Algebraic Geometry I

Here are some more problems on cubics! Assume everywhere that \( \text{char } \mathbb{k} \neq 2, 3 \).

1. (The group law on cubics.) Consider the cubic curve
   \[ E_\lambda : \ y^2 z = x(x - z)(x - \lambda z). \]
   (i) Consider two points \( P = [x_1, y_1, 1] \) and \( Q = [x_2, y_2, 1] \) on \( E_\lambda \). Prove that the sum \( P \oplus Q \) is \( [0, 1, 0] \) if \( x_1 = x_2, y_1 \neq y_2 \), and \( [x_3, y_3, 1] \) if \( x_1 \neq x_2 \), where:
   \[ x_3 = \left( \frac{y_1 - y_2}{x_1 - x_2} \right)^2 + 1 + \lambda - x_1 - x_2, \]
   \[ y_3 = \left( \frac{y_1 - y_2}{x_1 - x_2} \right)x_3 + \left( \frac{x_1 y_2 - y_1 x_2}{x_1 - x_2} \right). \]
   What are the corresponding formulas for \( P = Q \)?
   (ii) Show that if \( \lambda \in \mathbb{Q} \), then the set of points on the cubic curve with rational coordinates form an abelian group. You only need to check that if \( P, Q \) have rational coordinates, then so do \( -P \) and \( P \oplus Q \). (Note: The Mordell-Weil theorem states that this abelian group is finitely generated.)

2. Let \( C \subset \mathbb{P}^2 \) be a smooth cubic curve. Let \( P \in C \) be an inflection point. Show that there are exactly 4 tangents of \( C \) that pass through \( P \). Related question: How many torsion points of order 2 does a cubic have? (You may answer this also using problem 4.)

3. (Hesse configuration of points is unique up to the action of \( \text{PGL}_2 \).) Prove that if \( C, C' \subset \mathbb{P}^2 \) are smooth cubic curves, then there is an automorphism of \( \mathbb{P}^2 \) that takes the inflection points of \( C \) to the inflection points of \( C' \). (Hint: use problem 6 on Problem set 8.)

4. (Elliptic functions, the Weierstrass function and the parametrization of cubics. (This problem requires some background in complex analysis.)
   Let
   \[ L = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 \subset \mathbb{C} \]
   be a complex lattice, i.e., let \( \omega_1, \omega_2 \in \mathbb{C} \) be complex numbers with \( \omega_1/\omega_2 \notin \mathbb{R} \) and let
   \[ L = \{ m \omega_1 + n \omega_2 : \ m, n \in \mathbb{Z} \}. \]
   (i) Prove that the infinite series
   \[ \wp(z) = \frac{1}{z} + \sum_{\omega \in L \setminus \{0\}} \left( \frac{1}{(z - w)^2} - \frac{1}{\omega^2} \right) \]
   converges to a meromorphic function in \( z \) with poles of order 2 at the points of \( L \). This is called the Weierstrass function.
(ii) Prove that $\wp$ is an even function

$$\wp(-z) = \wp(z),$$

hence, its Laurent expansion near 0 contains only even powers of $z$. Prove that its derivative

$$\wp'(z) = -\sum_{\omega \in L} \frac{2}{(z-\omega)^3}$$

is an odd function.

(iii) Prove that $\wp$ is a meromorphic elliptic function:

$$\wp(z + w) = \wp(z)$$

for all $\omega \in L$, i.e., doubly periodic, with periods $\omega_1$ and $\omega_2$.

(iv) Use Liouville’s theorem to conclude that holomorphic doubly periodic functions are constant.

(v) Prove that $\wp$ satisfies a differential equation:

$$\wp'(z)^2 = c_3 \wp^3(z) + c_2 \wp(z)^2 + c_1 \wp(z) + c_0$$

for some constants $c_i$ that depend on $L$.

Hint: consider the meromorphic function

$$f(z) = \wp'(z)^2 - c_3 \wp^3(z) - c_2 \wp(z)^2 - c_1 \wp(z) - c_0.$$

Observe that $f(z)$ has a pole of order at most 6 at the origin, and only even powers of $z$ appear in the Laurent expansion. Prove that you can pick $c_0, c_1, c_2, c_3$ such that the Laurent coefficients of $z^{-6}, z^{-4}, z^{-2}, z^0$ in $f$ vanish. Conclude that $f$ is a holomorphic doubly periodic function. Then show that $f = 0$.

Note that one can actually show that $c_3 = 4, c_2 = 0$ and $c_1, c_0$ are given by the Eisenstein series

$$c_1 = -60 \sum_{\omega \in L \backslash \{0\}} \frac{1}{\omega^4}, \quad c_1 = -140 \sum_{\omega \in L \backslash \{0\}} \frac{1}{\omega^6}.$$

(vi) From (v) conclude that the point $(\wp(z), \wp'(z))$ lies on the cubic curve

$$y^2 = c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

for all values of $z$. In fact, it turns out that any point on the cubic curve above can be written as $(\wp(z), \wp'(z))$. Therefore, $z \mapsto (\wp(z), \wp'(z))$ gives a parametrization by elliptic functions. Note that we have seen that a cubic curve does not admit a parametrization by rational functions. It can be shown that the argument can be reversed, i.e., that any cubic can be parametrized by the Weierstrass function of some lattice $L$. Moreover, the group structure on $C$ arises from the group structure of $\mathbb{C}$. 