Random geometric graphs, Apollonian packings, number networks, and the Riemann hypothesis

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Random geometric graphs

- Discretizations of continuous spaces
- Riemannian
  - Nodes are “atoms” of space
  - Links reflect proximity
- Lorentzian
  - Nodes are “atoms” of spacetime
  - Links reflect causality
    - Random geometric graphs in non-pathological Lorentzian spacetimes define, up to the conformal factor, the spacetime itself (Hawking/Malament theorem)
Random Riemannian graphs

- Given $N$ nodes
- Hidden variables:
  - Node coordinates $\mathbf{\tilde{v}}$ in a Riemannian space
  - $\rho(\mathbf{\tilde{v}})$: uniform in the space
- Connection probability:
  $$\rho(\mathbf{\tilde{v}}, \mathbf{\tilde{v}}') = \Theta(R - d(\mathbf{\tilde{v}}, \mathbf{\tilde{v}}'))$$
- Results:
  - Clustering: Strong
  - Degree distribution: Power law (hyperbolic)
Lorentzian Random Riemannian graphs

- Given \( N \) nodes
- Hidden variables:
  - Node coordinates \( \hat{\kappa} \) in a Riemannian space
  - \( \rho(\hat{\kappa}) \): uniform in the space
- Connection probability:
  \[
  p(\hat{\kappa}, \hat{\kappa}') = \Theta(r - d(\hat{\kappa}, \hat{\kappa}'))
  \]
- Results:
  - Clustering: Strong
  - Degree distribution: Power law (de Sitter)
## Taxonomy of Riemannian spaces

<table>
<thead>
<tr>
<th>Property</th>
<th>Euclidean</th>
<th>Spherical</th>
<th>Hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curvature $K$</td>
<td>0</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>Parallel lines</td>
<td>1</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Triangles are</td>
<td>normal</td>
<td>thick</td>
<td>thin</td>
</tr>
<tr>
<td><strong>Shape of triangles</strong></td>
<td><img src="image" alt="triangle" /></td>
<td><img src="image" alt="spherical_triangle" /></td>
<td><img src="image" alt="hyperbolic_triangle" /></td>
</tr>
<tr>
<td>Sum of $\triangle$ angles</td>
<td>$\pi$</td>
<td>$&gt; \pi$</td>
<td>$&lt; \pi$</td>
</tr>
<tr>
<td>Circle length</td>
<td>$2\pi r$</td>
<td>$2\pi \sin \zeta r$</td>
<td>$2\pi \sinh \zeta r$</td>
</tr>
<tr>
<td>Disk area</td>
<td>$2\pi r^2/2$</td>
<td>$2\pi (1 - \cos \zeta r)$</td>
<td>$2\pi (\cosh \zeta r - 1)$</td>
</tr>
</tbody>
</table>
Taxonomy of Lorentzian spaces

<table>
<thead>
<tr>
<th>Curvature</th>
<th>0</th>
<th>&gt;0</th>
<th>&lt;0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>Minkowski</td>
<td>De Sitter</td>
<td>Anti-de Sitter</td>
</tr>
</tbody>
</table>

- De Sitter is dual to hyperbolic
- Random hyperbolic graphs $\xrightarrow{N\rightarrow\infty}$ Random de Sitter graphs
$ds^2 = dx^2 + dy^2 - dz^2$

$r_P = \tanh \frac{r_H}{2}$

$x^2 + y^2 - z^2 = -1$

$x^2 + y^2 < 1$

$ds^2 = dx^2 + dy^2 + dz^2$

$r_E = -iE(\text{i}r_H, 2)$
Degree distribution
\[ P(k) \sim k^{-\gamma} \]
\[ \gamma = k_B + 1 \]
\[ k_B = \frac{2}{\sqrt{-K}} \]

Clustering
\[ \bar{c}(k) \sim k^{-1} \text{ maximized} \]
Growing hyperbolic model

- New nodes $t = 1, 2, \ldots$ are coming one at a time
- Radial coordinate $r_t \sim \log t$
- Angular coordinate $\theta_t \in U[0, 2\pi]$
- Connect to $O(1)$ hyperbolically closest existing nodes; or
- Connect to all existing nodes at distance $\lesssim r_t$
The key properties of the model

• Angular coordinates are projections of properly weighted combinations of all the (hidden) similarity attributes shaping network structure and dynamics
• Infer these (hidden) coordinates using maximum-likelihood methods
• The results describe remarkably well not only the structure of some real networks, but also their dynamics
• Validating not only consequences of this dynamics (network structure), but also the dynamics itself
• Prediction of missing links using this approach outperforms best-performing methods
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\[ x^2 + y^2 - z^2 = -1 \]

\[ x^2 + y^2 - z^2 = +1 \]
Light cones

- L’s future light cone
- P’s past light cone

\[ t = \frac{x_0}{c} \]

- t = t_0
- t = 0

space x
Random Lorentzian graphs (causal sets)
\[ x^2 + y^2 - z^2 = -1 \]

\[ x^2 + y^2 - z^2 = +1 \]
Hyperbolic space

- $-z_0^2 + z_1^2 + \cdots z_{d+1}^2 = -b^2 = \frac{1}{K}$
- $z_0 = b \cosh \rho, \ z_i = b \omega_i \sinh \rho, \ \rho = \frac{r}{b}$
- $ds^2 = b^2 (d\rho^2 + \sinh^2 \rho \ d\Omega_d^2)$
- $dV = b^{d+1} \sinh^d \rho \ d\rho \ d\Phi_d \approx b \left(\frac{b}{2}\right)^d e^{d\rho} \ d\rho \ d\Phi_d$
De Sitter spacetime

- \(-z_0^2 + z_1^2 + \ldots z_{d+1}^2 = a^2 = \frac{1}{K}\)
- \(z_0 = a \sinh \tau, z_i = a \omega_i \cosh \tau, \tau = \frac{t}{a}\)
- \(ds^2 = a^2 \left( -d\tau^2 + \cosh^2 \tau \, d\Omega_d^2 \right)\)
- \(\sec \eta = \cosh \tau, \) conformal time
- \(ds^2 = a^2 \sec^2 \eta \left( -d\eta^2 + d\Omega_d^2 \right)\)
- \(dV = a^{d+1} \sec^{d+1} \eta \, d\eta \, d\Phi_d = a^{d+1} \cosh^d \tau \, d\tau \, d\Phi_d \approx a \left( \frac{a}{2} \right)^d \, e^{d\tau} \, d\tau \, d\Phi_d\)
Hyperbolic balls

• $\rho_0 = \frac{2}{d} \ln \frac{N_0}{v}$, current ball radius, i.e., the radial coordinate of new node $N_0$
• Hyperbolic distance from the new node
  \[
  x = b \cosh(\cosh \rho \cosh \rho_0 - \sinh \rho \sinh \rho_0 \cos \Delta \theta) \approx b (\rho + \rho_0 + 2 \ln \frac{\Delta \theta}{2})
  \]
• $x < b \rho_0$, connection condition
• $\rho + 2 \ln \frac{\Delta \theta}{2} < 0$, approximately
Past light cones

- $\Delta \theta < \Delta \eta$, connection condition
- $\Delta \theta < \eta_0 - \eta = \text{asec} \cosh \tau_0 - \text{asec} \cosh \tau \approx 2(e^{-\tau} - e^{-\tau_0})$, where $\eta_0, \tau_0$ is the current time
- $\Delta \theta < 2e^{-\tau}$, approximately
- $\rho + 2 \ln \frac{\Delta \theta}{2} < 0$, in the hyperbolic case
- The last two inequalities are the same iff $\tau = \frac{\rho}{2}$
- Mapping, two equivalent options:
  - $t = r; a = 2b$
  - $t = \frac{r}{2}; a = b$
Node density

- **Hyperbolic, non-uniform**
  
  \[ N = \nu e^{\frac{d\rho}{2}} \]
  
  \[ dV \approx b \left( \frac{b}{2} \right)^d e^{d\rho} d\rho \, d\Phi_d \]
  
  \[ dN = \delta e^{-\frac{d\rho}{2}} dV \]

- **De Sitter, uniform**

  \[ dV \approx a \left( \frac{a}{2} \right)^d e^{d\tau} d\tau \, d\Phi_d = b^{d+1} e^{\frac{d\rho}{2}} d\rho \, d\Phi_d \]

  \[ dN = \delta \, dV, \text{ where } \delta = \frac{dv}{2b^d+1\sigma_d}, \quad \sigma_d = \int d\Phi_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \]
Web of trust (PGP graph)

User A
Data A
Email address, PGP certificate

Trust
User A signs data B

User B
Data B
Email address, PGP certificate
Brain (fMRI graph)
De Sitter universe

- The universe is accelerating
- Dark energy (73%) is an explanation
- Remove all matter from the universe
  - Dark matter (23%)
  - Observable matter (4%)
- Obtain de Sitter spacetime
  - Solution to Einstein’s field equations for an empty universe with positive vacuum energy (cosmological constant $\Lambda > 0$)
The degree distribution $P(k)$ follows a power law relationship with $k$, given by $P(k) \sim k^{-2}$. The graph illustrates this trend across different datasets, including the Internet, Trust, Brain, and de Sitter networks.
Exponential random graphs

• Intuition:
  – Maximum-entropy graphs under the constraints that expected values of some graph observables are equal to given values

• Definition:
  – Set of graphs $G$ with probability measure

$$P(G) = \frac{e^{-\frac{H(G)}{k_B T}}}{Z}$$

where the partition function

$$Z = \sum_G e^{-\frac{H(G)}{k_B T}}$$

and

$$H(G) = \sum_i \omega_i O_i(G)$$

is the graph Hamiltonian with Lagrange multipliers $\omega_i$ maximizing the Gibbs entropy

$$S = -\sum_G P(G) \ln P(G)$$

under constraints

$$\sum_G P(G) O_i(G) = \langle O_i \rangle$$
RHGs as ERGs

- Hamiltonian of graph $G$ with adjacency matrix $A_{ij}$ is
  \[ H(G) = \sum_{i<j} \omega_{ij} A_{ij} \]
  where $\omega_{ij} = x_{ij} - R$, and $x_{ij}$ is the hyperbolic distance between nodes $i$ and $j$, and $R$ is the disk radius.

- In the limit $T \to \infty$, $k_B \to 0$, $k_B T = \gamma - 1$ (soft configuration model or random graphs with a given expected degree sequence)
  \[ \omega_{ij} \to \omega_i + \omega_j \]
  \[ H(G) \to \sum_i \omega_i k_i \]

- $H = xp$ is the Berry-Keating Hamiltonian conjectured to be related, after quantization, to the Polya-Hilbert Hermitian operator (unknown) containing non-trivial Riemann zeros
  - Some extensions of $H = xp$ yield the Hamiltonian of non-interacting fermions in de Sitter spacetime
  - Graph edges in hyperbolic graphs are non-interacting fermions with energy $x_{ij}$ (graph energy is $H(G)$, while $R$ is the chemical potential of the fermionic gas)
Euclidean Apollonian packing
Spherical Apollonian packing
Hyperbolic Apollonian packing
Descartes theorem

• Descartes quadratic form \( Q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 \)

• If \( a, b, c, d \) are curvatures of any four mutually tangent disks, then
  
  – \( Q(a, b, c, d) = 0 \) in any Euclidean covering
  – \( Q(a, b, c, d) = -4 \) in any spherical covering
  – \( Q(a, b, c, d) = 4 \) in any hyperbolic covering
Lorentz form

• There exists symmetric matrix \( J \), \( J^2 = I \), such that 
  \[
  L = \frac{1}{2} J Q J^T
  \]
  is 
  \[
  \hat{L}(T, X, Y, Z) = -T^2 + X^2 + Y^2 + Z^2
  \]
• That is, \( J \) maps integers curvatures of tangent disk quadruples to integer coordinates of points lying in 1+2-dimensional
  – light cone (Euclidean covering)
  – hyperbolic space (spherical covering)
  – de Sitter space (hyperbolic covering)
  in the 1+3-dimensional Minkowski space
• The point density (CDF) is asymptotically uniform in all cases
Apollonian questions

• Since the point density is uniform, hyperbolic Apollonian covering networks are asymptotically random de Sitter graphs with power-law degree distributions, strong clustering, etc.

• **What are their exact properties?**

• **What are timelike-separated (connected) versus spacelike-separated (disconnected) pairs of tangent disk quadruples?**
Symmetries

• The group of isometries of
  – $\mathbb{H}^{1,2}$ and $dS^{1,2}$ is the Lorentz group $SO(1,3)$
  – Apollonian covering is its subgroup $SO(1,3,\mathbb{Z})$

• $SO(1,3)$ is isomorphic to the conformal group $SL(2,\mathbb{C})$

• $SO(1,3,\mathbb{Z})$ is isomorphic to the modular group $SL(2,\mathbb{Z})$

• $SL(2, X)$ ($X = \mathbb{C}, \mathbb{Z}$) acts transitively on $PX^2$ by linear fractional transformations
  $$(a \ b) \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$
  – In particular, $SL(2, \mathbb{Z})$ acts transitively on (ir)reducible fractions
Uniformity of Farey numbers

- The distribution of Farey rationals on $[0,1]$ is very close to uniform.
- Let $f_k, k = 1, \ldots, N$, be a sequence of Farey numbers $\frac{p_k}{q_k}$ for some $Q \geq q_k$ ($N(Q) \sim 3Q^2/\pi^2$ is given by Euler’s totient function).
- Let $g_k = \frac{k}{N}$, and $d_k = |f_k - g_k|$.
- Franel-Landau theorem: the Riemann hypothesis is equivalent to

\[ \sum_k d_k = o \left( Q^{2+\epsilon} \right) \]

for any $\epsilon > 0$. 
Divisibility networks

• Two natural numbers are connected if one divides the other or if they are not coprime
• Degree distribution is a power law with exponent $\gamma = 2$
• Clustering is strong $\langle c \rangle = \frac{1}{2}$
• That is, the same as in random de Sitter graphs or hyperbolic Apollonian packings
• Many other important statistical structural properties also appear similar in experiments
Number-geometric questions

• Are there any (soft) embeddings of Farey or divisibility networks into (de Sitter) spaces?
• If such embeddings are found, then timelike distances will have clear interpretations
• But what are spacelike distances?? What is the hidden geometry of numbers???
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