Picard Groups of Affine Curves

Victor I. Piercey
University of Arizona
Math 518

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Abstract

We will develop a purely algebraic definition for the Picard group of an affine variety. We will then develop computational techniques for Picard groups of affine curves. If the curve is non-singular, the technique will require some geometry. However, in the singular case, we will find a purely algebraic tool. Along the way we will obtain an explicit description of the Picard group for every non-singular affine curve over $\mathbb{C}$ as well as affine curves with a single cusp or node singularity.

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1 Introduction

In algebraic geometry, one encounters invertible sheaves (or equivalently, line bundles) over an algebraic variety. Invertible sheaves over a fixed algebraic variety form a group under the tensor product called the Picard group. The elements of the Picard group are inherently geometric objects.

We will begin with a purely algebraic definition of invertible sheaves and the Picard group. The algebraic varieties will be replaced by commutative rings with identity. If the variety is affine, the corresponding ring is the coordinate ring. Invertible sheaves in this context are invertible modules. The invertible modules form a group under the tensor product which we will call the Picard group. We will proceed to develop methods to compute the Picard groups of coordinate rings of affine curves. When the curve is nonsingular, we will have to use some geometry. We will have an explicit description if the underlying field is $\mathbb{C}$. When the curve is singular, the technique is purely algebraic. It involves constructing a "Meier-Veitoris" sequence out of a commutative diagram called the "conductor square." We will illustrate this method by computing the Picard group explicitly for a curve with a cusp and a curve with a node.

2 Invertible Modules and the Picard Group

2.1 Basic Definitions

First we define what it means for a module over a ring to be invertible.

Definition 2.1. For a ring A, an A-module I is **invertible** if I is finitely generated and for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, $I_\mathfrak{p} \cong A_\mathfrak{p}$ as $A_\mathfrak{p}$-modules.

Remark 2.2. The condition that $I_\mathfrak{p} \cong A_\mathfrak{p}$ is often described by saying that I is locally free of rank 1.

If I is an ideal in A and is invertible as an A-module, we say that I is an **invertible ideal**. The notation I for the modules is chosen because we will see that every invertible module is isomorphic to an invertible ideal.

Before proceeding, we consider a couple of examples.

Example 2.3. If I is a free module of rank 1, then I is invertible. Note that the isomorphism class of the free module of rank 1 will play a special role shortly.

Example 2.4. Any principal ideal generated by a nonzero divisor is invertible.

Example 2.5. Let $A = \mathbb{Z}[\sqrt{-5}]$ and let $I = (2, 1 + \sqrt{-5})$. I claim I is not principle, but I is locally free of rank 1.

Suppose $I = (x)$ for some $x \in A$. Let the norm $N : A \to \mathbb{Z}$ be given by $N(a + b\sqrt{-5}) = a^2 + 5b^2$. Then $2 \in I$ and $N(2) = 4$, hence $N(x)$ divides 4. Since $1 + \sqrt{-5} \in I$ and $N(1 + \sqrt{-5}) = 6$, $N(x)$ divides 6. Therefore $N(x) = \pm 1$ or ± 2. Since I is a proper ideal, x is not a unit and therefore $N(x) \neq \pm 1$. Thus $N(x) = \pm 2$, and since $N(x) > 0$, it must be that $N(x) = 2$. But $a^2 + 5b^2 = 2$ requires $b = 0$, and there is no solution to $a^2 = 2$ in $\mathbb{Z}$. Therefore I is not principle.

Now we will show that I is locally free of rank 1. Note that I is a maximal ideal, since

$$\mathbb{Z}[\sqrt{-5}]/I \cong \mathbb{F}_2.$$

Now, if $I \not\subseteq \mathfrak{p}$, then $I \cap (A \setminus \mathfrak{p}) \neq \varnothing$ which implies $I_\mathfrak{p} = A_\mathfrak{p}$. Therefore we suppose $I \subset \mathfrak{p}$. But then $I = \mathfrak{p}$ since I is maximal. Since $3 \notin I$ (otherwise $1 \in I$), 3 is invertible in $A_\mathfrak{p}$. Therefore:

$$2 = \frac{(1 + \sqrt{-5})(1 - \sqrt{-5})}{3} \in \mathfrak{p}A_\mathfrak{p}.$$

It follows that $I_\mathfrak{p} = (1 + \sqrt{-5})_\mathfrak{p}$ which is principal and hence isomorphic to $A_\mathfrak{p}$ as $A_\mathfrak{p}$-modules.

Example 2.6. Let k be an algebraically closed field and let $A = k[x, y]/(y^2 - x^3 + x)$. The ring A is the coordinate ring for a non-singular, affine elliptic curve. Since the curve is non-singular, A is a Dedekind domain. I claim every maximal ideal $\mathfrak{m} \in \operatorname{Max}(A)$ is invertible. Since the only prime ideal $\mathfrak{p}$ in $\operatorname{Spec}(A)$ with $\mathfrak{m} \subset \mathfrak{p}$ is $\mathfrak{p} = \mathfrak{m}$, it follows that $\mathfrak{m}_\mathfrak{p} = A_\mathfrak{p}$ for all $\mathfrak{p} \neq \mathfrak{m}$. Furthermore, since A is a Dedekind domain, $A_\mathfrak{m}$ is a discrete valuation ring whose unique maximal ideal $\mathfrak{m}A_\mathfrak{m}$ is principal (see Proposition 9.2 in [AM69]). Thus the maximal ideals are invertible. On the other hand, note that these ideals are not principal. There is a geometric reason for this. It comes down to the fact that when this affine curve is embedded in projective space $\mathbb{P}^2_k$, the result is a curve of genus 1. If all maximal ideals were principal, then all ideals would be principal (since every ideal in a Dedekind domain is a finite product of primes). This would imply that the curve has genus zero. We will see a sketch of the last implication below.

Remark 2.7. Note that to check whether or not a finitely generated module I is invertible, it suffices to check that $I_\mathfrak{m} \cong A_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(A)$.

2.2 The Picard Group

The isomorphism classes of invertible modules from a group, called the Picard group. The operation is given by the tensor product. The identity is the isomorphism class of A as a module over itself. If I is an invertible module, its inverse is the dual module $I^* = \operatorname{Hom}_A(I, A)$. Hence invertible modules are modules that have "inverses" in this sense.

Before proving the theorem that justifies the statements above, we need a couple of preliminaries. First of all, given an A-module I, there is a natural map $\mu : I^* \otimes I \to A$ defined by $\varphi \otimes a \mapsto \varphi(a)$. The map μ is the unique homomorphism induced by the duality pairing on $I^* \times I$.

Let $K = K(A)$ denote the total ring of fractions. Since there is a natural map $A \to K(A)$, $K(A)$ is an A-module. The A-submodules of $K(A)$ are called **fractional ideals** of A. If I is a finitely generated fractional ideal of A, we write all the generators over a common denominator. This shows that I is isomorphic as an A-module to an ordinary ideal of A. For any set $I \subset K(A)$, define $I^{-1} := \{s \in K(A) \mid sI \subset A\}$.

Theorem 2.8. Let A be a Noetherian domain.

1. If I is an A-module, then I is invertible if and only if the natural map $\mu : I^* \otimes I \to A$ is an isomorphism.

2. Every invertible module is isomorphic to a fractional ideal of A. Every invertible fractional ideal contains a non-zero-divisor of A.

3. If $I, J \subset K(A)$ are invertible modules, then the natural map $I \otimes J \to IJ$ given by $s \otimes t \mapsto st$ is an isomorphism, as is the natural map $I^{-1}J \to \mathrm{Hom}_A(I, J)$ given by $t \mapsto \varphi_t$ where $\varphi_t(a) = ta$. In particular, $I^{-1} \cong I^*$.

4. If $I \subset K(A)$ is any A-submodule, then I is invertible if and only if $I^{-1}I = A$.

PROOF: First, suppose I is invertible. Then for any prime $\mathfrak{p} \in \mathrm{Spec}(A)$,

$$A_{\mathfrak{p}}^* \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} = \mathrm{Hom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}, A_{\mathfrak{p}}) = A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}$$

and μ localizes to the natural isomorphism

$$\mu_{\mathfrak{p}} : A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \to A_{\mathfrak{p}}.$$

Therefore, since the property of being an isomorphism is local, it follows that μ is an isomorphism.

Suppose now that μ is an isomorphism. Then there are some φ_i and a_i such that

$$\mu\left(\sum_{i=1}^{n} \varphi_i \otimes a_i\right) = 1.$$

Since μ is an isomorphism, for any prime $\mathfrak{p}$ we have an isomorphism $\mu_p : (I^* \otimes_A I)_{\mathfrak{p}} = I_{\mathfrak{p}}^* \otimes_{A_{\mathfrak{p}}} I_{\mathfrak{p}} \to A_{\mathfrak{p}}$. I claim that $I_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ where $I_{\mathfrak{p}}$ is generated by some a_i.

Since $\mu(\varphi_i \otimes a_i) = \sum \varphi_i(a_i) = 1$, and $\mathfrak{p}$ is a proper ideal of A, it follows that there is some i such that $\varphi_i(a_i) \notin \mathfrak{p}$. Therefore we can define

$$v = \frac{1}{\varphi_i(a_i)} \in A_{\mathfrak{p}}$$

and set $a = va_i \in A_{\mathfrak{p}}$. It follows that $(\varphi_i)_{\mathfrak{p}}(a) = 1$. Note that $(\varphi_i)_{\mathfrak{p}}(a) = 1$ implies a is not a zero devisor in $A_{\mathfrak{p}}$. Therefore if we denote the cyclic $A_{\mathfrak{p}}$-submodule of $I_{\mathfrak{p}}$ generated by a by $A_{\mathfrak{p}}a$ then $A_{\mathfrak{p}}a \cong A_{\mathfrak{p}}$ as $A_{\mathfrak{p}}$-modules and $\ker(\varphi_i)_{\mathfrak{p}} \cap A_{\mathfrak{p}}a = 0$. Now suppose for some $\alpha \in I_{\mathfrak{p}}$, $(\varphi_i)_{\mathfrak{p}}(\alpha) = \beta \neq 0 \in A_{\mathfrak{p}}$. Then $(\varphi_i)_{\mathfrak{p}}(\alpha - \beta a) = 0$, so $\alpha - \beta a \in \ker(\varphi_i)_{\mathfrak{p}}$. Thus it follows that $I_{\mathfrak{p}} = A_p a \oplus \ker(\varphi_i)_{\mathfrak{p}}$. Similarly, we can consider the dual element $\varphi_a \in \mathrm{Hom}_{A_{\mathfrak{p}}}(I_{\mathfrak{p}}^*, A_{\mathfrak{p}})$ defined by $\varphi_a(a) = 1$ and $\varphi_a(b) = 0$ for any $b \notin A_{\mathfrak{p}}a$. By the same argument as above, we have $I_{\mathfrak{p}}^* = A_{\mathfrak{p}}\varphi_i \oplus \ker(\varphi_a)$ with $A_{\mathfrak{p}}\varphi_i \cong A_{\mathfrak{p}}$. Thus:

$$I_{\mathfrak{p}}^* \otimes_{A_{\mathfrak{p}}} I_{\mathfrak{p}} = (A_{\mathfrak{p}}\varphi_i \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}a) \oplus (A_{\mathfrak{p}}\varphi_i \otimes \ker(\varphi_i)_{\mathfrak{p}}) \oplus \cdots .$$

Since

$$\mu_{\mathfrak{p}}(A_{\mathfrak{p}}\varphi_i \otimes_{A_{\mathfrak{p}}} \ker(\varphi_i)_{\mathfrak{p}}) = (\varphi_i)_{\mathfrak{p}}(\ker(\varphi_i)_{\mathfrak{p}}) = 0$$

and $\mu_{\mathfrak{p}}$ is an isomorphism, $A_{\mathfrak{p}}\varphi_i \otimes_{A_{\mathfrak{p}}} \ker(\varphi_i)_{\mathfrak{p}} = 0$. Since $A_{\mathfrak{p}}\varphi_i \cong A_{\mathfrak{p}}$, it follows that $\ker(\varphi_i)_{\mathfrak{p}} = 0$ and $(\varphi_i)_{\mathfrak{p}}$ is an isomorphism. Hence $I_{\mathfrak{p}} = A_{\mathfrak{p}}a \cong A_{\mathfrak{p}}$. Therefore I is locally free of rank 1. Note that a_i is mapped to a generator in $I_\mathfrak{p}$. It follows that the natural map $(a_1, \ldots, a_n) \to I$ is locally surjective, hence surjective. Thus I is generated by $\{a_1, \ldots, a_n\}$. Therefore I is finitely generated and locally free of rank 1. Thus I is invertible. This proves the first statement.

For the second statement, suppose I is an invertible module of A. Note that since A is a domain, $(0) \in \mathrm{Spec}A$ and $K = A_{(0)}$. Then by definition and the assumption that I is invertible:

$$I \otimes_A K = I \otimes_A A_{(0)} \cong I_{(0)} \cong A_{(0)} = K.$$

Therefore it suffices to find an embedding

$$I \hookrightarrow I \otimes_A K \cong I_{(0)}.$$

I claim that the natural map $\eta : I \to I_{(0)}$ is a monomorphism. First observe:

$$\begin{aligned} \eta(\alpha) = 0 \quad &\Leftrightarrow \quad \frac{\alpha}{1} = 0 \\ &\Leftrightarrow \quad u\alpha = 0 \text{ for some } u \in A \setminus \{0\}. \end{aligned}$$

Let $\mathfrak{m} \in \mathrm{Max}A$ be arbitrary. Localize at $\mathfrak{m}$ and view u and α in $A_\mathfrak{m}$. Then $u\alpha = 0$ in $I_\mathfrak{m} \cong A_\mathfrak{m}$. Since $A_\mathfrak{m}$ is a domain, $u = 0$ or $\alpha = 0$ in $A_\mathfrak{m}$. If $u = 0$ in $A_\mathfrak{m}$, then there is some element $u' \in A \setminus \mathfrak{m}$ such that $u'u = 0$. This is not possible, since $0 \in \mathfrak{m}$ implies $u' \neq 0$ while $u \neq 0$ in A and A being a domain implies that u is not a zero divisor. Thus $u \neq 0$ in $A_\mathfrak{m}$ and therefore $\alpha = 0$ in $A_\mathfrak{m}$. Since $\mathfrak{m} \in \mathrm{Max}A$ was arbitrary, it follows that $\alpha = 0$ in $I_\mathfrak{m}$ for every $\mathfrak{m}$ and thus $\alpha = 0$. Therefore η is injective and thus is an embedding. It follows that $I \cong \eta(A) \subset K$. This proves the first part of the second statement.

Now suppose $I \subset K$ is a finitely generated fractional ideal such that $I \cap A = \{0\}$. Let u be the product of the denominators of the generators of I. Then $u \in A$ and $uI \subset A \cap I = \{0\}$. Therefore, for every $\alpha \in I$, $u\alpha = 0$. Since $A \hookrightarrow K = A_{(0)}$ canonically, and $I \subset K$, then $u\alpha = 0$ for every $\alpha \in I$ is an equation in K. Since K is a field and $u \neq 0$, it follows that $I = (0)$. Since (0) is not invertible, it follows that I is not invertible. Therefore, by contrapositive logic, every invertible submodule of K contains a nonzero (hence nonzerodivisor) element of A. This completes the proof of the second statement.

Now if I and J are invertible modules, by the second statement we can assume that $I, J \subset K$. The map

$$I \otimes J \to IJ$$

given by $s \otimes t \mapsto st$ is an A-module homomorphism that is clearly surjective. Therefore it suffices to show that this map is injective. By exactness of localization, it suffices to show that this map is injective when localized at $\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}A$. Note that $K_\mathfrak{p} = K$. Therefore we wish to show that the map

$$I_\mathfrak{p} \otimes_{A_\mathfrak{p}} J_\mathfrak{p} \to (IJ)_\mathfrak{p} \subset K(A_\mathfrak{p}) \subset K$$

is injective. Since $A_\mathfrak{p}$ is a domain and $K(A_\mathfrak{p})$ is its field of fractions, we may assume A is a local ring.

If $(A, \mathfrak{m})$ is a local ring, then $A = A_\mathfrak{m}$. Therefore if I and J are invertible, $I = I_\mathfrak{m}$, $J = J_\mathfrak{m}$ and $A = A_\mathfrak{m}$ which implies that $I \cong A \cong J$ as A-modules. Therefore there are nonzero elements $s, t \in K$ such that $I = (s)$, $J = (t)$ and $IJ = (st)$. It follows that the desired map is injective, completing the first part of the third statement.

Now consider the map $I^{-1}J \to \mathrm{Hom}_A(I, J)$ given by $t \mapsto \varphi_t$ where $\varphi_t(a) = ta$. Note that $t \in I^{-1}J$ implies that

$$t = \sum a_i r_i s_i$$

with $a_i \in A$, $r_i \in I^{-1}$ and $s_i \in J$. Therefore:

$$ta = \left(\sum a_i r_i s_i\right) a = \sum a_i (r_i a) s_i.$$

Since $r_i \in I^{-1}$ and $a \in I$, $r_i a \in A$ and therefore $ta \in J$. It is clear that φ_t is a homomorphism. Now if $v \in A \cap I$ is nonzero (which exists by the second statement of the theorem proved above), then $tv \neq 0$ in K whenever $t \neq 0$. Thus $\varphi_t = 0$ if and only if $t = 0$ and our map is injective.

Therefore it suffices to show that the map $t \mapsto \varphi_t$ is surjective. It suffices to show this locally. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $\varphi \in \mathrm{Hom}_{A_\mathfrak{p}}(I_\mathfrak{p}, J_\mathfrak{p})$ be any homomorphism. Since I is invertible, $I_\mathfrak{p} \cong A_\mathfrak{p}$ and therefore $I_\mathfrak{p} = xA_\mathfrak{p}$ for some $x \in K(A)$. Let $w = \varphi(x)$. Note that if $\alpha = xa \in I_\mathfrak{p}$ (where $a \in A_\mathfrak{p}$), then $(1/x)(xa) = a \in A_\mathfrak{p}$ and therefore $1/x \in I_\mathfrak{p}^{-1}$. Then $w/x \in I_\mathfrak{p}^{-1} J_\mathfrak{p}$.[1] The claim is that $\varphi = \varphi_{w/x}$. Note that $\varphi_{w/x}(x) = w = \varphi(x)$. Since x is a generator of $I_\mathfrak{p}$, the claim follows. Therefore our map is surjective locally, and hence is surjective.

Finally, suppose $I \subset K$ is invertible. Then the map $\mu : I^* \otimes I \to A$ is an isomorphism, and by the third part proved above this map is the same as the multiplication map $I^{-1} \otimes I \to A$. It follows that $I^{-1}I \cong A$. Since $I^{-1}I \subset A$ it follows that $I^{-1}I = A$. This establishes one direction of the fourth statement.

Now suppose $I \subset K$ is an A-submodule with $I^{-1}I = A$. By localization, we can assume A is a local ring with maximal ideal $\mathfrak{m}$. It then suffices to show under this assumption, $I \cong A$. Since $I^{-1}I = A$, there is an element $v \in I^{-1}$ and an element $\alpha \in I$ such that $v\alpha \notin \mathfrak{m}$. Therefore $vI \not\subseteq \mathfrak{m}$. I claim that the map $v : I \to A$ given by $\alpha \mapsto v\alpha$ is an isomorphism. Since $vI \not\subseteq \mathfrak{m}$ implies $v \neq 0$, then $v\alpha = 0$ in $A \subset K$ implies $\alpha = 0$, so multiplication by v is injective. Now since $vI \not\subseteq \mathfrak{m}$, there is $\alpha \in I$ such that $v\alpha = y \notin \mathfrak{m}$. Since $\mathfrak{m}$ is the unique maximal ideal in the local ring A, it follows that $y \in A^\times$. Therefore we have $vy^{-1}\alpha = 1$ with $y^{-1}\alpha \in I$. It follows that if $a \in A$, then $v(ay^{-1}\alpha) = a$ with $ay^{-1}\alpha \in I$. Hence v is surjective. This completes the proof of the fourth statement and hence the theorem. ■

Corollary 2.9. The collection of isomorphism classes of invertible modules of A form a set.

PROOF: By the second part of Theorem 2.8, each isomorphism class of invertible modules of A has a representative as a submodule of K. Note that distinct invertible submodules of K may be isomorphic! In any case, the collection of isomorphism classes of invertible modules of A can be realized as a subset of the power set of K, and is therefore a set. ■

Corollary 2.10. The set of isomorphism classes of invertible modules of A is a group under $\otimes$ with identity element given by A (as a module over itself) and the inverse of I given by I^*.

PROOF: This is an immediate consequence of Theorem 2.8. ■

Definition 2.11. The **Picard group of A**, denoted by $\mathrm{Pic}(A)$, is the group of all isomorphism classes of invertible A-modules.

[1] I am tacitly asserting that $(I^{-1})_\mathfrak{p} = (I_\mathfrak{p})^{-1}$, which is true over a Noetherian ring.

Before considering Picard groups of rings of affine curves, we should look at the relationship between $\mathrm{Pic}(A)$ and invertible submodules of K. We will exploit this relationship in the next section. Observe that the collection of all invertible submodules of K (NOT isomorphism classes!) is a subset of the power set of K and is therefore a set. This set also forms a group with identity element given by A as a module over itself and where the inverse of I is given by I^{-1} (hence the notation!). All of this follows from Theorem 2.8. This allows us to give the following definition.

Definition 2.12. The group of invertible submodules of K is called the group of **Cartier divisors** and denoted by $C(A)$.

Definition 2.13. A **principal divisor** is an element of $C(A)$ of the form Au for $u \in K^\times$. Denote the set of principal divisors by $PC(A)$.

Remark 2.14. Note that if Au and Av are principal divisors, then $Au \otimes Av = A(uv)$, and $uv \in K^\times$. It follows that $PC(A)$ is a subgroup of $C(A)$.

Lemma 2.15. $PC(A) \cong K^\times/A^\times$.

PROOF: We clearly have a surjective map $K^\times \to PC(A)$ given by $u \mapsto Au$. Suppose $Au = Av$. Then there is $a \in A$ such that $u = av$. Similarly, there is $b \in A$ such that $v = bu$. It follows that $(ab)u = u$. Since u is a unit in K, it follows that $ab = 1$. Therefore u and v differ by a unit of A. Hence the kernel of our map is $A^\times$, completing the proof. ∎

The relationship between $C(A)$ and $\mathrm{Pic}(A)$ is given by the following.

Corollary 2.16. Let $\varphi : C(A) \to \mathrm{Pic}(A)$ be given by $\varphi(I) = [I]$ (where $[I]$ is the isomorphism class of the invertible module I).

1. The map φ is a surjective group homomorphism and $\ker \varphi = K^\times/A^\times$.

2. The group $C(A)$ is generated by the set of invertible ideals of A.

PROOF: It is clear that φ is a group homomorphism and surjectivity follows by part (2) of Theorem 2.8. Therefore it suffices by Lemma 2.15 to show that the kernel of φ is $PC(A)$. It is clear that if $Au \in PC(A)$, then $Au \cong A$ is an A-module under the map $u \mapsto 1$. Therefore one inclusion is obvious and it suffices to show the other inclusion. If I is any invertible module and $Au \in PC(A)$, then $(Au)I = u(AI) \subset uI$ on the one hand, while clearly $uI = (1 \cdot u)I \subset (Au)I$. Therefore $(Au)I = uI$ for any $Au \in PC(A)$. Accordingly, it suffices to show that if $I, J \in C(A)$ and $I \cong J$, then $J = uI$ for some $u \in K^\times$. But this is immediate from the third statement in Theorem 2.8, since any isomorphism $\psi : I \to J$ is an element of $\mathrm{Hom}_A(I, J)$ and thus can be realized as multiplication by a nonzero element $u \in I^{-1}J$ (since multiplication by zero would not be an isomorphism). Any nonzero element of $I^{-1}J$ is a nonzero element of K and therefore a unit of K. Thus $J = \psi(I) = uI$ with $u \in K^\times$. This proves the first statement.

For the second statement, suppose $I \subset K$ is an invertible fractional ideal. Then I^{-1} is an invertible fractional ideal by part (4) of Theorem 2.8, and therefore by the second part of Theorem 2.8, there is a nonzero element $a \in A \cap I^{-1}$. Since $a \in I^{-1}$, $aI \subset A$. We now have $I = aI \cdot (a)^{-1}$, where aI and $(a)^{-1}$ are invertible ideals of A. ∎

Corollary 2.17. $\mathrm{Pic}(A) \cong C(A)/PC(A)$.

PROOF: This is immediate from Corollaries 2.16 and 2.15. ∎

3 Picard Groups of Non-Singular Curves

We will now determine a method for computing Picard groups of nonsingular curves over an algebraically closed field k. If X is a nonsingular affine curve, its Picard group $\mathrm{Pic}(X)$ is the Picard group $\mathrm{Pic}(A)$ where A is the affine coordinate ring $k[x_1, \ldots, x_n]/I$ of the curve. We will be able to describe these Picard groups explicitly when the ground field is $\mathbb{C}$. To lay the ground work for this discussion, we begin with objects called Weil divisors. The Weil divisors form a group, and after taking a quotient we obtain a group called the class group of the curve. We will then relate the class group, the group of Cartier divisors, and the Picard group. This will allow us to write down a split short exact sequence that can be used to compute the Picard group of a nonsingular curve embedded in projective space. We can then pull this program back to obtain the Picard group of the affine curve.

3.1 Weil Divisors

We begin with the definition of a Weil divisor.

Definition 3.1. The **group of Weil divisors** of a ring A, denoted $\mathrm{Div}(A)$, is the free abelian group on the set of codimension one prime ideals of A. A **Weil divisor** is an element of $\mathrm{Div}(A)$.

Weil divisors and Cartier divisors are generally speaking very different. However, the following shows that when A is a Dedekind domain, they are actually the same. This will suffice for us since affine rings of nonsingular curves are Dedekind domain.

Lemma 3.2. Every nonzero proper ideal $I \subset A$ of a Dedekind domain A can be written uniquely as the product of finitely many prime ideals.

PROOF: See Theorem 3.7 in [J.S08]. ∎

Lemma 3.3. A product of invertible ideals in a Noetherian ring A is invertible.

PROOF: Let $I, J \subset A$ be invertible. View I and J as fractional ideals in $K(A)$. Then

$$(IJ)(IJ)^{-1} = (IJ)(J^{-1}I^{-1}) = I(JJ^{-1})I^{-1} = IAI^{-1} = II^{-1} = A.$$

Therefore IJ is invertible. ∎

Lemma 3.4. Every nonzero prime ideal in a Dedekind domain A is invertible.

PROOF: If $\mathfrak{p} \subset A$ is a nonzero prime ideal, since A is a Dedekind domain, $\mathfrak{p}$ is maximal. Hence for any other prime ideal $\mathfrak{q} \subset A$, $\mathfrak{p}A_\mathfrak{q} = A_\mathfrak{q}$. Since A is a Dedekind domain, $A_\mathfrak{p}$ is a discrete valuation ring and hence its unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$ is principal (see [AM69], Proposition 9.2). Therefore as $A_\mathfrak{p}$-modules, $\mathfrak{p}A_\mathfrak{p} \cong A_\mathfrak{p}$. Hence $\mathfrak{p}$ is invertible. ∎

Lemma 3.5. Every nonzero ideal in a Dedekind domain A is invertible.

PROOF: This is an immediate consequence of Lemma 3.2, Lemma 3.3 and Lemma 3.4. ∎

Proposition 3.6. If A is a Dedekind domain, the Cartier divisors $C(A)$ is the free abelian group on the set of maximal ideals of A.

PROOF: First of all, suppose $I \subset A$ is a proper invertible ideal. Then I is nonzero, so I is an element of the free abelian group generated by the nonzero prime ideals of A. Since any nonzero prime ideal is a maximal ideal in a Dedekind domain, it follows that I is an element of the free abelian group on the maximal ideals of A. Finally, every fractional ideal in $C(A)$ is an element of the free abelian group on maximal ideals of A by the second part of Corollary 2.16. The reverse inclusion is obvious by Lemma 3.3 and Lemma 3.4. ■

We now have the following immediate corollary.

Corollary 3.7. In A is a Dedekind domain, $C(A) = \mathrm{Div}(A)$.

PROOF: In a Dedekind domain, by definition the codimension 1 prime ideals are precisely the maximal ideals. The result therefore follows from Proposition 3.6. ■

As above, in the case where A is a Dedekind domain, principal divisors in $\mathrm{Div}(A)$ are the same as in $C(A)$.

Definition 3.8. Let A be a Dedekind domain. Then the group of Weil divisors modulo principal divisors is called the **class group** and denoted

$$\mathrm{Cl}(A) \cong \mathrm{Div}(A)/PC(A).$$

Two elements that are equivalent modulo $\mathrm{PC}(A)$ are said to be **linearly equivalent**.

We now have the following corollary.

Corollary 3.9. If A is a Dedekind domain, $\mathrm{Pic}(A) \cong \mathrm{Cl}(A)$.

PROOF: By the first part of Corollary 2.16, and by Corollary 3.7 since A is a Dedekind domain, we have:

$$\mathrm{Pic}(A) \cong C(A)/PC(A) = \mathrm{Div}(A)/PC(A) = \mathrm{Cl}(A)$$

as claimed. ■

Remark 3.10. Suppose A is a Dedekind domain. Then A is a unique factorization domain if and only if $\mathrm{Pic}(A)$ is trivial.

PROOF: First of all, it is clear that $PC(A)$ can be identified with the principal ideals of A. Therefore it is clear that in a Dedekind domain, A is a principal ideal domain if and only if $PC(A) = \mathrm{Div}(A)$, which holds if and only if $\mathrm{Cl}(A)$ is trivial. In addition, if A is a Dedekind domain, A is a principal ideal domain if and only if A is a unique factorization domain. See Proposition 3.18 on page 45 of [J.S08] for the proof. Finally, since $\mathrm{Pic}(A) \cong \mathrm{Cl}(A)$ for a Dedekind domain A, it follows that A is a uniform factorization domain if and only if $\mathrm{Cl}(A)$ is trivial if and only if $\mathrm{Pic}(A)$ is trivial. ■

3.2 Nonsingular Curves

We now start with a nonsingular curve X of genus g embedded into projective space $X \hookrightarrow \mathbb{P}^N$ for some N. This embedding allows the curve to satisfy a "completeness" hypothesis. So far we have only discussed $\mathrm{Pic}(A)$ for a ring A and by extension $\mathrm{Pic}(X)$ for an affine curve X. A projective curve does not correspond to a single coordinate ring. Instead, it has affine "patches" that have corresponding coordinate rings, which may not all be the same.[2] Therefore it is necessary to make sense of the Picard group in this situation. Taking a collection of invertible modules over the affine coordinate rings and gluing them together in a consistent manner forms a sheaf of modules that is locally free of rank one. Such a sheaf is called an **invertible sheaf**. Invertible sheaves are precisely line bundles over the curve. In the same way that invertible modules form a group, the invertible sheaves form a group under tensor product. The identity element is the trivial sheaf and the inverse is given by the dual. The group of isomorphism classes of invertible sheaves over X is now the Picard group $\mathrm{Pic}(X)$. Restricting to an affine part of the curve gives the Picard of the corresponding coordinate ring.

We will relate the Picard group $\mathrm{Pic}(X)$ to the divisor class group $\mathrm{Cl}(X)$ when X is nonsingular. To do so, we also need to make sense of Weil divisors on a projective curve. A Weil divisor can be thought of as a formal (finite) sum of integral multiples of points on the projective curve. The group of Weil divisors $\mathrm{Div}(X)$ is then the free abelian group on the points of the curve. Linear equivalence is defined a little differently from above. One defines a principal divisor to be the formal sum of zeros and poles of certain functions on the curve whose coefficients are the corresponding order (where the order of the zero or pole is determined by a certain discrete valuation on a certain function field on the curve). See Chapter II, Lemma 6.1 and definitions that follow in [Har77] for details.[3] We can then form the quotient $\mathrm{Cl}(X)$ as above. The class group is useful here because of the following result.

Lemma 3.11. There is a surjective group homomorphism $D : \mathrm{Div}(X) \to \mathbb{Z}$ that descends to a well defined group homomorphism $d : \mathrm{Cl}(X) \to \mathbb{Z}$.

PROOF: For a divisor $\sum n_i p_i$, define

$$D\left(\sum n_i p_i\right) = \sum n_i.$$

It is clear that this defines a group homomorphism. Surjectivity is also clear, since for any $n \in \mathbb{Z}$, $D(np) = n$. Finally, if (f) is a principal divisor, $D(f) = 0$. See Proposition 6.4 in Chapter II of [Har77]. Therefore if two divisors are linearly equivalent, they have the same image under D. ∎

Definition 3.12. The map d in Lemma 3.11 is called the **degree** map.

For a nonsingular projective curve X, we have $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$. Hence we have a degree map on $\mathrm{Pic}(X)$.

Definition 3.13. For an isomorphism class of invertible sheaves (line bundles) $\mathcal{L} \in \mathrm{Pic}(X)$, the **degree** of $\mathcal{L}$ is the image under d of the corresponding Weil divisor class. The isomorphism classes of invertible sheaves of degree zero are the kernel of the degree map and therefore form a subgroup of $\mathrm{Pic}(X)$ denoted by $\mathrm{Pic}^0(X)$.

Proposition 3.14. $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(X)$

PROOF: Since the degree map from $\mathrm{Pic}(X) \to \mathbb{Z}$ is surjective and its kernel is $\mathrm{Pic}^0(X)$, we have a short exact sequence of abelian groups

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, this short exact sequence splits and we obtain the desired result. ∎

[2]This actually describes the space of a scheme. The other data involved in a scheme involves keeping track of the coordinate rings of open (and hence affine) subsets. These coordinate rings form a sheaf, called the structure sheaf.

[3]The definitions are modeled after meromorphic functions on Riemann surfaces.

3.3 The Jacobian of a Riemann Surface and $\operatorname{Pic}(X)$ over $\mathbb{C}$

By Proposition 3.14, describing $\operatorname{Pic}(X)$ for a nonsingular projective curve X is reduced to describing $\operatorname{Pic}^0(X)$, the group of invertible sheaves (line bundles) of degree zero. This has a nice, explicit description if we assume the underlying ground field is $\mathbb{C}$. Over $\mathbb{C}$, a nonsingular projective curve X is a compact Riemann surface of genus g. To understand $\operatorname{Pic}^0(X)$ we take a brief detour and discuss the Jacobian of a Riemann surface.

Recall that for a Riemann surface X of genus g, the first singular homology group is given by

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

Recall that a holomorphic 1-form ω on X is closed. Accordingly, if $c = \partial A$ is a boundary in $H_1(X, \mathbb{Z})$, then by Stokes' theorem:

$$\int_c \omega = \int_{\partial A} \omega = \int_A \mathrm{d}\omega = 0.$$

Accordingly, denoting the vector space of holomorphic one-forms by $\Omega^1(X)$, for each $[c] \in H_1(X, \mathbb{Z})$ we have a well-defined linear functional:

$$\int_{[c]} : \Omega^1(X) \to \mathbb{C}.$$

This gives us an embedding $H_1(X, \mathbb{Z}) \hookrightarrow (\Omega^1(X))^*$.

Definition 3.15. The Jacobian of the curve X, denoted by $J(X)$, is defined as:

$$J(X) = (\Omega^1(X))^* / H_1(X, \mathbb{Z}).$$

Proposition 3.16. If X is a Riemann surface of genus g, then

$$J(X) \cong \mathbb{T}^{2g}$$

where the isomorphism is in the category of abelian groups.

PROOF: The dual $(\Omega^1(X))^*$ is a complex vector space of dimension g, the genus of X. Identify $(\Omega^1(X))^*$ with $\mathbb{C}^g$. The corresponding embedding $H_1(X, \mathbb{Z}) \hookrightarrow \mathbb{C}^g$ gives us $H_1(X, \mathbb{Z}) \cong \Lambda$ where $\Lambda \subset \mathbb{C}^g$ is a rank $2g$ lattice. Accordingly, as abelian groups:

$$J(X) \cong \mathbb{C}^g / \Lambda \cong \mathbb{R}^{2g} / \mathbb{Z}^{2g} \cong \mathbb{T}^{2g}$$

which is the desired result. ∎

A very deep theorem of Abel and Jacobi shows that $\operatorname{Pic}^0(X) \cong J(X)$. See Chapter VIII of [Mir95] for details. This gives us the following immediate corollary.

Corollary 3.17. If X is a nonsingular projective curve over $\mathbb{C}$ of genus g, then

$$\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{T}^{2g}.$$

PROOF: By Proposition 3.14, the Abel-Jacobi theorem, and Proposition 3.16:

$$\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \operatorname{Pic}^0(X) \cong \mathbb{Z} \oplus J(X) \cong \mathbb{Z} \oplus \mathbb{T}^{2g}.$$

Remark 3.18. As an interesting observation, note that if X is an elliptic curve, $\operatorname{Pic}(X) \cong \mathbb{Z} \oplus X$.

Remark 3.19. A curve of genus zero is called a **rational** curve. A rational curve over $\mathbb{C}$ embedded in projective space has Picard group $\operatorname{Pic}(X) \cong \mathbb{Z}$. Since $X \cong \mathbb{P}^1$, we can explicitly describe a generator for the Picard group. The generator is the so-called tautological bundle. This is the line bundle over $\mathbb{P}^1$ where the fiber over a point $\ell \in \mathbb{P}^1$ is the line in $\mathbb{C}^2$ represented by ℓ.

3.4 Non-Singular Affine Curves

Now suppose $A = k[x_1, \ldots, x_n]/I$ is the coordinate ring of a non-singular affine curve X. To compute $\operatorname{Pic}(A)$, we will "complete" the curve to a projective curve X', obtaining the Picard group $\operatorname{Pic}(X')$ as above, and determine how the Picard group responds when restricting to the affine part that we started with.

The first observation is that our coordinate ring gives us a closed embedding of our curve $X := \operatorname{Spec} A$ into $\mathbb{A}^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. Complete $\mathbb{A}^n$ to $\mathbb{P}^n$. This amounts to adding in the "hyperplane at infinity", which can be chosen to be the hyperplane in $\mathbb{P}^n$ given by the homogeneous equation $z_0 = 0$ where $\mathbb{P}^n$ is given homogeneous coordinates $[z_0 : z_1 : \cdots : z_n]$. The curve X is then completed using this embedding, and it requires the addition of finitely many points. Therefore, given our complete curve X' in $\mathbb{P}^n$, we can return to X by removing finitely many points. The question therefore becomes how the Picard group of X' changes when finitely many points are removed.

To see how this works, we exploit again the isomorphism between the Picard group $\operatorname{Pic}(X')$ and the class group $\operatorname{Cl}(X')$.

Proposition 3.20. Let X' be a nonsingular curve embedded in projective space $\mathbb{P}^n$. Let

$$Z = \{p_1, \ldots, p_k\} \subset X'$$

be a finite subset of the points of X and let $U = X' \setminus Z$. Then there is an exact sequence:

$$\mathbb{Z}^k \to \operatorname{Cl}(X') \to \operatorname{Cl}(U) \to 0.$$

PROOF: Map $\mathbb{Z}^k \to \operatorname{Cl}(X')$ by the map sending the i^{th} generator of $\mathbb{Z}^k$ to the divisor p_i. Then the image is the residue of the subgroup $\mathbb{Z}p_1 + \mathbb{Z}p_2 + \ldots + \mathbb{Z}p_k$ modulo principal divisors. Define the map $\operatorname{Cl}(X') \to \operatorname{Cl}(U)$ by mapping a divisor of the form $\sum n_i q_i$ on X to a divisor $\sum \widehat{n_i} q_i$ where $\widehat{n_i} = n_i$ if $q_i \in U$ and zero otherwise. Then this map is surjective, and the kernel is precisely the image of $\mathbb{Z}^k$. Therefore this sequence is exact. ∎

Corollary 3.21. Let A be the coordinate ring of an affine non-singular curve X of genus g over $\mathbb{C}$. Let k be the number of points added in its embedding into projective space. Then:

$$\operatorname{Pic}(A) \cong \mathbb{T}^{2g}/\mathbb{Z}^{k-1}.$$

PROOF: Let X' denote the image of X in projective space. By Corollary 3.17 we have

$$\operatorname{Pic}(X') \cong \mathbb{Z} \oplus \mathbb{T}^{2g}.$$

Let Z be the set of points of $X' \setminus X$. Then (since $\operatorname{Pic}(A) = \operatorname{Pic}(X)$) by Proposition 3.20, we have an exact sequence:

$$\mathbb{Z}^k \to \operatorname{Pic}(X') \to \operatorname{Pic}(A) \to 0$$

and hence

$$\operatorname{Pic}(A) \cong \operatorname{Pic}(X')/\mathbb{Z}^k \cong (\mathbb{Z} \oplus \mathbb{T}^{2g})/\mathbb{Z}^k \cong \mathbb{T}^{2g}/\mathbb{Z}^{k-1}$$

as desired. ∎

Remark 3.22. Over an arbitrary field k, the same argument shows that under the same hypotheses, $\operatorname{Pic}(A) \cong \operatorname{Pic}^0(X')/\mathbb{Z}^{k-1}$.

One consequence is the following.

Corollary 3.23. Let A be the coordinate ring of an affine non-singular curve X. Then A is a unique factorization domain if and only if the curve is rational.

PROOF: I will prove this only over $\mathbb{C}$. Let k be the number of points at infinity for the curve X and let g be the genus. Then it follows that

$$\mathrm{Cl}(A) \cong \mathrm{Pic}(X) \cong \mathbb{T}^{2g}/\mathbb{Z}^{k-1}.$$

If $g \neq 0$, then $\mathbb{T}^{2g}$ is uncountable and $\mathbb{Z}^{k-1}$ is countable and hence $\mathrm{Cl}(A)$ is nontrivial and hence A is not a unique factorization domain. Conversely, if $g = 0$ then $\mathrm{Cl}(A)$ is trivial and hence A is a unique factorization domain by Remark 3.10.

The proof over an arbitrary field k requires showing that for the completion X' of X in projective space, $\mathrm{Pic}^0(X)$ is uncountable. ■

4 Singular Affine Curves

We now turn to the case where X is a singular affine curve. At the end of this discussion we will compute $\mathrm{Pic}(X)$ where $X = \mathrm{Spec}A$ for $A = k[x,y]/(y^2 - x^3)$ and $\mathrm{Pic}(Y)$ where $Y = \mathrm{Spec}B$ for $B = k[x,y]/(y^2 - x^2(x+1))$. The first curve is a curve with a cusp and the second curve has a node.

The method used for non-singular curves no longer works in the singular case. In the non-singular case, the ring A is a Dedekind domain and we have isomorphisms with the divisor class group. Affine coordinate rings for singular curves are Noetherian domains, but they are not Dedekind domains. Therefore a lot of the work we did above fails. To compute Picard groups in this new context, we will be able to map $\mathrm{Pic}(A)$ to $\mathrm{Pic}(A_1)$, where A_1 is the normalization of A and corresponds to an affine non-singular curve. We will be able to do this by constructing the beginning of a "Meyer-Vietoris" sequence for Picard groups. This sequence will be the conclusion of a series of lemmata involving constructing projective curves over a fiber product.

4.1 Projective Modules over Fiber Products

As a preliminary step, I will show that an invertible A-module is projective.

Proposition 4.1. If I is an invertible A-module, then I is a projective A-module.

PROOF: Suppose M and N are A-modules and that

$$0 \to M \to N \to I \to 0$$

is a short exact sequence. Localize at an arbitrary prime $\mathfrak{p}$ to give us an exact sequence of $A_\mathfrak{p}$-modules

$$0 \to M_\mathfrak{p} \to N_\mathfrak{p} \to I_\mathfrak{p} \to 0.$$

Since I is invertible, $I_\mathfrak{p} \cong A_\mathfrak{p}$ and is therefore a projective $A_\mathfrak{p}$-module. Accordingly, for any $\mathfrak{p} \in \mathrm{Spec}A$,

$$I_\mathfrak{p} \cong M_\mathfrak{p} \oplus N_\mathfrak{p} = (M \oplus N)_\mathfrak{p}.$$

Accordingly $I \cong M \oplus N$ which implies the original exact sequence splits. It follows that I is a projective A-module. ∎

Proposition 4.1 is the motivation for the work we will do to construct projective modules over fiber products. For the next several results, we assume the following is a fiber-product diagram of commutative rings with identity:

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & A_1 \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \beta_1} \\ A_2 & \xrightarrow[\beta_2]{} & S \end{array}$$

We will also assume β_1 is surjective. The construction of fiber products in the category of commutative rings with identity provides an explicit description of A:

$$A = \{(a_1, a_2) \in A_1 \times A_2 \bigm| \beta_1(a_1) = \beta_2(a_2)\}.$$

Suppose P_i is an A_i-module for $i = 1, 2$ and we have an S-module isomorphism $\varphi : P_1 \otimes_{A_1} S \to P_2 \otimes_{A_2} S$. Define:

$$M(P_1, P_2, \varphi) := \{(p_1, p_2) \in P_1 \times P_2 \bigm| \varphi(p_1 \otimes 1) = p_2 \otimes 1\}.$$

Lemma 4.2. $M(P_1, P_2, \varphi)$ is an A-module.

PROOF: For $(p_1, p_2) \in M(P_1, P_2, \varphi)$ and $a \in A$, define

$$a \cdot (p_1, p_2) = (\alpha_1(a) \cdot p_1, \alpha_2(a) \cdot p_2).$$

Then

$$\varphi(\underbrace{\alpha_1(a) \cdot p_1}_{A_1-\text{action}} \otimes 1) = \underbrace{\beta_1(\alpha_1(a))\varphi(p_1 \otimes 1)}_{S-\text{action}} = \beta_2(\alpha_2(a))\varphi(p_1 \otimes 1) = \alpha_2(a) \cdot p_2 \otimes 1.$$

It is clear that the module axioms are satisfied. ∎

Our first goal is to show that if P_i are A_i-modules, then $M(P_1, P_2, \varphi)$ is a projective A-module. We will then be able to use this result to derive the desired exact sequence. The arguments below are from [Mil71].

Suppose $f : A \to A'$ is a ring homomorphism and M is an A-module. Define the following notation (inspired by [Mil71]):

$$f_\sharp M = A' \otimes_A M.$$

Then $f_\sharp M$ is naturally an A'-module. In fact, $f_\sharp$ induces an additive covariant functor from the category of A-modules to the category of A'-modules. It is clear that if M is projective, free, or finitely generated then $f_\sharp M$ is projective, free or finitely generated respectively. There is a natural A-linear map $f_* : M \to f_\sharp M$ defined by

$$f_*(m) = 1 \otimes m.$$

In this notation, the module constructed above is

$$M = M(P_1, P_2, \varphi)$$

where $\varphi : (\beta_1)_\sharp P_1 \to (\beta_2)_\sharp P_2$ is an isomorphism.

Remark 4.3. In this notation, in the category of A-modules, $M(P_1, P_2, \varphi)$ is the fiber product of P_1 and P_2 over $(\beta_2)_\sharp P_2$.

PROOF: For $i = 1, 2$, define $p_i : M(P_1, P_2, \varphi) \to P_i$ to be the natural projections. If we consider the following diagram:

$$\begin{array}{ccc} M(P_1, P_2, \varphi) & \xrightarrow{p_1} & P_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \varphi(\beta_1)_*} \\ P_2 & \xrightarrow[(\beta_2)_*]{} & (\beta_2)_\sharp P_2 \end{array}$$

the result is clear from the definition of $M(P_1, P_2, \varphi)$ and the construction of fiber products in the category of A-modules. ∎

We begin by assuming that P_i is free and finitely generated over A_i for $i = 1, 2$. Let $\{x_i\}$ be a basis for P_1 over A_1 and let $\{y_j\}$ be a basis for P_2 over A_2. Then clearly $\{(\beta_1)_* x_i\}$ is a basis for $(\beta_1)_\sharp P_1$ and $\{(\beta_2)_* y_j\}$ is a basis for $(\beta_2)_\sharp P_2$, both over S. Thus there are elements $s'_{ij} \in S$ such that

$$\varphi((\beta_1)_* x_i) = \sum_j s'_{ij} (\beta_2)_* y_j.$$

Since φ is an isomorphism, the matrix

$$T' = (s'_{ij})$$

is invertible. Let $(T')^{-1} = (s_{ij})$ denote the inverse, which represents φ^{-1}. Then:

$$\varphi^{-1}((\beta_2)_* y_i) = \sum_j s_{ij} (\beta_1)_* x_j.$$

Lemma 4.4. If the matrix T is the image under β_1 of an invertible matrix over A_1, then $M(P_1, P_2, \varphi)$ is free.

PROOF: Suppose $s_{ij} = \beta_1(c_{ij})$ (recall β_1 is assumed to be surjective) where the matrix (c_{ij}) is invertible. Define:

$$x'_i = \sum_j c_{ij} x_j \in A_1.$$

Since (c_{ij}) is invertible, $\{x'_i\}$ forms a basis for P_1 over A_1. Observe that

$$\begin{aligned} (\beta_1)_*(x'_i) &= (\beta_1)_* \left(\sum_j c_{ij} x_j \right) \\ &= \sum_j \beta_1(c_{ij}) (\beta_1)_* x_j \\ &= \sum_j s_{ij} (\beta_1)_* x_j \\ &= \varphi^{-1}((\beta_2)_* y_i). \end{aligned}$$

Therefore $\varphi((\beta_1)_* x'_i) = (\beta_2)_* y_i$ and thus

$$z_i = (x'_i, y_i) \in P_1 \times P_2$$

are elements of $M(P_1, P_2, \varphi)$. I claim that $\{z_i\}$ forms a basis for $M(P_1, P_2, \varphi)$ over A.

Suppose $q = (p_1, p_2) \in M(P_1, P_2, \varphi)$. Then $p_1 \in A_1$ and $p_2 \in A_2$. Thus there are elements $b_i \in A_1$ and $c_i \in A_2$ such that

$$\begin{aligned} p_1 &= \sum_i b_i x'_i \\ p_2 &= \sum_i c_i y_i. \end{aligned}$$

Moreover:

$$\sum_i \beta_1(b_i)\varphi((\beta_1)_* x'_i) = \varphi((\beta_1)_* p_1) = (\beta_2)_* p_2 = (\beta_2)_* \left(\sum_i c_i y_i\right) = \sum_i \beta_2(c_i)(\beta_2)_* y_i.$$

Since

$$\varphi((\beta_1)_* x'_i) = (\beta_2)_* y_i$$

and moreover $(\beta_2)_* y_i$ is a basis for $(\beta_2)_\sharp P_2$ as an S-module, it follows that for each i, $\beta_1(b_i) = \beta_2(c_i)$. Therefore we have

$$a_i = (b_i, c_i) \in A$$

and we can write

$$q = \sum a_i z_i.$$

Uniqueness is an immediate consequence of the fact that $\{x'_i\}$ forms a basis for P_1 and $\{y_i\}$ forms a basis for P_2. Therefore $\{z_i\}$ forms a basis for $M(P_1, P_2, \varphi)$ over A and $M(P_1, P_2, \varphi)$ is free over A. ∎

Corollary 4.5. Under the hypothesis of Lemma 4.4,

$$\operatorname{rank}(P_1) = \operatorname{rank}(P_2) = \operatorname{rank}(M(P_1, P_2, \varphi)).$$

PROOF: Since for $i = 1, 2$, P_i is free and finitely generated over A_i, $P_i \otimes_{A_i} S$ is free and finitely generated over S with the same rank as P_i. Since $P_1 \otimes_{A_1} S \cong P_2 \otimes_{A_2} S$:

$$\operatorname{rank}(P_1) = \operatorname{rank}(P_1 \otimes_{A_1} S) = \operatorname{rank}(P_2 \otimes_{A_2} S) = \operatorname{rank}(P_2).$$

The basis $\{z_i\}$ constructed for $M(P_1, P_2, \varphi)$ in the proof of Lemma 4.4 has the same cardinality as any basis for P_1 or P_2 and the desired result follows. ∎

If P_1 and P_2 are free, but φ does not come from an invertible matrix over A_1, $M(P_1, P_2, \varphi)$ is not necessarily free, but it is at least projective.

Lemma 4.6. Suppose P_1 and P_2 are free. Then $M(P_1, P_2, \varphi)$ is a projective A-module.

PROOF: First we define Q_1 to be a free module over A_1 with a basis $\{u_j\}$ in one-to-one correspondence with the basis $\{y_j\}$ for P_2. Similarly, define Q_2 to be a free module over A_2 with a basis $\{v_i\}$ in one-to-one correspondence with the basis $\{x_i\}$. Let

$$\psi : (\beta_1)_\sharp Q_1 \to (\beta_2)_\sharp Q_2$$

be the isomorphism of S-modules given by T defined above Lemma 4.4. It is clear that

$$M(P_1, P_2, \varphi) \oplus M(Q_1, Q_2, \psi) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, \varphi \oplus \psi)$$

and that the isomorphism $\varphi \oplus \psi$ corresponds with the matrix

$$\begin{pmatrix} T' & 0 \\ 0 & T \end{pmatrix}.$$

Observe that since $T' = T^{-1}$ and vice versa:

$$\begin{pmatrix} T' & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -T' & I \end{pmatrix} \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Since β_1 is surjective, there exist matrices T_1 and T_2 such that $\beta_1(T_1) = T$ and $\beta_1(T_2) = T'$ (where the map acts component-wise). It follows that the above equation can lift to the following:

$$\begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} I & T_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_2 & I \end{pmatrix} \begin{pmatrix} I & T_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Since the matrices on the right are invertible, so is the matrix on the left. Therefore the matrix

$$\begin{pmatrix} T' & 0 \\ 0 & T \end{pmatrix} = \beta_1 \begin{pmatrix} T_2 & 0 \\ 0 & T_1 \end{pmatrix}$$

and is the image of an invertible matrix. It follows from Lemma 4.4 that $M(P_1 \oplus Q_1, P_2 \oplus Q_2, \varphi \oplus \psi)$ is a free A-module. Therefore, since

$$M(P_1 \oplus Q_1, P_2 \oplus Q_2, \varphi \oplus \psi) \cong M(P_1, P_2, \varphi) \oplus M(Q_1, Q_1, \psi)$$

it follows that $M(P_1, P_2, \varphi)$ is a projective A-module. ■

Now we generalize to the case where P_1 and P_2 are merely projective. We will also assume that P_1 and P_2 are finitely generated.

Lemma 4.7. There are projective modules Q_i over A_i such that $P_i \oplus Q_i$ are finitely generated free modules over A_i, for $i = 1, 2$, and so that $(\beta_1)_\sharp Q_1 \cong (\beta_2)_\sharp Q_2$.

PROOF: Since P_i are finitely generated and projective over A_i, there exist projective modules N_i over A_i such that $P_i \oplus N_i$ are finitely-generated free A-modules. Therefore there are positive integers r and s such that

$$\begin{aligned} P_1 \oplus N_1 &\cong A_1^r \\ P_2 \oplus N_2 &\cong A_2^s. \end{aligned}$$

Define $P' = (\beta_1)_\sharp P_1$. Then $P' \cong (\beta_2)_\sharp P_2$. Since $(\beta_1)_\sharp$ and $(\beta_2)_\sharp$ preserve direct sums, it follows that:

$$P' \oplus (\beta_1)_\sharp N_1 = (\beta_1)_\sharp P_1 \oplus (\beta_1)_\sharp N_1 \cong (\beta_1)_\sharp (P_1 \oplus N_1) \cong (\beta_1)_\sharp (A_1^r) \cong S^r$$

and

$$P' \oplus (\beta_2)_\sharp N_2 \cong (\beta_2)_\sharp P_2 \oplus (\beta_2)_\sharp N_2 \cong (\beta_2)_\sharp (P_2 \oplus N_2) \cong (\beta_2)_\sharp (A_2^s) \cong S^s.$$

Define the following:

$$\begin{aligned} Q_1 &= N_1 \oplus A_1^s \\ Q_2 &= N_2 \oplus A_1^r. \end{aligned}$$

Then it is clear that $P_i \oplus Q_i$ are finitely generated, free A_i-modules for $i = 1, 2$. Therefore, Q_i are projective A_i-modules. Moreover, by associativity and commutativity of the direct sum:

$$(\beta_1)_\sharp Q_1 \cong ((\beta_1)_\sharp N_1) \oplus S^s \cong (\beta_1)_\sharp N_1 \oplus P' \oplus (\beta_2)_\sharp N_2 \cong ((\beta_2)_\sharp N_2) \oplus S^r \cong (\beta_2)_\sharp Q_2.$$

This is the desired result. ∎

We are now in a position to prove the results we need to derive the Meyer-Vietoris sequence for Picard groups.

Theorem 4.8. If P_i are finitely-generated, projective A_i modules, then $M(P_1, P_2, \varphi)$ is a finitely-generated, projective A-module.

PROOF: Choose Q_i for $i = 1, 2$ as in Lemma 4.7 and choose an isomorphism

$$\psi : (\beta_1)_\sharp Q_1 \cong (\beta_2)_\sharp Q_2.$$

By Lemma 4.6, since $P_i \oplus Q_i$ are free A_i-modules, the module

$$M(P_1, P_2, \varphi) \oplus M(Q_1, Q_2, \psi) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, \varphi \oplus \psi)$$

is projective and therefore $M(P_1, P_2, \varphi)$ is projective. Since we can choose Q_i such that $P_i \oplus Q_i$ are finitely generated A_i-modules, by the proof of Lemma 4.4 and Lemma 4.6 it follows that $M(P_1, P_2, \varphi)$ is finitely generated. ∎

Theorem 4.9. Every projective A-module is isomorphic to $M(P_1, P_2, \varphi)$ for some P_1, P_2 and φ.

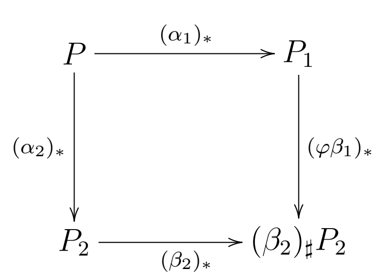
PROOF: Let P be a projective A-module. Define $P_i = (\alpha_i)_\sharp P$ for $i = 1, 2$. Then each P_i is a projective A_i-module. Since $\beta_1 \alpha_1 = \beta_2 \alpha_2$, there is a natural isomorphism

$$\varphi : (\beta_1)_\sharp P_1 \to (\beta_2)_\sharp P_2$$

such that

$$\varphi(\beta_1)_*(\alpha_1)_* = (\beta_2)_*(\alpha_1)_*.$$

Therefore we have the following commutative diagram (of abelian groups):



We can identify P with the subgroup of $P_1 \times P_2$ such that

$$(\varphi\beta_1)_*(p_1) = (\beta_2)_*(p_2).$$

Accordingly, P is the fiber product of P_1 and P_2 over $(\beta_2)_\sharp P_2$. By Remark 4.3, $P \cong M(P_1, P_2, \varphi)$. ■

Theorem 4.10. Given projective modules P_1 and P_2 and an isomorphism $\varphi : (\beta_1)_\sharp P_1 \to (\beta_2)_\sharp P_2$, let $M = M(P_1, P_2, \varphi)$. Then $P_i \cong (\alpha_i)_\sharp M$.

PROOF: Recall the definition of $M(P_1, P_2, \varphi)$:

$$M(P_1, P_2, \varphi) := \{(p_1, p_2) \in P_1 \times P_2 \,\big|\, \varphi(p_1 \otimes 1) = p_2 \otimes 1\}.$$

Viewing P_1 as an A-module via the map α_1, there is a natural A-linear map $f : M \to P_1$ given by projection. This induces a natural A_1-linear map $g : (\alpha_1)_\sharp M \to P_1$ given by $g(1 \otimes m) = f(m)$. Since the argument does not depend on the index 1 or 2, it suffices to prove g is an isomorphism. In the special case where the hypotheses of Lemma 4.4 are satisfied, this is clear since both $(\alpha_1)_\sharp M$ and P_1 are free over A_1 with the same number of generators and g takes one basis to the other. In the general case, choosing Q_1 and Q_2 as in Lemma 4.7, we obtain the A-module

$$M(P_1 \oplus Q_1, P_2 \oplus Q_2, g \oplus h)$$

which satisfies the hypotheses of Lemma 4.4, hence $g \oplus h$ is an isomorphism. It follows that g is an isomorphism. ■

4.2 The Conductor Square

To apply this machinery to come up with a computational technique for Picard groups, we begin with a lemma that determines when $M(P_1, P_2, \varphi)$ and $M(P_1', P_2', \varphi')$ are isomorphic. We continue to use the same setup.

Lemma 4.11. Suppose P_i and P_i' are invertible A_i modules for $i = 1, 2$. Then $M(P_1, P_2, \varphi) \cong M(P_1', P_2', \varphi')$ if and only if there are isomorphisms $\psi_i : P_i \to P_i'$ for $i = 1, 2$ such that

$$\varphi = (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi' \circ (\psi_1 \otimes \mathrm{Id}_S).$$

PROOF: Define the following notation:

$$\begin{aligned} M &= M(P_1, P_2, \varphi) \\ M' &= M(P_1', P_2', \varphi') \end{aligned}$$

Suppose we have isomorphisms ψ_i as in the statement. Define $f : M \to M'$ by

$$f(p_1, p_2) = (\psi_1(p_1), \psi_2(p_2)).$$

Then we have the following:

$$\varphi'(\psi_1(p_1) \otimes 1) = (\psi_2 \otimes \mathrm{Id}_S) \circ \varphi(p_1 \otimes 1) = (\psi_2 \otimes \mathrm{Id}_S)(p_2 \otimes 1) = \psi_2(p_2) \otimes 1.$$

This shows that $f(M) \subset M'$. Since ψ_i are isomorphisms, it is clear that f is bijective and A-linear and hence f is an isomorphism.

Now suppose we have an isomorphism $f : M \to M'$. First of all, I claim that given $p_1 \in P_1$, there is a unique $p_2 \in P_2$ such that $(p_1, p_2) \in M$. Existence is clear, since given $p_1 \in P_1$, there is $p_2 \in P_2$ such that $\varphi(p_1 \otimes 1) = p_2 \otimes 1$ (since φ is an isomorphism of S-modules). Now suppose there is $p_2, q_2 \in P_2$ such that $\varphi(p_1 \otimes 1) = p_2 \otimes 1 = q_2 \otimes 1$. Then by bilinearity we have

$$(p_2 - q_2) \otimes 1 = 0.$$

We can localize this equation. Then we have for any prime $\mathfrak{p}$, in the localization since $(P_2)_\mathfrak{p} \cong (A_2)_\mathfrak{p}$,

$$0 = (p_2 - q_2) \otimes 1 = (p_2 - q_2)(1 \otimes 1)$$

which implies $p_2 - q_2 = 0$. Since this holds for all primes, it follows that $p_2 - q_2 = 0$ in P_2 and hence $p_2 = q_2$. This establishes uniqueness. Observe that this holds for P_2 (using φ^{-1}) as well as P_1' and P_2'.

It follows that we can define an inclusion $\iota_i : P_i \to M$ and a projection $\pi_i' : M' \to P_i'$ and that the composition $\psi_i = \pi_i' \circ f \circ \iota_i$ is an isomorphism $P_i \to P_i'$.

Next, by construction, given $p_1 \in P_2$ there is a unique $p_2 \in P_2$ and (since f is an isomorphism) a unique $(p_1', p_2') \in M'$ such that:

$$\psi_1(p_1) = \pi_1' f \iota_1(p_1) = \pi_1' f(p_1, p_2) \pi_1'(p_1', p_2') = p_1'$$

and moreover $\psi_2(p_2) = p_2'$. Accordingly:

$$\begin{aligned}
(\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi' \circ (\psi_1 \otimes \mathrm{Id}_S)(p_1 \otimes 1) &= (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi'(\psi_1(p_1) \otimes 1) \\
&= (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi'(p_1' \otimes 1) \\
&= (\psi_2^{-1} \otimes \mathrm{Id}_S)(p_2' \otimes 1) \\
&= p_2 \otimes 1 \\
&= \varphi(p_1 \otimes 1).
\end{aligned}$$

Since $\{p \otimes 1\}$ for $p \in P_1$ generates $P_1 \otimes_A S$ as an S-module, it follows that

$$\varphi = (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi' \circ (\psi_1 \otimes \mathrm{Id}_S)$$

as desired. ∎

Lemma 4.12. If P_i, Q_i are projective A_i-modules for $i = 1, 2$ and $\varphi : P_1 \otimes_A S \to P_2 \otimes_A S$, $\psi : Q_1 \otimes_A S \to Q_2 \otimes_A S$ are isomorphisms, then

$$M(P_1, P_2, \varphi) \otimes_A M(Q_1, Q_2, \psi) \cong M(P_1 \otimes_{A_1} Q_1, P_2 \otimes_{A_2} Q_2, \varphi \otimes \psi).$$

PROOF: Given $((p_1, p_2), (q_1, q_2)) \in M(P_1, P_2, \varphi) \times M(Q_1, Q_2, \psi)$, define the following map:

$$\xi((p_1, p_2), (q_1, q_2)) = (p_1 \otimes q_1, p_2 \otimes q_2).$$

Then note that

$$(\varphi \otimes \psi)((p_1 \otimes q_1) \otimes 1_S) = (p_2 \otimes q_2) \otimes 1_S$$

and therefore ξ is a map

$$\xi : M(P_1, P_2, \varphi) \times M(Q_1, Q_2, \psi) \to M(P_1 \otimes_{A_1} Q_1, P_2 \otimes_{A_2} Q_2, \varphi \otimes \psi).$$

It is clear that ξ is bilinear, hence ξ induces a unique linear map

$$M(P_1, P_2, \varphi) \otimes_A M(Q_1, Q_2, \psi) \to M(P_1 \otimes_{A_1} Q_1, P_2 \otimes_{A_2} Q_2, \varphi \otimes \psi)$$

that is clearly bijective. ■

In order to define the sequence below, we need a little bit of algebraic K-theory.

Definition 4.13. For a commutative ring A with identity, $K_0(A)$ is the quotient of the free abelian group on isomorphism classes of finitely generated projective modules over A modulo the relation $[P] + [Q] = [P \oplus Q]$.

The group $K_0(A)$ can be made into a commutative ring with identity where multiplication is given by the tensor product and the identity is given by $[A]$ (viewed as a module over itself).

Proposition 4.14. The Picard group $\mathrm{Pic}(A)$ embeds into $K_0(A)$ as the group of units.

PROOF: By Lemma 4.1, $\mathrm{Pic}(A)$ embeds into $K_0(A)$ as a set and by part 1 of Theorem 2.8, $\mathrm{Pic}(A)$ actually embeds into the group of units $K_0(A)^\times$. Now suppose $[M] \in K_0(A)^\times$. Then there is $[N] \in K_0(A)$ such that

$$M \otimes_A N \cong A.$$

Let $\mathfrak{p} \in \mathrm{Spec} A$ be arbitrary. Localizing at $\mathfrak{p}$ gives

$$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \cong A_\mathfrak{p}.$$

Since M and N are projective, they are locally free. Therefore there are positive integers k and l such that

$$M_\mathfrak{p} \cong A_\mathfrak{p}^k$$

and

$$N_\mathfrak{p} \cong A_\mathfrak{p}^l.$$

Accordingly,

$$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} \cong A_\mathfrak{p}^{kl} \cong A_\mathfrak{p}$$

which implies $k = l = 1$. Therefore M and N are locally free of rank 1. Since M and N are finitely generated, they are invertible A-modules. Thus $[M] \in \mathrm{Pic}(A)$. ■

Let us now define the maps that will be used in our exact sequence. First of all, if $f : A \to B$ is any homomorphism of commutative rings with identity (requiring $f(1) = 1$) and $a \in A^\times$ is a unit, then $f(a)$ is a unit. Consequently, α_i (for $i = 1, 2$) induces a map $A^\times \to A_i^\times$ which (by an abuse of notation) we will also denote by α_i. The same goes for the β_i's. Define

$$\iota : A^\times \to A_1^\times \oplus A_2^\times$$

by

$$\iota(a) = (\alpha_1(a), \alpha_2(a)).$$

Define

$$j : A_1^\times \oplus A_2^\times \to S^\times$$

by

$$j(a_1, a_2) = \beta_1(a_1)(\beta_2(a_2))^{-1}.$$

Finally, for each $i = 1, 2$, define a map $\delta_i : \mathrm{Pic}(A) \to \mathrm{Pic}(A_i)$ by

$$\delta_i(I) = A_i \otimes_A I.$$

Then $\delta_i(I)$ is a finitely generated A_i-module. By properties of localization, given a prime $\mathfrak{p} \in \mathrm{Spec} A_i$ and $\mathfrak{q} \in \mathrm{Spec} A$ such that $\alpha_i^{-1}(\mathfrak{p}) = \mathfrak{q}$, since I is invertible we have:

$$(A_i \otimes_A I)_\mathfrak{p} = (A_i)_\mathfrak{p} \otimes_{A_\mathfrak{q}} I_\mathfrak{q} \cong (A_i)_\mathfrak{p} \otimes_{A_\mathfrak{q}} A_\mathfrak{q} \cong (A_i)_\mathfrak{p}.$$

It follows that $\delta_i(I)$ really is an element of $\mathrm{Pic}(A_i)$. It is clear that δ_i is a group homomorphism, since if $I, J \in \mathrm{Pic}(A)$, by associativity of the tensor product:

$$\begin{aligned}
\delta_i(I)\delta_i(J) &= [(I \otimes_A A_i) \otimes_{A_i} (J \otimes_A A_i)] \\
&= [I \otimes_A (A_i \otimes_{A_i} (J \otimes_A A_i))] \\
&= [I \otimes_A (J \otimes_A A_i)] \\
&= [(I \otimes_A J) \otimes_A A_i] \\
&= \delta(IJ).
\end{aligned}$$

Finally, define

$$\delta : \mathrm{Pic}(A) \to \mathrm{Pic}(A_1) \oplus \mathrm{Pic}(A_2)$$

by $\delta = (\delta_1, \delta_2)$.

Finally, I add the additional assumption on the fiber square that α_1 is injective (this has been unnecessary thusfar, but will be satisfied in the conductor square below).

Theorem 4.15. There is a homomorphism

$$\partial : S^\times \to \mathrm{Pic}(A)$$

such that following sequence is exact:

$$0 \longrightarrow A^\times \xrightarrow{\iota} A_1^\times \oplus A_2^\times \xrightarrow{j} S^\times \xrightarrow{\partial} \mathrm{Pic}(A) \xrightarrow{\delta} \mathrm{Pic}(A_1) \oplus \mathrm{Pic}(A_2).$$

PROOF: First, since α_1 is injective it is clear that ι is injective. Next I show that $\ker j = \mathrm{Im} i$. One containment is obvious. Now suppose $j(a_1, a_2) = 1$. This means that $\beta_1(a_1)(\beta_2(a_2))^{-1} = 1$. Thus $\beta_1(a_1) = \beta_2(a_2)$. Since A is the fiber product of A_1 and A_2 over S, there is $a \in A$ such that $\alpha_i(a) = a_i$ which precisely says that $\iota(a) = (a_1, a_2)$. Hence $\ker j = \mathrm{Im} j$.

Now we define the connecting homomorphism ∂. Let $P_i = A_i$ viewed as modules over A_i for $i = 1, 2$. Let $s \in S^\times$. Then s induces an S-module isomorphism $\varphi_s : A_1 \otimes_{A_1} S \to A_2 \otimes_{A_2} S$ given by $\varphi(1_{A_1} \otimes 1_S) = s(1_{A_2} \otimes 1_S)$ (since $1_{A_i} \otimes 1_S$ generates $A_i \otimes_A S$ as an S-module). Therefore we can form the module $M_s = M(A_1, A_2, \varphi_s)$. By Theorem 4.8, $[M_s] \in K_0(A)$.

First I claim that

$$M_{st} \cong M_s \otimes_A M_t.$$

Starting from the right hand side, by Lemma 4.12, we have :

$$M_s \otimes_A M_t \cong M(A_1 \otimes_{A_1} A_1, A_2 \otimes_{A_2} A_2, \varphi_s \otimes \varphi_t) \cong M(A_1, A_2, \varphi_s \otimes \varphi_t) \cong M(A_1, A_2, \varphi_{st}) = M_{st}$$

(since it is clear that $\varphi_s \otimes \varphi_t = \varphi_{st}$). In addition, note that

$$M_1 = M(A_1, A_2, \varphi_{1_S}) \cong A.$$

Therefore:

$$M_s \otimes M_{s^{-1}} \cong M_{ss^{-1}} \cong M_1 \cong A.$$

Therefore in $K_0(A)$ as a ring, $[M_s][M_{s^{-1}}] = [A]$ and $[M_s]$ is a unit. Accordingly, by Proposition 4.14, $[M_s] \in \mathrm{Pic}(A)$ for every $s \in S^\times$. Define $\partial : S^\times \to \mathrm{Pic}(A)$ by $\partial(s) = [M_s]$. Then we have

$$\partial(st) = [M_{st}] = [M_s \otimes_A M_t] = [M_s] \otimes_A [M_t] = \partial(s)\partial(t).$$

Thus ∂ is a group homomorphism.

Now note that if $s = j(a_1, a_2) = \beta_1(a_1)\beta_2(a_2)^{-1}$ we have

$$M_s = M_{\beta_1(a_1)\beta_2(a_2)^{-1}} \cong M_{\beta_1(a_1)} \otimes_A M_{\beta_2(a_2)}.$$

By the proof of Lemma 4.4, since A_1 and A_2 are free modules of rank one and $\varphi_{\beta(a_1)}$ as a one-by-one matrix is the image of the one-by-one matrix over A_1 given by (a_1) (and likewise for $\varphi_{\beta(a_2^{-1})}$), it follows that $M_{\beta_1(a_1)} \cong A$ and $M_{\beta_2(a_2)^{-1}} = M_{\beta_2(a_2^{-1})} \cong A$.[4] Accordingly, if $s \in \mathrm{Im}(j)$, then

$$\partial(s) = [M_s] = [A \otimes_A A] = [A]$$

which implies $s \in \ker \partial$.

On the other hand, suppose $\partial(s) = [M_s] = [A]$. This means, in particular, that we have

$$M(A_1, A_2, \varphi_s) \cong M(A_1, A_2, \varphi_1).$$

By Lemma 4.11, there are isomorphisms $\psi_i : A_i \to A_i$ such that:

$$\varphi_1 = (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi_s \circ (\psi_1 \otimes \mathrm{Id}_S). \tag{4.1}$$

Note that for each $i = 1, 2$, $1_{A_i} \otimes 1_S$ generates $A_i \otimes_{A_i} S$ as an S-module. Therefore, we plug $1_{A_1} \otimes 1_S$ into equation (4.1). On the right hand side, we have

$$\varphi_1(1_{A_1} \otimes 1_S) = 1_{A_2} \otimes 1_S.$$

[4] Lemma 4.4 could just as easily been proved using β_2 instead of β_1. In fact, this is how Milnor proves the lemma in [Mil71].

Therefore:

$$\begin{aligned}
1_{A_2} \otimes 1_S &= (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi_s \circ (\psi_1 \otimes \mathrm{Id}_S)(1_{A_1} \otimes 1_S) \\
&= (\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi_s(\psi_1(1_{A_1}) \otimes 1_S) \\
&= \beta_1(\psi_1(1_{A_1}))(\psi_2^{-1} \otimes \mathrm{Id}_S) \circ \varphi_s(1_{A_1} \otimes 1_S) \\
&= s\beta_1(\psi_1(1_{A_1}))(\psi_2^{-1} \otimes \mathrm{Id}_S)(1_{A_2} \otimes 1_S) \\
&= s\beta_1(\psi_1(1_{A_1}))(\psi_2^{-1}(1_{A_2}) \otimes 1_S) \\
&= s\beta_1(\psi_1(1_{A_2}))\beta_2(\psi_2^{-1}(1_{A_2}))(1_{A_2} \otimes 1_S).
\end{aligned}$$

Since $1_{A_2} \otimes 1_S$ is a generator, it follows that

$$s\beta_1(\psi_1(1_{A_2}))\beta_2(\psi_2^{-1}(1_{A_2})) = 1_S. \tag{4.2}$$

Note that since ψ_i is an isomorphism for each i, $\psi_i(1_{A_i})$ must be a generator for A_i as an A_i-module, and therefore must be a unit. Therefore, let $a_1 = (\psi_1(1_{A_1}))^{-1}$ and let $a_2 = \psi_2^{-1}(1_{A_2})$. Then equation (4.2) reads:

$$s\beta_1(a_1^{-1})\beta_2(a_2) = 1$$

which implies (since β_i is a ring homomorphism for each $i = 1, 2$) that

$$s = (\beta_1(a_1^{-1}))^{-1}(\beta_2(a_2))^{-1} = \beta_1(a_1)(\beta_2(a_2))^{-1} = j(a_1, a_2).$$

It follows that $s \in \mathrm{Im}(j)$. Therefore $\ker \partial = \mathrm{Im}(j)$.

Finally we show that $\ker \delta = \mathrm{Im}(\partial)$. Suppose $[I] \in \mathrm{Im}(\partial)$. Then $[I] = [M_s]$ for some $s \in S^\times$. Therefore $I \cong M(A_1, A_2, \varphi_s)$. By Theorem 4.10, $I \otimes_{A_i} A_i \cong A_i$, it follows that $\delta([I]) = ([A_1], [A_2])$ which implies $[I] \in \ker \delta$.

On the other hand, suppose $\delta([I]) = ([A_1], [A_2])$. By Theorem 4.9, there are projective A_i-modules P_i and an S-module isomorphism $\varphi : P_1 \otimes_A S \to P_2 \otimes_A S$ such that $[I] = [M(P_1, P_2, \varphi)]$. Therefore, by Theorem 4.10 and by the hypothesis on $\delta([I])$:

$$P_i \cong I \otimes_A A_i \cong A_i.$$

Therefore

$$I \cong M(A_1, A_2, \varphi)$$

for some S-module isomorphism $\varphi : A_1 \otimes_A S \to A_2 \otimes_A S$. As an S-module, $A_i \otimes_A S$ is generated by $1_{A_i} \otimes 1_S$. Therefore there is some $s \in S$ such that $\varphi(1_{A_1} \otimes 1_S) = s(1_{A_2} \otimes 1_S)$. Since φ is an isomorphism, it follows that $s \in S^\times$. Therefore $\varphi = \varphi_s$. Hence

$$[I] = [M(A_1, A_2, \varphi_s)] = [M_s] = \partial(s)$$

which implies $[I] \in \mathrm{Im}(\partial)$. Thus $\mathrm{Im}(\partial) = \ker \delta$. This completes the proof. ∎

Finally, we put together a special fiber square that is useful in computations for affine singular curves. Given the coordinate ring of an affine singular curve A, let A_1 be the normalization of A. Then $A \hookrightarrow A_1$ and A_1/A is an A-module.

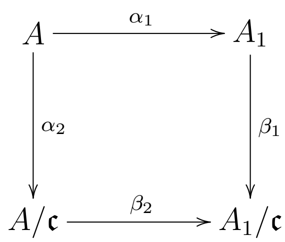
Definition 4.16. The annihilator in A of the A-module A_1/A is called the **conductor** of A and denoted $\mathfrak{c}$.

Lemma 4.17. Let $\iota : A \hookrightarrow A_1$ be the natural inclusion. Then $\mathfrak{c}^e = \mathfrak{c}$.

PROOF: By definition, $\mathfrak{c}A_1 \subset A$. If $f \in \mathfrak{c}^e = (\iota(\mathfrak{c})) = (\mathfrak{c})$, then there are $f_i \in \mathfrak{c}$ and $g_i \in A_1$ such that $f = \sum f_i g_i$. For each i, $f_i g_i \in A$, so $f_i g_i \in A \cap \mathfrak{c}^e = \mathfrak{c}$ for each i. Hence $f \in \mathfrak{c}$. ■

Let $A_2 = A/\mathfrak{c}$ and $S = A_1/\mathfrak{c}$. Let $\alpha_1 : A \hookrightarrow A_1$ and $\beta_2 : A_2 \hookrightarrow S$ be the natural inclusions and let $\alpha_2 : A \to A_2$ and $\beta_1 : A_1 \to S$ be the natural projections.

Proposition 4.18. The map β_1 is surjective and the following diagram:



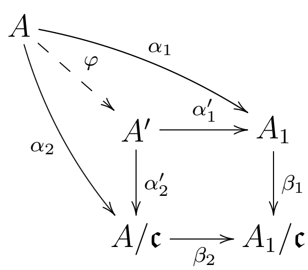
is a fiber square.

PROOF: It is clear that the square above commutes, and it is obvious that β_1 is surjective. Note as well that α_2 is surjective, α_1 is injective.

Let A' denote the fiber product of A_1 and $A/\mathfrak{c}$ over β_1 and β_2 (with maps $\alpha_1' : A' \to A_1$ and $\alpha_2' : A' \to A/\mathfrak{c}$). Then A' is explicitly described as

$$A' = \{(a_1, [a]) \in A_1 \times A/\mathfrak{c} \mid \beta_1(a_1) = \beta_2([a])\}.$$

Since the diagram above commutes, by the universal property of fiber products there is a unique ring homomorphism $\varphi : A \to A'$ such that the following diagram commutes:



This map can be described explicitly:

$$\varphi(a) = (\alpha_1(a), \alpha_2(a)).$$

I claim that φ is an isomorphism. Since φ is a ring homomorphism, it suffices to show that φ is bijective. Since α_1 is injective, φ must be injective. Now suppose $(a_1, [a]) \in A'$. Then $\beta_1(a_1) = \beta_2([a])$, which implies that:

$$a_1 \equiv a \mod \mathfrak{c}^e$$

which, by Lemma 4.17 implies that

$$a_1 - a \in \mathfrak{c}^e = \mathfrak{c} \subset A.$$

This implies that $a_1 = \iota(a)$ and hence $(a_1, [a]) = \varphi(a)$. Therefore φ is surjective. ■

Definition 4.19. The square in Proposition 4.18 is called the **conductor square**.

The following corollary is an immediate consequence of Theorem 4.15 and Proposition 4.18.

Corollary 4.20. The following is an exact sequence:

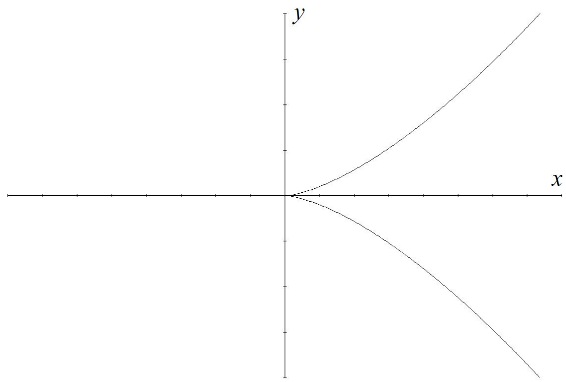
$$0 \to A^\times \to A_1^\times \oplus (A/\mathfrak{c})^\times \to (A_1/\mathfrak{c})^\times \to \mathrm{Pic}(A) \to \mathrm{Pic}(A_1) \oplus \mathrm{Pic}(A/\mathfrak{c}).$$

4.3 The Cusp and the Node

First we consider the cusped curve. Let k be a field and consider the ring

$$A = k[t^2, t^3] = k[x, y]/(y^2 - x^3).$$

This is a singular curve in $\mathbb{A}_k^2$ with a cusp at the origin. When $k = \mathbb{R}$, the picture as follows:



Lemma 4.21. The normalization of A is given by $A_1 = k[t]$ and the conductor is given by $\mathfrak{c} = (t^2, t^3) \subset A$.

PROOF: Denote $A = k[\overline{x}, \overline{y}]$. Then $\overline{x}, \overline{y}$ are clearly integral over A. Moreover, since $\overline{x}^2 = \overline{y}^3$, in the field of fractions $K(A)$, we have

$$(\overline{x}/\overline{y})^2 = \overline{y}. \tag{4.3}$$

Therefore $\overline{x}/\overline{y}$ is integral over A. It follows that A is not integrally closed. Consider the ring

$$C = k[\overline{x}, \overline{y}, \overline{x}/\overline{y}].$$

Since

$$\overline{x} = \overline{y}\,(\overline{x}/\overline{y})$$

it follows that

$$C = k[\overline{y}, \overline{x}/\overline{y}].$$

On the other hand, by equation (4.3) we have

$$C = k[\overline{x}/\overline{y}] = k[t].$$

Thus C is a minimal integrally closed subring of $K(A)$ that contains A, and therefore is the integral closure of A. Hence $A_1 = k[t]$. Note that $t^2 = \overline{y}$ and $t^3 = \overline{x}$.

It is clear that $\overline{y} \in \mathfrak{c}$ since $\overline{y}t = \overline{x} \in A$. Moreover,

$$\overline{x}t = \overline{x}^2/\overline{y} = \overline{y}^3/\overline{y} = \overline{y}^2 \in A.$$

Therefore $\overline{x} \in \mathfrak{c}$. Hence $(\overline{x}, \overline{y}) \subset \mathfrak{c}$. On the other hand, by the correspondence principle $(\overline{x}, \overline{y})$ is a maximal ideal in A. The entire ring A cannot be the conductor, since otherwise $1 \cdot A_1 \subset A$ which implies A is integral. Therefore $\mathfrak{c} = (\overline{x}, \overline{y}) = (t^2, t^3)$ as claimed ∎

Lemma 4.22. For the conductor square for A, we have $A_1^\times = k^\times, (A/\mathfrak{c})^\times = k^\times$. There is an embedding of the additive group of k, k_+, into $(A_1/\mathfrak{c})^\times$ such that $(A_1/\mathfrak{c})^\times = k^\times \oplus k_+$.

PROOF: First of all, by Lemma 4.21, $A_1 = k[t]$ and therefore $(k[t])^\times = k^\times$. Again, by Lemma 4.21, $A/\mathfrak{c} = k$ and hence $(A/\mathfrak{c})^\times = k^\times$.

Finally, note that in $k[t]$, $t^3 \in (t^2)$ and therefore $\mathfrak{c} = (t^2)$ in A_1. Thus $A_1/\mathfrak{c} = k[t]/(t^2)$. Therefore if $[f] \in A_1/\mathfrak{c}$, there is a representative f such that $f(t) = a + bt, b \in k$. Suppose $[f]$ is a unit with inverse $g(t) = c + dt$. Since $t^2 = 0$ we have:

$$1 = (a + bt)(c + dt) = ac + (bc + da)t.$$

Therefore $a, c \in k^\times$ and $bc = -da$. Hence we can rewrite f as $f = a(1 + b/at)$. If we let $u = b/a$, we have $f = a(1 + ut)$. Therefore

$$(A_1/\mathfrak{c})^\times = \{a(1 + ut) \mid a \in k^\times, u \in k\}.$$

Define a map $\iota : k_+ \hookrightarrow (A_1/\mathfrak{c})^\times$ by $\iota(u) = 1 + ut$ (which is clearly injective). Then:

$$\iota(u)\iota(v) = (1 + ut)(1 + vt) = 1 + (u + v)t = \iota(u + v).$$

Therefore ι embeds k_+ into $(A_1/\mathfrak{c})^\times$. It remains to show that

$$(A_1/\mathfrak{c})^\times \cong k^\times \oplus k_+.$$

Define

$$\zeta : (A_1/\mathfrak{c})^\times \to k^\times \oplus k_+$$

by $\zeta(a(1 + ut)) = (a, u)$. Then:

$$\zeta(a(1 + ut)b(1 + vt)) = \zeta(ab(a + (u + v)t)) = (ab, u + v) = (a, u) \cdot (b, v) = \zeta(a(1 + ut))\zeta(b(1 + vt)).$$

Hence ζ is a group homomorphism. It is clearly bijective and hence an isomorphism. This completes the proof. ∎

Proposition 4.23. $\operatorname{Pic}(A) = k_+$.

PROOF: Using the Meyer-Vietoris sequence, the following is exact:

$$0 \to A^\times \to k^\times \times k^\times \to k^\times \oplus k_+ \to \operatorname{Pic}(A) \to \operatorname{Pic}(k[t]) \oplus \operatorname{Pic}(k).$$

Clearly $\operatorname{Pic}(k) = 0$ and since $k[t]$ is a principal ideal domain, $\operatorname{Pic}(k[t]) = 0$. Therefore we have the following exact sequence:

$$0 \to A^\times \to k^\times \times k^\times \to k^\times \oplus k_+ \to \operatorname{Pic}(A) \to 0.$$

The image of $k^\times \times k^\times$ in $k^\times \oplus k_+$ is the factor $k^\times$, since the map takes units in k to themselves sitting inside the unit group of $k[x]/(x^2)$, which gives the factor of $k^\times$, whereas k_+ sits in $(k[x]/(x^2))^\times$ as degree one polynomials of the form $1 + ut$. Hence the image lies in the $k^\times$-factor. On the other hand, given a unit $s \in k^\times$, s is the image of $(s, 1)$. So the entire factor is the image. Since the map $\partial : k^\times \oplus k_+ \to \operatorname{Pic}(A)$ is surjective, and its kernel is the image of $k^\times \times k^\times$, it follows that

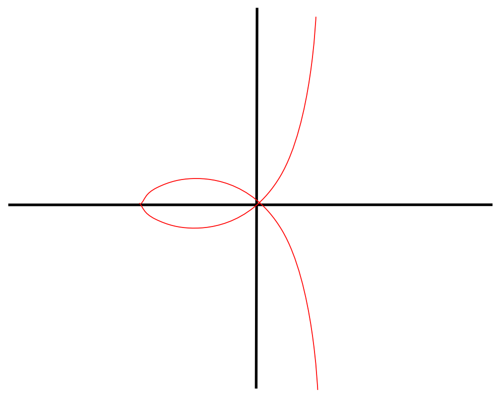
$$\operatorname{Pic}(A) \cong (k^\times \oplus k_+)/\ker \partial = (k^\times \oplus k_+)/k^\times = k_+.$$

This proves the proposition. ■

Now we consider a nodal curve. Consider the ring

$$B = k[t^2 - 1, t^3 - t] = k[x, y]/(y^2 - x^2(x + 1)).$$

This is a singular curve in $\mathbb{A}^2_k$ with a node at the origin. For simplicity (in one step below), I will assume $\mathrm{char}(k) \neq 2$. When $k = \mathbb{R}$, the picture as follows:



Lemma 4.24. The normalization of B is given by $B_1 = k[t]$ and the conductor is given by $\mathfrak{c} = (t^2 - 1, t^3 - t)$.

PROOF: Denote $B = k[\overline{x}, \overline{y}]$. Then $\overline{x}, \overline{y}$ are clearly integral over B. Moreover, in the field of fractions $K(B)$, we have

$$(\overline{y}/\overline{x})^2 = \overline{x} + 1. \tag{4.4}$$

Therefore $\overline{y}/\overline{x}$ is integral over B. It follows that B is not integrally closed. Consider the ring

$$C = k[\overline{x}, \overline{y}, \overline{y}/\overline{x}].$$

Since

$$\overline{y} = \overline{x}(\overline{y}/\overline{x})$$

it follows that

$$C = k[\overline{x}, \overline{y}/\overline{x}].$$

On the other hand, by equation (4.4) we have

$$C = k[\overline{y}/\overline{x}] = k[t].$$

Thus C is a minimal integrally closed subring of $K(B)$ that contains B, and therefore is the integral closure of B. Hence $B_1 = k[t]$. Note that $t^2 - 1 = \overline{x}$ and $t^3 - t = \overline{x}t = \overline{y}$.

It is clear that $\overline{x} \in \mathfrak{c}$ since $\overline{x}t = \overline{y} \in B$. Moreover,

$$\overline{y}t = \overline{y}^2/\overline{x} = \overline{x}^2(\overline{x} + 1)/\overline{x} = \overline{x}(\overline{x} + 1) \in B.$$

Therefore $\overline{y} \in \mathfrak{c}$. Hence $(\overline{x}, \overline{y}) \subset \mathfrak{c}$. Therefore since by maximality of the ideal $(\overline{x}, \overline{y})$, $\mathfrak{c} = (\overline{x}, \overline{y}) = (t^2 - 1, t^3 - t)$ as claimed. ■

Lemma 4.25. For the conductor square for B, we have $B_1^\times = k^\times$, $(B/\mathfrak{c})^\times = k^\times$ and $(B_1/\mathfrak{c})^\times = k^\times \oplus k^\times$.

PROOF: Since $B_1 = k[t]$, $(B_1)^\times = k^\times$. Since $B/\mathfrak{c} = k[t^2-1, t^3-t]/(t^2-1, t^3-t) = k$, $(B/\mathfrak{c})^\times = k^\times$. Since $t \in B_1$, $t^3 - t \in (t^2-1)$ and therefore $\mathfrak{c} = (t^2-1)$ in B_1. Since I assumed $\mathrm{char}(k) \neq 2$, $(t-1)$ and $(t+1)$ are coprime in B_1. Hence $(t^2-1) = (t-1) \cap (t+1)$ and by the Chinese Remainder Theorem:

$$k[t]/(t^2-1) = k[t]/[(t-1) \cap (t+1)] \cong k[t]/(t-1) \times k[t]/(t+1) \cong k \times k.$$

Since $(R \times S)^\times \cong R^\times \times S^\times$ for any rings R and S, it follows that $(B_1/\mathfrak{c})^\times \cong k^\times \oplus k^\times$ as claimed. ∎

Proposition 4.26. $\mathrm{Pic}(B) = k^\times$.

PROOF: Using the Meyer-Vietoris sequence, the following is exact:

$$0 \to B^\times \to k^\times \oplus k^\times \to k^\times \oplus k^\times \to \mathrm{Pic}(B) \to \mathrm{Pic}(k[t]) \oplus \mathrm{Pic}(k).$$

Again, $\mathrm{Pic}(k) = 0$ and $\mathrm{Pic}(k[t]) = 0$. Therefore we have the following exact sequence:

$$0 \to B^\times \to k^\times \oplus k^\times \to k^\times \oplus k^\times \to \mathrm{Pic}(B) \to 0.$$

The image of $k^\times \oplus k^\times$ in $k^\times \oplus k^\times$ is one factor of $k^\times$. This is because the units in both B_1 and $B/\mathfrak{c}$ are the respective images under α_1 and α_2 of the units in k sitting in B. Hence since the conductor square commutes, their images in $B_1/\mathfrak{c}$ must be the same. Moreover, any unit $(s, 1) \in k^\times \oplus k^\times$ is the image under j of $(s, 1)$. Hence the image of j is one factor of $k^\times$. Since the map $\partial : k^\times \oplus k^\times \to \mathrm{Pic}(B)$ is surjective, and its kernel is the image of $k^\times \otimes k^\times$, it follows that

$$\mathrm{Pic}(B) \cong (k^\times \oplus k^\times)/\ker \partial = (k^\times \oplus k^\times)/k^\times = k^\times.$$

This proves the proposition. ∎

The geometric story for the node can be described intuitively. Think of the elements of the Picard group as line bundles. The normalization of B corresponds to an affine, non-singular rational curve $Y_1 \approx \mathbb{A}^1_k = \mathrm{Spec}(k[t])$. Since $\mathrm{Pic}(Y_1)$ is trivial, the only line bundle over Y_1 is the trivial line bundle $Y_1 \times k$. The map from B to B_1 corresponds to a map from the corresponding curve Y to Y_1 that separates the node on X into two distinct points p_1, p_2 on Y_1. Every line bundle over Y corresponds to some identification of the fibers through p_1 and p_2 of the trivial line bundle over Y_1. There is a one-to-one correspondence between the possible identifications of the fibers and the transition maps in a local trivialization. The transition maps are in a one-to-one correspondence with maps from the trivializing open neighborhood of the node into the general linear group over k with rank one. This is precisely $k^\times$. In other words, every transition map is given by multiplication by some unit $u \in k^\times$, hence $\mathrm{Pic}(Y) \cong k^\times$.

Remark 4.27. If one completes the cusped curve in projective space, the resulting Picard group is $\mathbb{Z} \oplus k_+$, whereas the completion of the nodal curve has Picard group $\mathbb{Z} \oplus k^\times$. The interested reader can see part (b) of Exercise 6.9 in Chapter II, Section 6 of [Har77]. In this exercise, one proves that if X is the completion of the cusped curve, there is a short exact sequence

$$0 \to k_+ \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0$$

and in the case where Y is the completion of the nodal curve there is a short exact sequence

$$0 \to k^\times \to \mathrm{Pic}(Y) \to \mathbb{Z} \to 0.$$

Since both short exact sequences end in $\mathbb{Z}$, which is a projective $\mathbb{Z}$-module, they split giving the descriptions of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ above.

5 Conclusion

We have developed a purely algebraic definition for the Picard group of an affine variety and developed techniques for its computation in the case of affine curves. When the curve is non-singular, we had to defer to some geometry. However, in the singular case, the Meier-Vietoris sequence and the conductor square provided a purely algebraic technique for computation.

The kernel of the natural map $\mathrm{Pic}(A) \to \mathrm{Pic}(A_1)$, where A_1 is the normalization of A, is an invariant of the singularities of a singular affine curve. When the field is an infinite, perfect domain, triviality of this kernel is equivalent to direct-sum cancelation holding for finitely-generated, torsion free A-modules. When k is algebraically closed, triviality of the kernel implies A is a Dedekind domain and the curve is non-singular. When k is not algebraically closed, if the kernel is trivial and A is not a Dedekind domain then the curve has exactly one singularity. See [Wie89] for proofs of these results and for further discussion.

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