Complex networks in quantum gravity and cosmology

Dmitri Krioukov
NU

M. Kitsak (NU), R. Sinkovits (UCSD/SDSC)
David Rideout & David Meyer
UCSD/Math

Marian Boguñá
U. Barcelona

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Random geometric graphs

- Given $N$ nodes
- Hidden variables: Node coordinates $\mathbf{k}$ in a space $\rho(\mathbf{k})$: uniform in the space
- Connection probability: $p(\mathbf{k}, \mathbf{k}') = \Theta(R - d(\mathbf{k}, \mathbf{k}'))$
- Results:
  - Clustering: Strong
  - Degree distribution:
    - Euclidean or spherical space: Poisson
    - Hyperbolic space: Power law

\[
\mathbf{k} \quad d(\mathbf{k}, \mathbf{k}') < R \quad \mathbf{k}'
\]
## Taxonomy of spaces

<table>
<thead>
<tr>
<th>Property</th>
<th>Euclidean</th>
<th>Spherical</th>
<th>Hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curvature $K$</td>
<td>0</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>Parallel lines</td>
<td>1</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Triangles are normal</td>
<td>normal</td>
<td>thick</td>
<td>thin</td>
</tr>
<tr>
<td>Shape of triangles</td>
<td><img src="image" alt="triangle" /></td>
<td><img src="image" alt="triangle" /></td>
<td><img src="image" alt="triangle" /></td>
</tr>
<tr>
<td>Sum of $\triangle$ angles</td>
<td>$\pi$</td>
<td>$&gt; \pi$</td>
<td>$&lt; \pi$</td>
</tr>
<tr>
<td>Circle length</td>
<td>$2\pi r$</td>
<td>$2\pi \sin \zeta r$</td>
<td>$2\pi \sinh \zeta r$</td>
</tr>
<tr>
<td>Disk area</td>
<td>$2\pi r^2/2$</td>
<td>$2\pi (1 - \cos \zeta r)$</td>
<td>$2\pi (\cosh \zeta r - 1)$</td>
</tr>
</tbody>
</table>
Taxonomy of Lorentzian spaces

<table>
<thead>
<tr>
<th>Curvature</th>
<th>0</th>
<th>&gt;0</th>
<th>&lt;0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space</td>
<td>Minkowski</td>
<td>De Sitter</td>
<td>Anti-de Sitter</td>
</tr>
</tbody>
</table>

- Problem: find **dual** to hyperbolic
- Solution: **de Sitter**
Random hyperbolic graphs

- Given $N$ nodes
- Hidden variables:
  - Node coordinates $\tilde{\kappa}$ in a hyperbolic space
  - $\rho(\tilde{\kappa})$: uniform in the space
- Connection probability:
  $p(\tilde{\kappa}, \tilde{\kappa}') = \Theta(R - d(\tilde{\kappa}, \tilde{\kappa}'))$
- Results:
  - Clustering: Strong
  - Degree distribution:
    Power law ($\gamma = 3$)
    --- $\gamma = 2$
Main theorem

- **Growing** random hyperbolic graphs $N \to \infty$ to Random de Sitter graphs.
Random geometric graphs

• Are “quantizations” (?)
• No, discretizations of continuous space(time)s
• Riemannian
  • Nodes are “atoms” of space
  • Links reflect proximity
• Lorentzian
  • Nodes are “atoms” of spacetime
  • Links reflect causality
Taxonomy of networks in quantum gravity

• Networks in space
  – Loop quantum gravity/spin networks
  – Quantum graphity

• Networks in timespace
  – Causal dynamic triangulations
  – Causal sets
    • The causal set of a non-pathological spacetime defines, up to the conformal factor, the spacetime itself (Hawking/Malament theorem)
Outline

- Random hyperbolic networks
- Random de Sitter networks
- Real graphs and simulations
\[ ds^2 = dx^2 + dy^2 - dz^2 \]

\[ r_P = \tanh \frac{r_H}{2} \]

\[ x^2 + y^2 - z^2 = -1 \]

\[ x^2 + y^2 < 1 \]

Hyperboloid model

Poincaré model

\[ ds^2 = dx^2 + dy^2 + dz^2 \]

\[ r_E = -iE(\text{ir}_H, 2) \]
Degree distribution
\[ P(k) \sim k^{-\gamma} \]
\[ \gamma = k_B + 1 \]
\[ k_B = \frac{2}{\sqrt{-K}} \]

Clustering
\[ \bar{c}(k) \sim k^{-1} \]
maximized
Growing hyperbolic model

- New nodes $t = 1, 2, \ldots$ are coming one at a time
- Radial coordinate $r_t \sim \log t$
- Angular coordinate $\theta_t \in U[0, 2\pi]$
- Connect to $\Theta(1)$ hyperbolically closest existing nodes; or
- Connect to all existing nodes at distance $\leq r_t$
The key properties of the model

• Angular coordinates are projections of properly weighted combinations of all the (hidden) similarity attributes shaping network structure and dynamics
• Infer these (hidden) coordinates using maximum-likelihood methods
• The results describe remarkably well not only the structure of some real networks, but also their dynamics
• Validating not only consequences of this dynamics (network structure), but also the dynamics itself
• Prediction of missing links using this approach outperforms best-performing methods
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\[ x^2 + y^2 - z^2 = -1 \]

\[ x^2 + y^2 - z^2 = +1 \]
Light cones

- L’s future light cone
- P’s past light cone
- \( t = \frac{x_0}{c} \)
Random Lorentzian graphs (causal sets)
\[ x^2 + y^2 = \pm 1 \]
Hyperbolic space

- \(-z_0^2 + z_1^2 + \cdots z_{d+1}^2 = -b^2 = \frac{1}{K}\)
- \(z_0 = b \cosh \rho, \ z_i = b \omega_i \sinh \rho, \ \rho = \frac{r}{b}\)
- \(ds^2 = b^2 (d\rho^2 + \sinh^2 \rho \, d\Omega_d^2)\)
- \(dV = b^{d+1} \sinh^d \rho \, d\rho \, d\Phi_d \approx b \left(\frac{b}{2}\right)^d e^{d\rho} \, d\rho \, d\Phi_d\)
De Sitter spacetime

- $-z_0^2 + z_1^2 + \cdots + z_{d+1}^2 = a^2 = \frac{1}{K}$
- $z_0 = a \sinh \tau$, $z_i = a \omega_i \cosh \tau$, $\tau = \frac{t}{a}$
- $ds^2 = a^2 (-d\tau^2 + \cosh^2 \tau \, d\Omega_d^2)$
- $\sec \eta = \cosh \tau$, conformal time
- $ds^2 = a^2 \sec^2 \eta \left(-d\eta^2 + d\Omega_d^2\right)$
- $dV = a^{d+1} \sec^{d+1} \eta \, d\eta \, d\Phi_d = a^{d+1} \cosh^d \tau \, d\tau \, d\Phi_d \approx a \left(\frac{a}{2}\right)^d e^{d\tau} \, d\tau \, d\Phi_d$
Hyperbolic balls

- \( \rho_0 = \frac{2}{d} \ln \left( \frac{N_0}{\nu} \right) \), current ball radius, i.e., the radial coordinate of new node \( N_0 \)
- Hyperbolic distance from the new node
  \( x = \)
  \( b \, \text{acosh}(\cosh \rho \cosh \rho_0 - \sinh \rho \sinh \rho_0 \cos \Delta \theta) \approx b \left( \rho + \rho_0 + 2 \ln \frac{\Delta \theta}{2} \right) \)
- \( x < b \rho_0 \), connection condition
- \( \rho + 2 \ln \frac{\Delta \theta}{2} < 0 \), approximately
Past light cones

- $\Delta \theta < \Delta \eta$, connection condition
- $\Delta \theta < \eta_0 - \eta = \text{asec} \cosh \tau_0 - \text{asec} \cosh \tau \approx 2(e^{-\tau} - e^{-\tau_0})$, where $\eta_0, \tau_0$ is the current time
- $\Delta \theta < 2e^{-\tau}$, approximately
- $\rho + 2 \ln\frac{\Delta \theta}{2} < 0$, in the hyperbolic case
- The last two inequalities are the same iff
  \[ \tau = \frac{\rho}{2} \]
- Mapping, two equivalent options:
  - $t = r; a = 2b$
  - $t = \frac{r}{2}; a = b$
Node density

- **Hyperbolic, non-uniform**
  
  \[- N = νe^{\frac{dρ}{2}}\]

  \[- dV \approx b \left(\frac{b}{2}\right)^d e^{dρ} dρ dΦ_d \]

  \[- dN = \delta e^{-\frac{dρ}{2}} dV \]

- **De Sitter, uniform**

  \[- dV \approx a \left(\frac{a}{2}\right)^d e^{dτ} dτ dΦ_d = b^{d+1} e^{\frac{dρ}{2}} dρ dΦ_d \]

  \[- dN = \delta dV, \text{ where } \delta = \frac{dv}{2b^{d+1}\sigma_d}, \quad \sigma_d = \int dΦ_d = \frac{2π^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \]
Web of trust (PGP graph)

User A
Data A
Email address, PGP certificate

User B
Data B
Email address, PGP certificate

Trust
User A signs data B
De Sitter universe

• The universe is accelerating
• Dark energy (73%) is an explanation
• Remove all matter from the universe
  – Dark matter (23%)
  – Observable matter (4%)
• Obtain de Sitter spacetime
  – Solution to Einstein’s field equations for an empty universe with positive vacuum energy (cosmological constant $\Lambda > 0$)
$P(k) \sim k^{-2}$
The graph shows the clustering coefficient $c(k)$ as a function of node degree $k$. The data for different networks (Internet, Trust, Brain, de Sitter) are plotted on a logarithmic scale. A power-law relationship is observed, $c(k) \sim k^{-1}$, indicating that the clustering coefficient decreases as the node degree increases, following a power-law decay.
Conclusion

• The large-scale structure and dynamics of complex networks and causal sets in de Sitter universes are asymptotically identical
  – Growing hyperbolic graphs
    • Radial coordinates: popularity
    • Angular coordinates: similarity
  – Growing de Sitter graphs
    • Popularity \(\Rightarrow\) time
    • Similarity \(\Rightarrow\) space
    • Growth dynamics \(\Rightarrow\) spacetime expansion

• Two options:
  – This equivalence is a coincidence
    • Compute the probability of this coincidence 😊
  – This equivalence is not a coincidence: possible explanation:
    • Spacetime expansion \(\Rightarrow\) Einstein’s equations \(\Rightarrow\) Least Einstein-Hilbert action
    • Einstein-Hilbert action has recently been found for causal sets
    • No action for growing networks (either preferential attachment or hyperbolic graphs)

Are there any common fundamental laws from which Einstein’s equations (known) and complex network equations (unknown) both follow as particular cases?
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