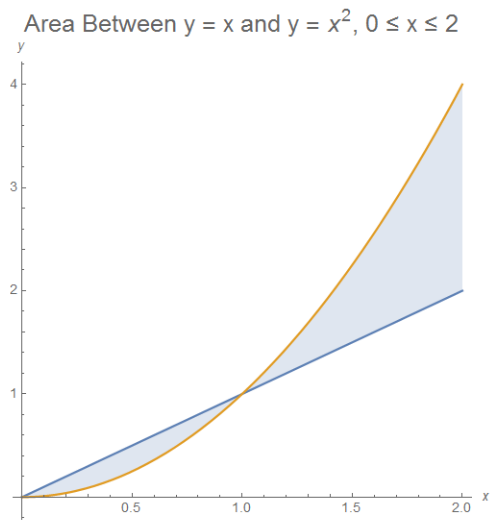
- Example: Evaluate $\int_0^{\ln 2} \sqrt{e^x - 1}\,dx$.
  - Since the integrand is $\sqrt{e^x - 1}$ we try substituting the complicated argument $u = e^x - 1$, so that $e^x = u + 1$.
  - Then $du = e^x\,dx$, and since there is no e^x term already present in the integral, we must create one. With $u = e^x - 1$, if $x = 0$ then $u = 0$ and if $x = \ln 2$ then $u = 1$.
  - Then $\int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \int_0^{\ln 2} \sqrt{e^x - 1} \cdot \dfrac{1}{e^x} \cdot e^x\,dx = \int_0^1 \sqrt{u} \cdot \dfrac{1}{u+1}\,du = \int_0^1 \dfrac{\sqrt{u}}{u+1}\,du$.
  - It is not clear how to evaluate this new integral. Since the difficulty is the term $\sqrt{u}$ in the numerator, we can try substituting for it by setting $w = \sqrt{u}$, so that $w^2 = u$.
  - Then the differential is given by $2w\,dw = du$, and $u = 0$ corresponds to $w = 0$ and $u = 1$ corresponds to $w = 1$. We get

$$\begin{aligned} \int_0^1 \frac{\sqrt{u}}{u+1}\,du &= \int_0^1 \frac{w}{w^2+1} \cdot 2w\,dw = \int_0^1 2\frac{w^2}{w^2+1}\,dw = \int_0^1 2\left[1 - \frac{1}{w^2+1}\right]\,dw \\ &= 2w - 2\tan^{-1} w\,\Big|_{w=0}^{1} = \boxed{2 - \frac{\pi}{2}}. \end{aligned}$$

  - Remark: Notice that we have $w = \sqrt{u} = \sqrt{e^x - 1}$, so we could have made the direct substitution $u = \sqrt{e^x - 1}$ at the beginning without using the intermediate variable u. However, it is quite a bit harder to see that this substitution would succeed at the outset!

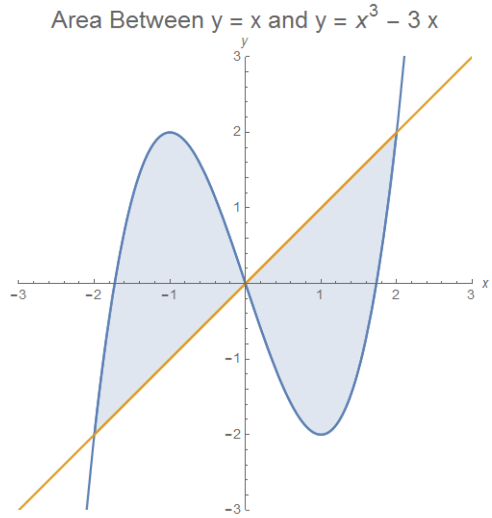
4.4 Areas

- Using integrals, we can compute areas under curves: if $f(x)$ is nonnegative, then the area under $y = f(x)$ and above the x-axis between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.
- Example: Find the area underneath $y = x$ and above the x-axis, between $x = 0$ and $x = 2$.
  - As an integral, this area is $\int_0^2 x\,dx = \frac{x^2}{2}\Big|_{x=0}^{2} = \frac{2^2}{2} - \frac{0^2}{2} = \boxed{2}$.
- Example: Find the area underneath $y = 2e^x + \sin(x)$ and above the x-axis, between $x = 0$ and $x = 2\pi$.
  - As an integral, this area is given by $\int_0^{2\pi} [2e^x + \sin(x)]\,dx$.
  - As an indefinite integral, we have $\int [2e^x + \sin(x)]\,dx = 2\int e^x\,dx + \int \sin(x)\,dx = 2e^x - \cos(x) + C$.
  - Then $\int_0^{2\pi} [2e^x + \sin(x)]\,dx = (2e^x - \cos(x))\Big|_{x=0}^{2\pi} = \left[2e^{2\pi} - 1\right] - \left[2e^0 - 1\right] = \boxed{2e^{2\pi} - 2}$.
- To find the area between two curves $y = f(x)$ and $y = g(x)$ for x in an interval $[a, b]$, we need only subtract the area under the lower curve from the area under the upper curve. Note that if the curves cross, the region of integration needs to be broken into multiple pieces. To ensure nothing is missed, follow these steps:
  - Step 1: Graph both curves on the region of integration.
  - Step 2: Find any points of intersection, and identify which function is larger on each interval.
  - Step 3: Integrate [larger] − [smaller] on each interval, and sum the results.
  - Note that if we want the (finite) area enclosed between the two curves, the region of integration is the interval between the leftmost and rightmost intersection points. Otherwise, the region of integration must be given explicitly.
- Example: Find the area of the region between the graphs of $y = x$ and $y = x^2$ from $x = 0$ to $x = 2$.
  - By plotting the graphs of both functions, we can see that they cross once on $[0, 2]$:

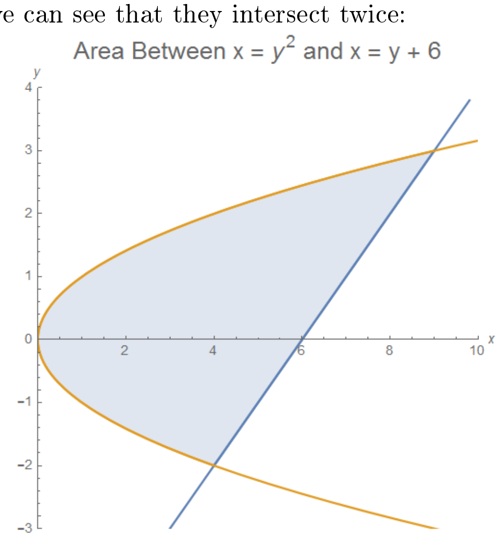


  - We can find the intersection point by solving the equations $y = x$ and $y = x^2$: this yields $x^2 = x$ so that $x = 0$ or $x = 1$. Thus, the intersection point is $(1, 1)$.
  - On the interval $[0, 1]$ we see that $y = x$ is the upper curve, while on $[1, 2]$ we see that $y = x^2$ is the upper curve.
  - Therefore, the desired area is $A = \int_0^1 \left[x - x^2\right]\,dx + \int_1^2 \left[x^2 - x\right]\,dx$.
  - Evaluating each integral yields $A = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^{1} + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_{x=1}^{2} = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \boxed{1}$.

- Example: Find the finite area enclosed between the graphs of $y = x^3 - 3x$ and $y = x$.
  - By plotting both functions, we can see that they intersect three times:

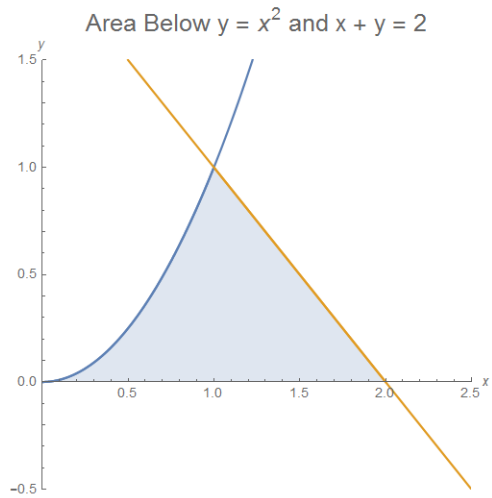


  - Solving explicitly for the intersection points of $y = x$ and $y = x^3 - 3x$ yields $x^3 - 4x = 0$, so that $x(x-2)(x+2) = 0$. Hence the graphs intersect when $x = -2, 0, 2$, at the three points $(-2, -2)$, $(0, 0)$, and $(2, 2)$.
  - Therefore, the interval of interest is $[-2, 2]$. We see from the graphs that $y = x^3 - 3x$ is the upper curve on $[-2, 0]$, and that $y = x$ is the upper curve on $[0, 2]$.
  - The desired area is then $A = \int_{-2}^{0} \left[(x^3 - 3x) - x\right] \, dx + \int_{0}^{2} \left[x - (x^3 - 3x)\right] \, dx$.
  - Evaluating each integral yields $A = \left(\dfrac{x^4}{4} - 2x^2\right)\Big|_{x=-2}^{0} + \left(2x^2 - \dfrac{x^4}{4}\right)\Big|_{x=0}^{2} = (0 - (-4)) + (4 - 0) = \boxed{8}$.
- In some situations, instead of integrating along the x-axis, we may wish to compute areas by integrating along the y-axis.
  - The procedure is the same as before, but with the roles of the coordinates x and y interchanged, and with "area under the curve above the x-axis" replaced with "area to the left of the curve and to the right of the y-axis".
  - Then to find the area between two curves $x = f(y)$ and $x = g(y)$ for y in an interval $[c, d]$, we need only subtract the area "left" of the rightmost curve from the area "left" of the leftmost curve.
- Example: Find the area of the finite region between the curves $x = y^2$ and $x = y - 6$.
  - By graphing both functions we can see that they intersect twice:



- We may find the intersection points by solving the equations $x = y^2$ and $x = y + 6$ simultaneously: this yields $y^2 = y + 6$ so that $(y+2)(y-3) = 0$ and thus $y = -2, 3$.
- Hence the intersection points are $(4, -2)$ and $(9, 3)$. So the range of integration is $[-2, 3]$, and on this range we can see that the line $x = y + 6$ lies to the right of the parabola $x = y^2$.
- The desired area is then $A = \int_{-2}^{3} \left[(y+6) - y^2\right] dy = (\frac{1}{2}y^2 + 6y - \frac{1}{3}y^3)\,|_{y=-2}^{3} = \boxed{\frac{125}{6}}$.
- We will remark that it is also possible to calculate the area of this region by using integration along the x-axis. However, this requires breaking the region into two pieces, because the upper and lower curves change as we move through the region.
- Explicitly, for $0 \le x \le 4$, the upper curve is $y = \sqrt{x}$ and the lower curve is $y = -\sqrt{x}$, while for $4 \le x \le 9$ the upper curve is $y = \sqrt{x}$ and the lower curve is $y = x - 6$.
- The desired area is then $A = \int_0^4 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_4^9 [\sqrt{x} - (x-6)]\,dx$. Evaluating both integrals will eventually confirm the same area of $\frac{125}{6}$ calculated above.

- Example: Find the area of the region in the first quadrant that is below the graphs of $y = x^2$ and $x + y = 2$.
  - By graphing the functions, we can see that the region is as follows:



  - By solving the equations we can identify the intersection point of $y = x^2$ with $x + y = 2$ as $(1, 1)$.
  - If we integrate along the x-axis, then we must break the region into two pieces, since the upper curve changes from $y = x^2$ to $y = 2 - x$ at $x = 1$.
  - The desired area is $A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{1}{3} + 2 - \frac{3}{2} = \boxed{\frac{5}{6}}$.
  - Alternatively, if we integrate along the y-axis, we can evaluate the area with a single integral.
  - The two curves are $x = \sqrt{y}$ and $x = 2 - y$, so the area is $A = \int_0^1 [(2-y) - \sqrt{y}]\,dy = 2 - \frac{1}{2} - \frac{2}{3} = \boxed{\frac{5}{6}}$

4.5 Arclength, Surface Area, Volume, Moments

- The general principle in many applications of integration (measuring area, surface area, volume, mass, etc.) is: slice the desired object up into many small pieces, evaluate the desired quantity on the small pieces, and then add up the results. This will be a Riemann sum for the integral we wish to find.
- Using differentials, we can simplify many of the calculations. The procedure is to determine the differential of the quantity we are interested in (in terms of differentials of quantities we know), and then integrate.

4.5.1 Arclength

- Arclength: If $x(t)$ and $y(t)$ are differentiable and their derivatives are continuous, then the arclength s of the curve parametrized by $(x(t), y(t))$ from $t = a$ to $t = b$ is $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$.
  - Note: The letter s is traditionally used to denote arclength. The arclength of a curve (being a length) is always nonnegative.
  - If the curve is given explicitly as $y = f(x)$, one can take $x = t$ and $y = f(t)$ in the formula to get $s = \int_a^b \sqrt{1 + f'(t)^2}dt$.
  - The idea behind the proof is to slice up the curve into small pieces, approximate each piece with a line segment, and add up the lengths. As we shrink the size of the pieces, the sum turns into an integral.
  - Proof: Over a small time interval Δt, the curve is closely approximated by a line segment joining $(x(t), y(t))$ to $(x(t + \Delta t), y(t + \Delta t))$.
  - The length Δs of this line segment is $[\Delta s]^2 = [x(t + \Delta t) - x(t)]^2 + [y(t + \Delta t) - y(t)]^2 = [\Delta x]^2 + [\Delta y]^2$.
  - Taking the limit as $\Delta t \to 0$ turns the differences into differentials: $[ds]^2 = [dx]^2 + [dy]^2$, so $ds = \sqrt{(dx)^2 + (dy)^2}$.
  - Dividing both sides by dt gives $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = \sqrt{x'(t)^2 + y'(t)^2}$.
  - Then integrating $\dfrac{ds}{dt}$ from $t = a$ to $t = b$ shows that the arclength is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$, as desired.
- Example: Find the arclength of the curve $x = \cos(t)$, $y = \sin(t)$ from $t = 0$ to $t = 2\pi$.
  - We have $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$, so $x'(t)^2 + y'(t)^2 = \sin^2(t) + \cos^2(t) = 1$.
  - Then the arclength $s = \int_0^{2\pi} 1\, dt = \boxed{2\pi}$.
  - Remark: Of course, we already know that the circumference of the unit circle is 2π.
- Example: Find the arclength of the curve $x = t^3 - 3t$, $y = 3t^2$ from $t = 1$ to $t = 2$.
  - We have $x'(t) = 3t^2 - 3$ and $y'(t) = 6t$, so $x'(t)^2 + y'(t)^2 = \left(9t^4 - 18t^2 + 9\right) + 36t^2 = 9t^4 + 18t^2 + 9 = (3t^2 + 3)^2$.
  - Then the arclength $s = \int_1^2 (3t^2 + 3)\, dt = \left(t^3 + 3t\right)\big|_{t=1}^2 = 14 - 4 = \boxed{10}$.
- We will remark that it is easy to extend these formulas to find the arclength of a parametric curve in 3-dimensional space as well.

4.5.2 Surface Area, Surfaces of Revolution

- Surfaces of Revolution: The graph of the differentiable function $y = f(x)$ between $x = a$ and $x = b$ is rotated about the x-axis, creating a surface. Using integration, we can show that the area of this surface is $A = \int_a^b 2\pi f(x) \cdot \sqrt{1 + f'(x)^2}\, dx$.
  - The idea behind the proof is to slice the surface into horizontal strips, and add up their areas.
  - Proof: Over a small interval $[x, x + \Delta x]$, each strip is a thin conical slice whose radius of revolution is the distance to the axis and whose outer length is a piece of the arclength of the curve.
  - So the differential area of each strip is $dA = 2\pi \cdot [\text{radius}] \cdot [\text{arclength}]$.
  - The radius is $f(x)$, and for the arclength, we approximate the curve with a line segment as in the calculation for the arclength of a curve to obtain $\sqrt{(dx)^2 + (dy)^2}$.
  - The differential of the surface area is therefore $dA = 2\pi f(x) \cdot \sqrt{(dx)^2 + (dy)^2}$, so $\dfrac{dA}{dx} = 2\pi f(x) \cdot \sqrt{1 + (f'(x))^2}$. Integrating from $x = a$ to $x = b$ then yields the formula.

- Example: Find the area of the surface formed by revolving the portion of the graph of $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.
  - We have $a = 0$, $b = 1$, and $f(x) = x^3$ with $f'(x) = 3x^2$, so the formula gives $A = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx$.
  - Now we substitute $u = 1 + 9x^4$: we have $du = 36x^3 dx$; also, $x = 0$ gives $u = 1$, and $x = 1$ gives $u = 10$.
  - So we get $A = \int_1^{10} \frac{\pi}{18} \sqrt{u}\, du = \left(\frac{\pi}{18} \cdot \frac{2}{3} u^{3/2}\right) \Big|_{u=1}^{10} = \frac{\pi}{27} \cdot \left(10^{3/2} - 1^{3/2}\right) = \boxed{\frac{\pi}{27}(10^{3/2} - 1)}$.

- Example: Find the area of the surface formed by revolving the portion of the graph of $y = \sqrt{r^2 - x^2}$ between $x = a$ and $x = b$ about the x-axis. Then find the surface area of a sphere of radius r.
  - We have $f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$, so the formula gives $A = \int_a^b 2\pi \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_a^b 2\pi \sqrt{r^2}\, dx = \boxed{2\pi r(b - a)}$.
  - The sphere of radius r corresponds to taking $b = r$ and $a = -r$, since the graph of $y = \sqrt{r^2 - x^2}$ is just the upper half of a circle of radius r. Plugging into the formula shows that the surface area is $A = \boxed{4\pi r^2}$, which (of course) agrees with the formula from elementary geometry.
  - Remark: Notice that the surface area $A = 2\pi r(b - a)$ only depends on the value $b - a$, which yields the interesting consequence that the surface area of a "slice" of a sphere by two parallel planes only depends on the distance between the planes.

4.5.3 Volume, Solids of Revolution

- General Volume Formula: If a solid running from $x = a$ to $x = b$ has cross-sectional area $A(x)$ at x, then the total volume of the solid is given by $V = \int_a^b A(x) dx$.
  - The justification of this formula is similar to that used for the area underneath a curve: the idea is to approximate a volume using Riemann sums, where instead of rectangles, we use the cross-sections to form "boxes".

- Example: A cut log running from $x = 0$ to $x = 1$ has square cross-sections. If the cross-section at x is a square with side length $1 + \sqrt{x}$, find the volume of the log.
  - Since the cross-sections are squares, we have $A(x) = (1 + \sqrt{x})^2 = 1 + 2\sqrt{x} + x$.
  - Then $V = \int_0^1 (1 + 2\sqrt{x} + x)\, dx = \left(x + 2 \cdot \frac{2}{3} x^{3/2} + \frac{1}{2} x^2\right) \Big|_{x=0}^{1} = \left(1 + \frac{4}{3} \cdot 1^{3/2} + \frac{1}{2} \cdot 1^2\right) - 0 = \boxed{\frac{17}{6}}$.

- Solids of Revolution: The area between the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is rotated about an axis, and we wish to find the volume of the resulting solid. Via the general volume formula, all that is needed is to choose an axis of integration, find the area of each "cross-section", and then integrate.
  - Disc method: If the axis of integration is the same as the axis of rotation, then the cross-sections are discs with radius equal to the distance of the function to the axis. When the axis of rotation is the x-axis, the radius is just the value of the function, and the volume is given by $V = \int_a^b \pi \cdot f(x)^2 dx$.
  - If instead of the area under a curve we are interested in the area between two curves $y = f(x)$ and $y = g(x)$, we need only subtract the volume generated by the smaller function from the volume generated by the larger function. If $f(x) \geq g(x)$ on $[a, b]$, then we obtain the formula $V = \int_a^b \pi \cdot \left[f(x)^2 - g(x)^2\right]\, dx$. This formula is sometimes referred to as the washer method.
  - Shell method: If the axis of integration is perpendicular to the axis of rotation, the cross-sections are cylindrical shells, instead, and the "shell area" will be equal to 2π times the distance to the axis times the height of the shell. In the particular case where the axis of rotation is the y-axis, then the volume will be given by $V = \int_a^b 2\pi x \cdot f(x) dx$.
  - Note that, by rewriting the function and (if necessary) choosing to integrate along the y-axis instead of the x-axis, it is possible to use either the disc method or the shell method for any given problem.

- Example: The area under the curve $y = \sqrt{3x^2+1}$ between $x = 1$ and $x = 4$ is rotated about the x-axis. Find the volume of the resulting solid.
  - The easiest axis of integration is the x-axis, since the function is a function of x. Since the axis of rotation is the same as the axis of integration, we use the disc method: $a = 1$, $b = 4$, and $f(x) = \sqrt{3x^2+1}$.
  - Then $V = \int_1^4 \pi \cdot \left[\sqrt{3x^2+1}\right]^2 \, dx = \int_1^4 \pi \cdot (3x^2+1) \, dx = \pi \cdot \left(x^3 + x\right) \big|_{x=1}^4 = 68\pi - 2\pi = \boxed{66\pi}$.

- Example: The area under the curve $y = \sqrt{3x^2+1}$ between $x = 1$ and $x = 4$ is rotated about the y-axis. Find the volume of the resulting solid.
  - The easiest axis of integration is the x-axis, since the function is a function of x. Since the axis of rotation is perpendicular to the axis of integration, we use the shell method: $a = 1$, $b = 4$, and $f(x) = \sqrt{3x^2+1}$.
  - This gives $V = \int_1^4 \pi \cdot x \cdot \left[\sqrt{3x^2+1}\right] \, dx$.
  - To evaluate this integral we substitute $u = 3x^2+1$ so that $du = 6x\,dx$. Since $x = 1$ gives $u = 4$ and $x = 4$ gives $u = 49$, we see that $V = \int_4^{13} \frac{\pi}{6} \cdot \sqrt{u} \, du = \frac{\pi}{6} \cdot \frac{2}{3} u^{3/2} \big|_{x=4}^{49} = \frac{\pi}{9} \cdot \left[49^{3/2} - 4^{3/2}\right] = \frac{\pi}{9} \cdot [343 - 8] = \boxed{\frac{335\pi}{9}}$.

- Example: The area under the curve $y = \sqrt{r^2 - x^2}$ between $x = a$ and $x = b$ is rotated about the x-axis. Find the volume of the resulting solid, and then calculate the volume of a sphere of radius r.
  - The easiest axis of integration is the x-axis, since the function is a function of x. Since the axis of rotation is the same as the axis of integration, we use the disc method with $f(x) = \sqrt{r^2 - x^2}$.
  - Then $V = \int_a^b \pi \cdot \left[\sqrt{r^2 - x^2}\right]^2 \, dx = \int_a^b \pi \cdot (r^2 - x^2) \, dx = \boxed{\pi r^2 (b-a) - \pi \frac{b^3 - a^3}{3}}$.
  - The sphere of radius r corresponds to taking $b = r$ and $a = -r$, since the graph of $y = \sqrt{r^2 - x^2}$ is just the upper half of a circle of radius r. Plugging into the formula shows that the volume is $V = 2\pi r^3 - \frac{2}{3}\pi r^3 = \boxed{\frac{4}{3}\pi r^3}$, which (of course) agrees with the formula from elementary geometry.

4.5.4 Center of Mass, Moments

- The center of mass of a physical object is its "balancing point", where, if the object is supported only at that point, gravity will not cause it to tip over.
  - The center of mass is also called the centroid of an object.
  - Physically, an object behaves as if all its mass is concentrated at its "center of mass" (hence the name): the forces exerted by an object are the same as an equivalent point mass concentrated at the object's center of mass.
- Using calculus, if we are given formulas for the shape of an object, we can find the location of its center of mass.
  - We would, in real life, be most interested in the physical properties of 3-dimensional objects with variable densities.
  - However, the calculations to find centers of masses for such objects are extremely difficult to do with a single integration in one variable.
    - These types of problems are a common application of multivariable calculus: they are far easier to work out using integration in several variables.
  - Since we are using single-variable integration, we can only discuss the problems of a 1-dimensional rod with variable density, and a 2-dimensional plate with density depending only on one coordinate.
- Center of Mass and Moment Formulas (rod): We are given a 1-dimensional rod of variable density $\rho(x)$ between $x = a$ and $x = b$.

- The total mass M is given by $M = \int_a^b \rho(x)dx$.
- The first moment M_0 is given by $M_0 = \int_a^b x\rho(x)dx$.
- The center of mass $\bar{x}$ is given by the ratio of the first moment to the total mass: $\bar{x} = \dfrac{M_0}{M} = \dfrac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx}$. (Note that the center of mass $\bar{x}$ is the average value of the x-coordinate over the rod.)

- Example: Find the mass and center of mass of the rod whose endpoints are at $x = 1$ and $x = 4$ and whose density at x is $\rho(x) = \sqrt{x}$.
  - The mass is given by $M = \int_1^4 \sqrt{x}\, dx = \dfrac{2}{3}x^{3/2}\,\big|_{x=1}^4 = \dfrac{2}{3} \cdot \left[4^{3/2} - 1^{3/2}\right] = \dfrac{2}{3} \cdot 7 = \boxed{\dfrac{14}{3}}$.
  - The first moment is $M_0 = \int_1^4 x \cdot \sqrt{x}\, dx = \dfrac{2}{5}x^{5/2}\,\big|_{x=1}^4 = \dfrac{2}{5} \cdot \left[4^{5/2} - 1^{5/2}\right] = \dfrac{2}{5} \cdot 31 = \dfrac{62}{5}$.
  - The center of mass is located at $\bar{x} = \dfrac{M_0}{M} = \dfrac{62/5}{14/3} = \boxed{\dfrac{91}{35}}$.

- Center of Mass and Moment Formulas (plate): We are given a 2-dimensional plate with density function $\rho(x)$ which depends on x but not y, whose upper boundary has equation $y = g(x)$ and whose lower boundary has equation $y = f(x)$, between $x = a$ and $x = b$.
  - The total mass is given by $M = \int_a^b \rho(x) \cdot [g(x) - f(x)]\, dx$.
  - The first moments are given by $M_y = \int_a^b x\cdot\rho(x)\cdot[g(x) - f(x)]\, dx$ and $M_x = \int_a^b \dfrac{1}{2}\cdot\rho(x)\cdot\left[g(x)^2 - f(x)^2\right]\, dx$.
  - The moment M_y is the average value of x over the plate, and the moment M_x is the average value of y over the plate. (The traditional notation for this is, unfortunately, very confusing.)
  - The center of mass $(\bar{x}, \bar{y})$ has coordinates $\bar{x} = \dfrac{M_y}{M}$ and $\bar{y} = \dfrac{M_x}{M}$.

- Example: Find the center of mass of the thin plate with uniform density 1 whose shape is the quarter-disc $x^2 + y^2 \le 1$ with $x \ge 0$ and $y \ge 0$.
  - The quarter-disc runs from $x = 0$ to $x = 1$, the lower boundary is the x-axis with equation $y = 0$, and the upper boundary is the curve $y = \sqrt{1 - x^2}$.
  - Since the density is 1, the mass of the disc is simply its area, which is $\dfrac{\pi}{4}$. As an integral the mass would be $\int_0^1 \sqrt{1 - x^2}\, dx$, but (as we have seen) this integral is rather complicated to evaluate.
  - The moments are $M_y = \int_0^1 x \cdot \sqrt{1 - x^2}\, dx$ and $M_x = \int_0^1 \dfrac{1}{2} \cdot (1 - x^2)\, dx$.
  - For M_y we substitute $u = 1 - x^2$, with $du = -2x$. We see $x = 0$ gives $u = 1$ and $x = 1$ gives $u = 0$, so the substitution gives $M_y = \int_1^0 \left(-\dfrac{1}{2}\right) \sqrt{u}\, du = \int_0^1 \dfrac{1}{2}u^{1/2}\, du = \dfrac{1}{2} \cdot \dfrac{2}{3} \cdot u^{3/2}\,\big|_{u=0}^1 = \dfrac{1}{3}$.
  - For M_x we have $M_x = \int_0^1 \dfrac{1}{2} \cdot (1 - x^2)\, dx = \dfrac{1}{2} \cdot \left(x - \dfrac{x^3}{3}\right)\big|_{x=0}^1 = \dfrac{1}{2} \cdot \dfrac{2}{3} - 0 = \dfrac{1}{3}$.
  - Then $\bar{x} = \dfrac{1/3}{\pi/4} = \dfrac{4}{3\pi}$ and $\bar{y} = \dfrac{1/3}{\pi/4} = \dfrac{4}{3\pi}$, so the center of mass is $\boxed{\left(\dfrac{4}{3\pi}, \dfrac{4}{3\pi}\right)}$.

Well, you're at the end of my handout. Hope it was helpful.