FACTORIZATION HOMOLOGY OF PUNCTURED SURFACES

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Abstract. These are the notes of a talk given on the RTG Graduate Research Seminar on factorization homology along surfaces and quantum groups in Fall 2019. We use the Barr-Beck theory explained by Vasya Krylov in [Kry] to compute the factorization homology of a punctured surface. We mostly follow the exposition of [BBJ18].

1. Consequences of Barr-Beck theorem

1.1. **Rigid abelian tensor categories.** Recall the following important theorem that was explained in Vasily's talk [Kry].

Theorem 1.1. *Let $\mathcal{A}$ be a rigid abelian tensor category in* Rex, *and let $\mathcal{M} \in$ Rex be an abelian $\mathcal{A}$-module category with an $\mathcal{A}$-progenerator $m \in \mathcal{M}$. Let $\mathrm{act}_m : \mathcal{A} \to \mathcal{M}$ be the functor corresponding to $\mathcal{A}$ action on m, and act_m^R be its right adjoint. Set $T = \mathrm{act}_m^R \circ \mathrm{act}_m$. Then act_m^R induces an equivalence of $\mathcal{A}$-module categories $\mathcal{M} \simeq \underline{\mathrm{End}}(m)\text{-mod}_{\mathcal{A}}$, where $\mathcal{A}$ acts on the right by multiplication.*

The following theorem was stated by Matej in the end of his talk.

Theorem 1.2. *Let $\mathcal{A}$ be a rigid abelian tensor category, and let $\mathcal{M}$, $\mathcal{N}$ be right and left module categories, respectively, with $\mathcal{A}$-progenerators $m \in M$ and $n \in N$. Then we have an equivalence of categories*
$$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \simeq \underline{\mathrm{End}}(m)\text{-mod}_{\mathcal{N}} \simeq (\underline{\mathrm{End}}(m) - \underline{\mathrm{End}}(n))\text{-bimod}_{\mathcal{A}}.$$

Proof. The second assertion directly follows from Theorem 1.1, so we will focus on th first one.

Consider the category $\mathcal{M}^\vee \simeq \mathrm{mod}_{\mathcal{A}}\text{-}\underline{\mathrm{End}}(m)$. We have a natural evaluation functor $\mathcal{M}^\vee \boxtimes \mathcal{M} \to \mathcal{A}$ given by the relative tensor product of right and left $\underline{\mathrm{End}}(m)$-modules in $\mathcal{A}$. In addition, we have the functors $\mathrm{Vect} \to \mathcal{M}$ and $\mathrm{Vect} \to \mathcal{M}^\vee$ given by $\underline{\mathrm{Hom}}(\bullet, m)$ and $\underline{\mathrm{Hom}}(m, \bullet)$ correspondingly. Therefore we get a coevaluation functor $\mathrm{Vect} \simeq \mathrm{Vect} \boxtimes \mathrm{Vect} \to \mathcal{M} \boxtimes \mathcal{M}^\vee \to \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{M}^\vee$.

The evaluation map $\mathcal{M}^\vee \boxtimes \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \to \mathcal{N}$ induces an equivalence $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \simeq \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}^\vee, \mathcal{N})$. Composing with the evaluation ev_m at m we get a functor $G : \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \to \mathcal{N}$. Since m is a pro-generator, if $\mathrm{ev}_m(f) = 0$ for some $f \in \mathrm{Fun}_{\mathcal{A}}(\mathcal{M}^\vee, \mathcal{N})$, then $f \equiv 0$. Since $\mathcal{N}$ is abelian, [Kry, Lemma 3.62] implies that ev_m is conservative, and therefore G is conservative. Since m is projective, G preserves coequalizers. Therefore, by Barr-Beck theorem [Kry, Theorem 3.59], $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \simeq \underline{\mathrm{End}}(m)\text{-mod}_{\mathcal{N}}$. □

In this section we will apply the monadicity theorem stated above to braided tensor categories, tensor products and dominant functors.

Definition 1.3. *A functor $F : \mathcal{A} \to \mathcal{B}$ is called **dominant** if every object of $\mathcal{B}$ appears as a sub-object (equivalently using rigidity, quotient) of an object in the image of F.*

Lemma 1.4. *[BN11, Lemma 2.1] A tensor functor $F : \mathcal{A} \to \mathcal{B}$ is dominant if, and only if, its right adjoint F^R is faithful. i.e. $\mathcal{B}$ should be generated under colimits by the image of $\mathcal{A}$.*

Lemma 1.5. *Let $\mathcal{M}$, $\mathcal{N}$ be the categories with a structure of $\mathcal{A}$-module, and $F : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-module dominant functor. Suppose that $m \in \mathcal{M}$ is an $\mathcal{A}$-generator of $\mathcal{M}$. Then $F(m)$ is an $\mathcal{A}$-generator of $\mathcal{N}$.*

Proof. Since F is an $\mathcal{A}$-module functor, we have $F \circ \mathrm{act}_m(a) = F(m \otimes a) = F(m) \otimes a = \mathrm{act}_{F(m)}(a)$. The isomorphism proven above implies that $\mathrm{act}^R_{F(m)} = F^R \circ \mathrm{act}^R_m$. Since m is an $\mathcal{A}$-generator of $\mathcal{M}$, act^R_m is faithful. Since F is dominant, by Lemma 1.4, F^R is faithful. Therefore $\mathrm{act}^R_{F(m)}$ is faithful, and $F(m)$ is an $\mathcal{A}$-generator of $\mathcal{N}$. $\square$

Proposition 1.6. *Let $\mathcal{M}$ be an abelian $\mathcal{A}$-module category, $F : \mathcal{A} \to \mathcal{B}$ a dominant tensor functor, and $m \in \mathcal{M}$ a $\mathcal{A}$-progenerator. Then $m \boxtimes_{\mathcal{A}} 1_{\mathcal{B}}$ is a $\mathcal{B}$-progenerator of $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B}$, and we have an equivalence of $\mathcal{B}$-module categories,*

$$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B} \simeq F(\underline{\mathrm{End}}(m))\text{-}\mathrm{mod}_{\mathcal{B}}\,.$$

Proof. We apply Theorem 1.2 for $\mathcal{N} = \mathcal{B}$, where the structure of $\mathcal{A}$-module on $\mathcal{B}$ is given using F. $\square$

1.2. **Braided monoidal categories.** From now on, suppose that $\mathcal{A}$ is a rigid braded tensor category. Then the multiplication functor $\mathcal{A}^{\boxtimes n} \to \mathcal{A}$ is a tensor functor.

Proposition 1.7. *For any n, the tensor unit $1_{\mathcal{A}}$ is a progenerator for the n-fold right regular action on $\mathcal{A}$.*

Proof. 1) To show that $1_{\mathcal{A}}$ is a progenerator, we need to show that $\mathrm{act}^R_{1_{\mathcal{A}}}$ is faithfull or, equivalently, that $\mathrm{act}_{1_{\mathcal{A}}}$ is dominant. Indeed, $X \simeq X \otimes 1_{\mathcal{A}}$, so $\mathrm{act}_{1_{\mathcal{A}}}$ is dominant.

2) To show that $1_{\mathcal{A}}$ is $\mathcal{A}$-projective, we need to show that $\mathrm{act}^R_{1_{\mathcal{A}}}$ preserves colimits. We refer for this fact to [BBJ18, Proposition 3.12]. $\square$

For $n = 2$, consider $\mathfrak{T}_{\mathcal{A}} := T(\underline{\mathrm{End}}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}}))$, where $T : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ is the tensor product. We have the following description of this object.

Proposition 1.8.

$$\mathfrak{T}_{\mathcal{A}} = \left(\bigoplus_{V \in \mathcal{A}} V^* \otimes V \right) / \langle \mathrm{Im}(\mathrm{id}_{W^*} \otimes \phi - \phi^* \otimes \mathrm{id}_V) | \phi : V \to W \rangle$$

The composition $(V^* \otimes V) \otimes (W^* \otimes W) \overset{\sigma_{V^* \otimes V, W^*}}{\to} (W^* \otimes V^*) \otimes (V \otimes W) \overset{\iota_{V \otimes W}}{\to} \mathfrak{T}_{\mathcal{A}}$ induces a multiplication structure on $\mathfrak{T}_{\mathcal{A}}$.

2. Factorization homology of punctured surfaces

2.1. **Moduli algebra.** Let S be a punctured surface, together with a choice of an interval along the boundary.

Recall that the embedding $\emptyset \to S$ induces a functor $\mathrm{Vect} \to \int_S \mathcal{A}$, and we defined the quantum structure sheaf $\mathcal{O}_{S,\mathcal{A}}$ to be the image of $k \in \mathrm{Vect}$.

Definition 2.1. *The moduli algebra of S is $A_S = \underline{\mathrm{End}}_{\mathcal{A}}(\mathcal{O}_{S,\mathcal{A}})$, where the $\mathcal{A}$-action on $\int_S \mathcal{A}$ is given by the chosen interval.*

The following proposition underlines the importance of A_S.

Proposition 2.2. 1) *The quantum structure sheaf $\mathcal{O}_{S,\mathcal{A}}$ is an $\mathcal{A}$-progenerator of $\int_S \mathcal{A}$.*

2) *We have an equivalence of categories $\int_S \mathcal{A} \simeq A_S\text{-}\mathrm{mod}_{\mathcal{A}}$.*

Proof. First, we note that 2) follows from 1) immediately using Theorem 1.1. To prove 1) we will first need a lemma.

Lemma 2.3. *Let S be a punctured surface, and $i : D^2 \to S$ be an embedding of a disk. Then the induced functor $i_* : \mathcal{A} = \int_{D^2} \mathcal{A} \to \int_S \mathcal{A}$ is dominant.*

Proof. The factorization homology $\int_S \mathcal{A}$ is defined as a colimit over all embeddings of a disjoint union of disks in S. Let Γ stand for the corresponding diagram, so that $\int_S \mathcal{A} = \varinjlim_\Gamma \mathcal{A}^{\boxtimes k}$. Any such embedding $\sqcup_i D_i^2 \hookrightarrow S$ can be factored through a bigger disk $\sqcup_i D_i^2 \hookrightarrow D^2 \hookrightarrow S$, and any two such embeddings give rise to isomorphic functors on the level of factorization homology, cause S is path connected. Therefore $\int_S \mathcal{A}$ is generated under colimits by the image of i_*. $\square$

Since the functor Vect $\to \int_S \mathcal{A}$ factors through $\mathcal{A}$, corresponding to the embedding $i : D^2 \to S$ of a small disk, we have $i_*(1_\mathcal{A}) = \mathcal{O}_{S,\mathcal{A}}$. Lemma 2.3, Lemma 1.5 and Proposition 1.7 imply that $\mathcal{O}_{S,\mathcal{A}}$ is an $\mathcal{A}$-generator of $\int_S \mathcal{A}$.

Analogously to the proof of Lemma 1.5, to show that $\mathcal{O}_{S,\mathcal{A}}$ is an $\mathcal{A}$-projective, it is enough to show that i_*^R preserves finite colimits. By construction, $i_*^R \in \mathrm{Fun}(\int_S \mathcal{A}, \mathcal{A}) = \mathrm{Fun}(\varinjlim_\Gamma \mathcal{A}^{\boxtimes k}, \mathcal{A}) = \varprojlim_\Gamma (\mathcal{A}^{\boxtimes k}, \mathcal{A})$. Filtered limits commute with finite colimits, so it is enough to check that each of the corresponding functors $\mathcal{A}^{\boxtimes k} \to \mathcal{A}$ is cocontinious. Every such map factors through the tensor product $\mathcal{A}^{\boxtimes k} \to \mathcal{A}$, so it is enough to check it for each map $\mathcal{A} \to \mathcal{A}$, where it easily follows from the construction of that map. $\square$

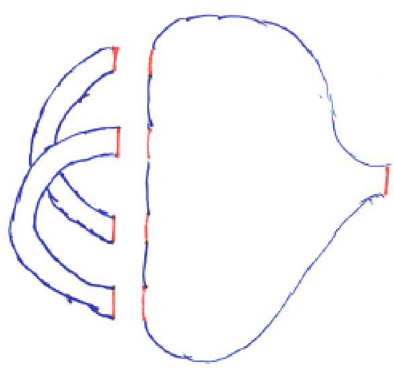
The main goal of the section is to give an explicit description of $\mathcal{A}_S$. To do that, we will first fix some data determining the surface S, namely, gluing pattern P.

2.2. Gluing pattern and the algebra a_P.

Definition 2.4. *A gluing pattern is a bijection* $P : (1, 1', 2, 2', \dots, g, g') \leftrightarrow (1, 2, \dots, 2g)$, *such that* $P(i) < P(i')$ *for all* i.

Given a gluing pattern P, we can construct a punctured surface $\Sigma(P)$ with a marked boundary interval in the following way.

We begin with a disk D^2 with a $2g+1$ boundary intervals numbered from 0 to $2g$, and then glue g handles by gluing marked intervals of the i-th handle H_i with $P(i)$ and $P(i')$-th intervals of the disk.



Plan:

1) To define an algebra structure a_P on $\mathfrak{T}_\mathcal{A}^{\otimes g}$ depending on the gluing pattern P;
2) To show a Morita equivalence between a_P and $\mathcal{A}_{\Sigma(P)}$ by computing the factorization homology.

Definition 2.5. *We say that the handles* H_i *and* H_j *for* $i < j$ *are*

- *positively linked if* $P(i) < P(j) < P(i') < P(j')$;
- *negatively linked if* $P(j) < P(i) < P(j') < P(i')$;
- *positively nested if* $P(j) < P(i) < P(i') < P(j')$;
- *negatively nested if* $P(i) < P(j) < P(j') < P(i')$;
- *positively unlinked if* $P(i) < P(i') < P(j) < P(j')$;
- *negatively unlinked if* $P(j) < P(j') < P(i) < P(i')$.

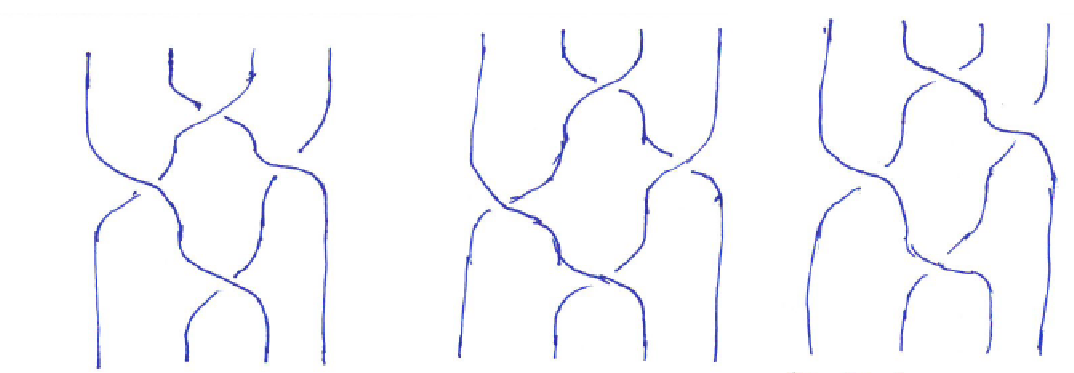
Example 2.6. *Suppose that $P = (1, 3, 1', 2, 2', 3)$. Then*
- *H_1 and H_2 are positively unlinked;*
- *H_1 and H_3 are positively linked;*
- *H_2 and H_3 are negatively nested.*

Definition 2.7. *We define the crossing morphisms $L, N, U : \mathfrak{T}_{\mathcal{A}} \otimes \mathfrak{T}_{\mathcal{A}} \to \mathfrak{T}_{\mathcal{A}} \otimes \mathfrak{T}_{\mathcal{A}}$ as the following compositions:*

$$L: \quad \begin{tikzcd} V^* \otimes V \otimes W^* \otimes W \arrow[r, "1\otimes\sigma_{V,W^*}\otimes 1"] & V^* \otimes W^* \otimes V \otimes W \arrow[d, "\sigma_{V^*,W^*}\otimes\sigma_{V,W}"] \\ W^* \otimes W \otimes V^* \otimes V & W^* \otimes V^* \otimes W \otimes V \arrow[l, "1\otimes\sigma^{-1}_{V^*,W}\otimes 1"] \end{tikzcd} .$$

$$N: \quad \begin{tikzcd} V^* \otimes V \otimes W^* \otimes W \arrow[r, "1\otimes\sigma_{V,W^*}\otimes 1"] & V^* \otimes W^* \otimes V \otimes W \arrow[d, "\sigma^{-1}_{V^*,W^*}\otimes\sigma_{V,W}"] \\ W^* \otimes W \otimes V^* \otimes V & W^* \otimes V^* \otimes W \otimes V \arrow[l, "1\otimes\sigma^{-1}_{V^*,W}\otimes 1"] \end{tikzcd} .$$

$$U: \quad \begin{tikzcd} V^* \otimes V \otimes W^* \otimes W \arrow[r, "1\otimes\sigma_{V,W^*}\otimes 1"] & V^* \otimes W^* \otimes V \otimes W \arrow[d, "\sigma_{V^*,W^*}\otimes\sigma_{V,W}"] \\ W^* \otimes W \otimes V^* \otimes V & W^* \otimes V^* \otimes W \otimes V \arrow[l, "1\otimes\sigma_{V^*,W}\otimes 1"] \end{tikzcd} .$$



Linked crossing L　　Nested crossing N　　Unlinked crossing U

Remark 2.8. *Since $\mathfrak{T}_{\mathcal{A}} \in \mathcal{A}$, we have a natural braiding $\mathfrak{T}_{\mathcal{A}} \otimes \mathfrak{T}_{\mathcal{A}} \to \mathfrak{T}_{\mathcal{A}} \otimes \mathfrak{T}_{\mathcal{A}}$. It coincides with the constructed morphism U.*

No we are ready to define the algebra a_P. We set $a_P = \mathfrak{T}_{\mathcal{A}}^{\otimes g}$, and will endow it with an algebra structure. Let $\mathfrak{T}_{\mathcal{A}}^{(i)}$ denote the i-th copy of $\mathfrak{T}_{\mathcal{A}}$ in $\mathfrak{T}_{\mathcal{A}}^{\otimes g}$. Then we have $a_P \otimes a_P = \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \dots \mathfrak{T}_{\mathcal{A}}^{(g)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \dots \mathfrak{T}_{\mathcal{A}}^{(g)}$. We have a well defined product on each copy, $m^{(i)} : \mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(i)} \to \mathfrak{T}_{\mathcal{A}}^{(i)}$. The algebra structure is given by "braiding", i.e. a morphism $\mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \dots \mathfrak{T}_{\mathcal{A}}^{(g)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \dots \mathfrak{T}_{\mathcal{A}}^{(g)} \to \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \dots \mathfrak{T}_{\mathcal{A}}^{(g)} \otimes \mathfrak{T}_{\mathcal{A}}^{(g)}$. To construct such morphism, it is enough to construct a morphism $\mathfrak{T}_{\mathcal{A}}^{(j)} \otimes \mathfrak{T}_{\mathcal{A}}^{(i)} \to \mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)}$ for all $1 \leq i < j \leq g$. We define it as $L^{\pm 1}$ if the handles H_i,

H_j are $\pm$ linked, $N^{\pm 1}$ if the handles H_i, H_j are $\pm$ nested and $U^{\pm 1}$ if the handles H_i, H_j are $\pm$ unlinked.

Example 2.9. *Suppose that $P = (1, 3, 1', 2, 2', 3)$. Then the algebra structure on a_P is given in the following way.*

$$\begin{tikzcd}
\mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \arrow[r, "1\otimes 1\otimes L\otimes 1\otimes 1"] & \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \arrow[d, "1\otimes U\otimes 1\otimes 1\otimes 1"] \\
\mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \arrow[d, "m^{(1)}\otimes m^{(2)}\otimes m^{(3)}"] & \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} \arrow[l, "1\otimes 1\otimes N^{-1}\otimes 1\otimes 1"'] \\
\mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \mathfrak{T}_{\mathcal{A}}^{(3)} &
\end{tikzcd}.$$

2.3. **The main result.** The following is the main result of the talk.

Theorem 2.10. *Let $\mathcal{A}$ be an abelian rigid balanced braided tensor category in Pr_c. We have a Morita equivalence between algebras $A_{\Sigma(P)}$ and a_P, and an equivalence of categories,*

$$\int_S \mathcal{A} \simeq a_P\text{-mod}_{\mathcal{A}} \simeq A_{\Sigma(P)}\text{-mod}_{\mathcal{A}}$$

Remark 2.11. *In fact, we have an equivalence of the algebras a_P and $A_{\Sigma(P)}$. To prove it one has to use the pointing on the categories a_P-mod$_{\mathcal{A}}$ and $A_{\Sigma(P)}$-mod$_{\mathcal{A}}$ given by the quantum structure sheaf.*

Proof. We want to compute the factorization homology over $\Sigma(P)$. We will use the excision for the disk with marked intervals D^2 and the union of handles $\bigsqcup H_i$.

For the disk, we have $\int_{D^2} \mathcal{A} \simeq \mathcal{A}$, but the markings on D induce the structure of $\mathcal{A}^{\boxtimes 2g} - \mathcal{A}$ bimodule on $\int_{D^2} \mathcal{A}$, where we assume that 0-th interval is on the right, and $2g$ others on the left. Note that the action of $\mathcal{A}^{\boxtimes 2g}$ naturally factors through the tensor functor $\mathcal{A}^{\boxtimes 2g} \to \mathcal{A}$. We denote the category $\mathcal{A}$ together with a structure of $\mathcal{A}^{\boxtimes 2g} - \mathcal{A}$ bimodule by ${}_{2g}\mathcal{A}_{\mathcal{A}}$.

For every handle H_i, we have $\int_{H_i} \mathcal{A} \simeq \mathcal{A}$, and the two marked intervals give $\int_{H_i} \mathcal{A}$ a structure of a right $\mathcal{A} \boxtimes \mathcal{A}$-module. Then $\int_{\bigsqcup_i H_i} \mathcal{A} \simeq \mathcal{A}^{\boxtimes g}$ with a structure of a right $\mathcal{A}^{\boxtimes 2g}$ module. Note that on the i-th copy of $\mathcal{A}^{(i)}$ in $\mathcal{A}^{\boxtimes g}$ there is an action of $\mathcal{A}^{P(i)} \boxtimes \mathcal{A}^{P(i')}$ inside $\mathcal{A}^{\boxtimes 2g}$. We denote the resulting category by $\mathcal{A}^P$. In other words, we have $(a_1 \boxtimes a_2 \boxtimes \ldots \boxtimes a_g) \boxtimes (b_1 \boxtimes b_2 \boxtimes \ldots \boxtimes b_{2g}) = (a_1 \otimes b_{P(1)} \otimes b_{P(1')}) \boxtimes \ldots \boxtimes (a_g \otimes b_{P(g)} \otimes b_{P(g')})$.

The excision property implies that

$$\int_S \mathcal{A} \simeq \mathcal{A}^P \boxtimes_{\mathcal{A}^{\boxtimes 2g}} {}_{2g}\mathcal{A}_{\mathcal{A}} \tag{*}$$

.

Let $\tau_P \in S_{2g}$ be the permutation obtained by the precomposing P with the map $\{1, 2, \ldots, 2g\} \to \{1, 1', 2, 2', \ldots, g, g'\}$ given by $2k \to k$, $2k-1 \to k'$. Applying the corresponding functor (defined by braiding) τ_P to an object $\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^{\boxtimes g} \in \mathcal{A}^{\boxtimes 2n}$, we get an algebra denoted by $\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P$.

Note that Proposition 1.7 and Theorem 1.1 imply that $\int_{H_i} \mathcal{A} \simeq \underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})$-mod$_{\mathcal{A}^{\boxtimes 2}}$. And therefore $\mathcal{A}^P = \int_{\bigsqcup_i H_i} \mathcal{A} \simeq \underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P$-mod$_{\mathcal{A}^{\boxtimes 2g}}$, and $\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P$ is an $\mathcal{A}^{\boxtimes 2g}$-progenerator of $\mathcal{A}^P$.

We apply Theorem 1.2 for the dominant tensor functor $T : \mathcal{A}^{\boxtimes 2g} \to \mathcal{A}$ to * and get

$$\int_S \mathcal{A} \simeq T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)\text{-mod}_{\mathcal{A}} \tag{**}$$

By Proposition 2.2, we get a Morita equivalence between A_S and $T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)$. It remains to show the isomorphism $T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P) \simeq a_P$, where a_P is the algebra constructed in Section 2.2.

Let us denote the subalgebra $\underline{\mathrm{End}}(1_{\mathcal{A}_{P(i)}\mathcal{A}_{P(i')}}$ of $T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)$ by $\mathfrak{T}_{\mathcal{A}}^{(i,i')}$, and set $\mathfrak{T}_{\mathcal{A}}^{(i)} = T(\mathfrak{T}_{\mathcal{A}}^{(i,i')})$. Note that by construction we have $\mathfrak{T}_{\mathcal{A}}^{(i)} \simeq \mathfrak{T}_{\mathcal{A}}$, where $\mathfrak{T}_{\mathcal{A}} = T(\underline{\mathrm{End}}_{\mathcal{A}^{\boxtimes 2}}(1_{\mathcal{A}})$ as in Section 1.2. The multiplication map $m : \mathfrak{T}_{\mathcal{A}}^{(1)} \otimes \mathfrak{T}_{\mathcal{A}}^{(2)} \otimes \ldots \otimes \mathfrak{T}_{\mathcal{A}}^{(g)} \to T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)$ is an isomorphism on objects. Therefore it is enough to compute the pairwise cross relations between factors in a_P and $T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)$.

Suppose that $i < j$. Note that $\mathfrak{T}_{\mathcal{A}}^{(i,i')} \otimes \mathfrak{T}_{\mathcal{A}}^{(j,j')} = \mathfrak{T}_{\mathcal{A}}^{(j,j')} \otimes \mathfrak{T}_{\mathcal{A}}^{(i,i')}$, because these two algebras of endomorphisms occupy different factors.

We have well-defined diagrams that send $\{i, i', j, j'\} \to \{P(i), P(i'), P(j), P(j')\}$, and $\{j, j', i, i'\} \to \{P(i), P(i'), P(j), P(j')\}$. Let $J_{ij} : \mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)} \to T^4(\mathfrak{T}_{\mathcal{A}}^{(i,i')} \otimes \mathfrak{T}_{\mathcal{A}}^{(j,j')})$ and $J_{ji} : \mathfrak{T}_{\mathcal{A}}^{(j)} \otimes \mathfrak{T}_{\mathcal{A}}^{(i)} \to T^4(\mathfrak{T}_{\mathcal{A}}^{(j,j')} \otimes \mathfrak{T}_{\mathcal{A}}^{(i,i')})$ be the maps given using these diagrams.

Then we have a commutative diagram.

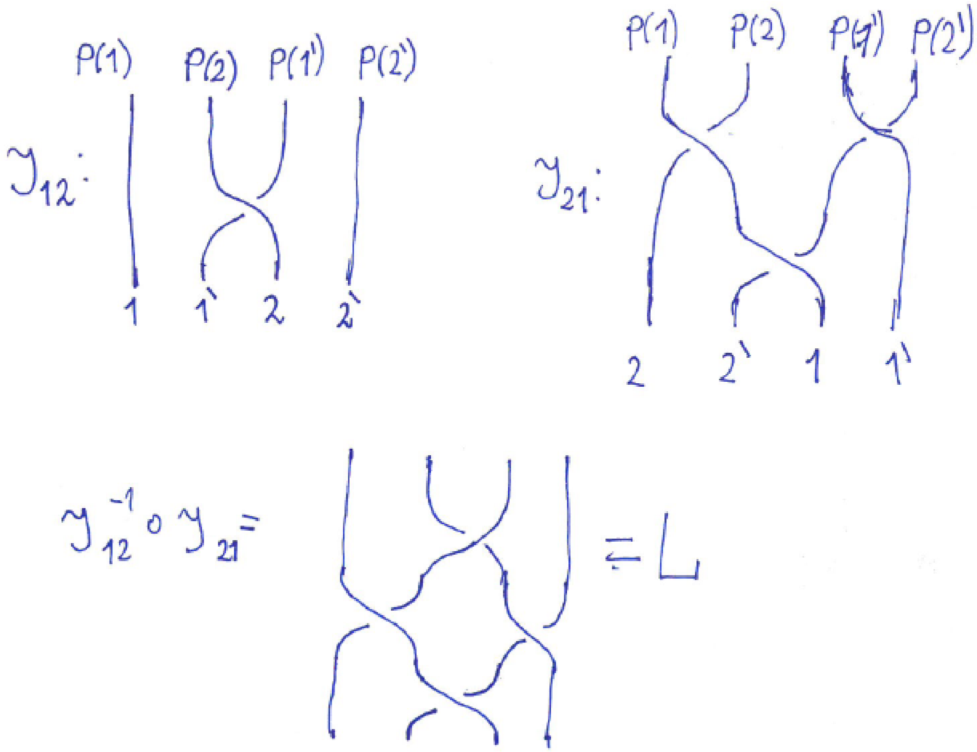
$$\begin{tikzcd}
 & T^4(\mathfrak{T}_{\mathcal{A}}^{(i,i')} \otimes \mathfrak{T}_{\mathcal{A}}^{(j,j')}) \arrow[rr, leftrightarrow, "="] \arrow[ddr, "T^4(m)"] & & T^4(\mathfrak{T}_{\mathcal{A}}^{(j,j')} \otimes \mathfrak{T}_{\mathcal{A}}^{(i,i')}) \arrow[ddl, "T^4(m)"'] & \\
\mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)} \arrow[ur, "J_{ij}"] \arrow[drr, "m"'] & & & & \mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)} \arrow[ul, "J_{ji}"'] \arrow[dll, "m"] \\
 & & a_P & &
\end{tikzcd}$$

We have $m|_{\mathfrak{T}_{\mathcal{A}}^{(j)} \otimes \mathfrak{T}_{\mathcal{A}}^{(i)}} = m|_{\mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)}} \circ J_{12}^{-1} J_{21}$. We want to show that $m|_{\mathfrak{T}_{\mathcal{A}}^{(j)} \otimes \mathfrak{T}_{\mathcal{A}}^{(i)}} = m|_{\mathfrak{T}_{\mathcal{A}}^{(i)} \otimes \mathfrak{T}_{\mathcal{A}}^{(j)}} \circ C$, where C is defined from the gluing pattern P as in Section 2.2, i.e. we want to prove $C = J_{12}^{-1} J_{21}$. That will prove an isomorphism of algebras a_P and $T(\underline{\mathrm{End}}(1_{\mathcal{A}\mathcal{A}})^P)$. We will check this isomorphism in the next section.

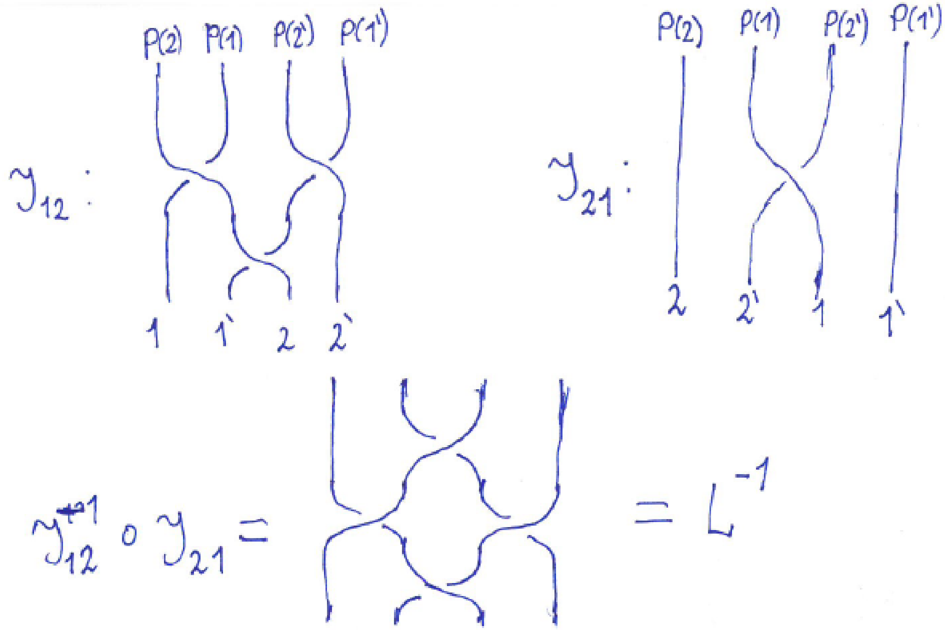
□

2.4. **Computing the braiding for every pair of habdles.** Let us check the isomorphism $C = J_{12}^{-1} J_{21}$ for all cases of relation between H_i and H_j.

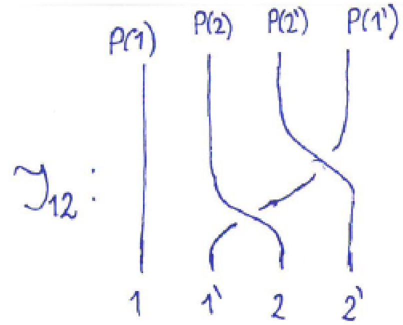
1) H_i and H_j are positively linked. $P : \{1, 1', 2, 2'\} \to \{1, 3, 2, 4\}$.

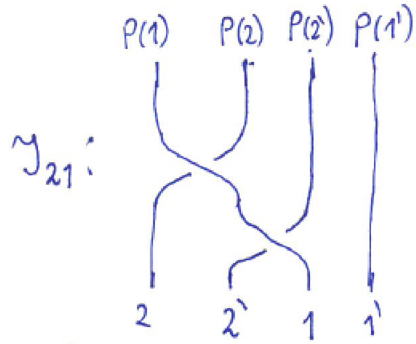


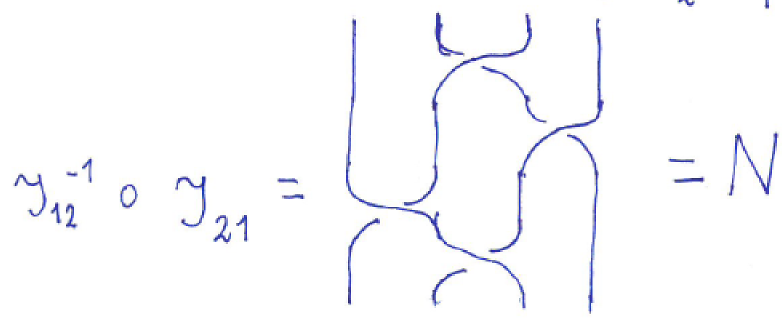
2) H_i and H_j are negatively linked. $P : \{1, 1', 2, 2'\} \to \{2, 4, 1, 3\}$.



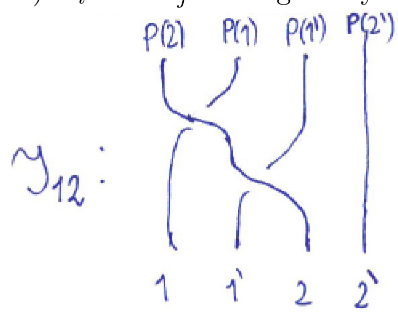
3) H_i and H_j are positively nested. $P : \{1, 1', 2, 2'\} \to \{1, 4, 2, 3\}$.

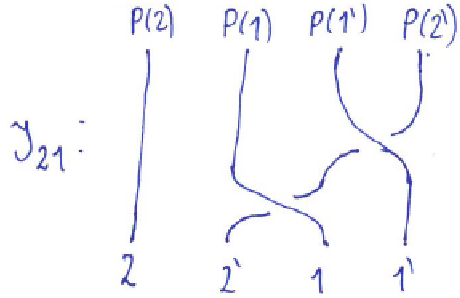


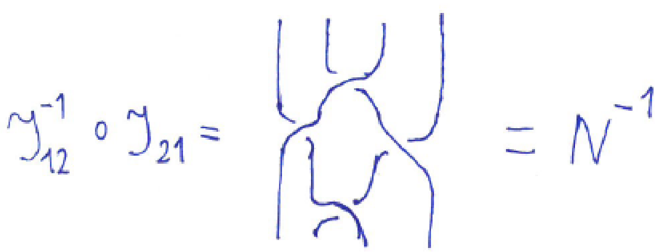




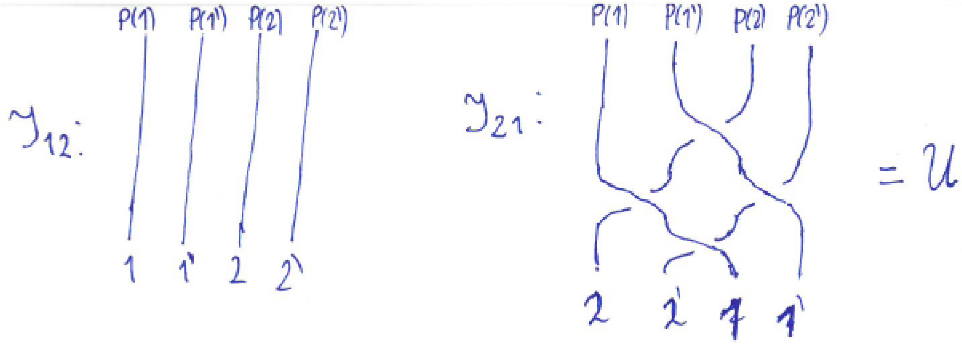
4) H_i and H_j are negatively nested. $P : \{1, 1', 2, 2'\} \to \{2, 3, 1, 4\}$.





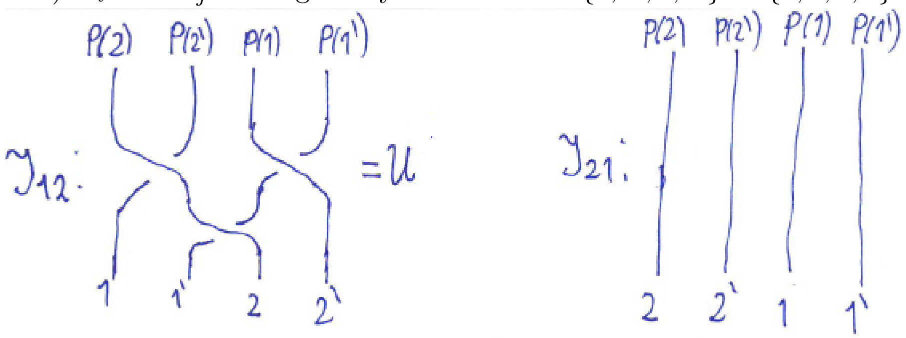


5) H_i and H_j are positively unlinked. $P : \{1, 1', 2, 2'\} \to \{1, 2, 3, 4\}$.



$$\mathcal{J}_{12}^{-1} \circ \mathcal{J}_{21} = \mathcal{J}_{21} = \mathcal{U}$$

6) H_i and H_j are negatively unlinked. $P : \{1, 1', 2, 2'\} \to \{3, 4, 1, 2\}$.



$$\mathcal{J}_{12}^{-1} \circ \mathcal{J}_{21} = \mathcal{J}_{12}^{-1} = \mathcal{U}^{-1}$$

References

[BBJ18] David Ben-Zvi, Adrien Brochier, and David Jordan. "Integrating quantum groups over surfaces". In: *Journal of Topology* 11.4 (Aug. 2018), pp. 874–917. DOI: `10.1112/topo.12072`. URL: `https://doi.org/10.1112/topo.12072`.

[BN11] A. Bruguieres and S. Natale. "Exact Sequences of Tensor Categories". In: *International Mathematics Research Notices* (Jan. 2011). DOI: `10.1093/imrn/rnq294`. URL: `https://doi.org/10.1093/imrn/rnq294`.

[Kry] Vasily Krylov. "2-categories and Barr-Beck for module categories". In: (). URL: `https://web.northeastern.edu/brwilliams/wp-content/uploads/2019/10/2-categories-talk-vasya.pdf`.

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