Algebraic Geometry I - Problem Set 8

Write up solutions to three of the problems (write as legibly and clearly as you can, preferably in LaTeX).

1. (Intersection Multiplicities.) Let \( C = V(f) \) and \( D = V(g) \) be two distinct curves in \( \mathbb{A}^2 \). Recall that the multiplicity of intersection \( m_p(C, D) \) of \( C \) and \( D \) at \( p \) is defined as the dimension of the \( k \)-vector space \( \mathcal{O}_{\mathbb{A}^2, p}/<f, g> \). Prove that:
   (a) \( m_p(C, D) \geq m_p(C) \cdot m_p(D) \).
   (b) If \( p \in C \), for all but finitely many lines \( L \) through \( p \), \( m_p(C, L) = m_p(C) \).
   (c) If \( C = C_1 \cup \ldots \cup C_r \) and \( D = D_1 \cup \ldots \cup D_s \) are decompositions into irreducible components, then
      \[
      m_p(C, D) = \sum_{i,j} m_p(C_i, D_j)
      \]

2. Consider the space \( \mathbb{P}^5 \) parametrizing all conics in \( \mathbb{P}^2 \) as follows: to any \( [a, b, c, d, e, f] \in \mathbb{P}^5 \) we associate the symmetric quadratic form whose matrix of coefficients in given by:
   \[
   \begin{pmatrix}
   a & b & c \\
   b & d & e \\
   c & e & f
   \end{pmatrix}
   \]
   In other words, it corresponds to the conic with equation:
   \[
   ax_0^2 + 2bx_0x_1 + 2cx_0x_2 + dx_1^2 + 2ex_1x_2 + fx_2^2 = 0
   \]
   (Assume that the characteristic is not 2.)
   Let \( \Sigma \), respectively \( \Gamma \), be the locus parametrizing conics for which the corresponding quadratic polynomial is reducible, respectively the square of a linear form (i.e., this is the locus of matrices of rank \( \leq 2 \), resp. rank \( \leq 1 \)).
   (a) Prove that \( \Gamma \) is the image of the Veronese map \( v_2 : \mathbb{P}^2 \to \mathbb{P}^5 \), where \( \mathbb{P}^2 \) denotes the dual \( \mathbb{P}^2 \), i.e., the space \( G(1, 2) \) of lines in \( \mathbb{P}^2 \) (a line \( ax_0 + bx_1 + cx_2 = 0 \) corresponds in \( \mathbb{P}^2 \) to the point \( [a, b, c] \)).
   (b) Prove that \( \Sigma \) is a cubic hypersurface.
   (c) Prove that \( \Sigma \) is the secant variety to \( \Gamma \), i.e., the closure of the union of all the lines in \( \mathbb{P}^5 \) that intersect \( \Gamma \) in at least 2 points.

3. Let \( C = V_+(F) \subset \mathbb{P}^2 \) be a smooth curve (\( F \) is a homogeneous polynomial in \( k[X, Y, Z] \)). Consider the morphism:
   \[
   \phi_C : C \to \mathbb{P}^2, \quad p \mapsto (\frac{\partial f}{\partial x_0}(p), \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p)).
   \]
   The image \( \phi(C) \) is called the dual curve of \( C \).
   (a) Find a geometric description of \( \phi \). What does it mean geometrically that \( \phi(p) = \phi(q) \) for two distinct points \( p, q \in C \)?
   (b) If \( C \) is a conic, prove that its dual \( \phi(C) \) is also a conic.
   (c) For any five lines in \( \mathbb{P}^2 \) in general position (what does this mean?) show that there is a unique conic in \( \mathbb{P}^2 \) that is tangent to these five lines.
4. Assume $\text{char } \overline{k} \neq 2,3$. Prove that a smooth plane cubic $C$ has nine distinct inflection points. Equivalently prove that $m_p(C, \text{Hess}(C)) = 1$ for every inflection point $p$ of $C$. (Hint: you may assume that the inflection point $p$ is $[0,0,1]$ and the tangent line to $C$ at $p$ is $y = 0$. Then prove that if $f = f(x,y)$ is the local equation of $C$ then in the local ring $\mathcal{O}_{k^2,p}$, we have $\langle f, H(x,y,1) \rangle = \langle x, y \rangle$. Note that $H(x,y,1) = yp(x,y) + H(x,0,1)$ and $H(x,0,1) = xv$, where $v$ is a unit in $\mathcal{O}_{k^2,p}$. Alternatively, you may use the Weierstrass normal form of the cubic to prove this.)

5. (*The $j$-invariant.*) Prove that $j(\lambda) = j(\mu)$ if and only if $E_\lambda \cong E_\mu$.
   Hint: prove that $j(\lambda) = j(\mu)$ if and only if
   \[ \mu \in \{ \lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \} . \]

6. Let $C$ be a smooth cubic and let $O \in C$. Consider the group structure $(C, \oplus)$ that has $O$ as the identity. Prove that:
   (i) The inflection points of $C$ correspond to the torsion points of order 3.
   (ii) If $P, Q$ are inflection points and $R$ is the third point of intersection of the line $\overline{PQ}$ with $C$, then $R$ is an inflection point.
   (iii) Prove that if $O \in C$ is an inflection point, then the nine inflection points form a subgroup of $C$ that is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. 