Lets remember what we did last time: (2,1) - category

$\underline{Rex}$: finitely co-complete, k-linear categories w/ right exact functors & k-linear nat. transforms

$\underline{Pr_c}$: Compactly generated presentable categories w/ compact & co-continuous functors & k-linear nat. transforms

$$\text{ind} : \underline{Rex} \rightleftarrows \underline{Pr_c} : \text{comp}$$

$$\mathcal{C} \longmapsto \text{Ind}(\mathcal{C})$$

$$\text{comp}(\mathcal{D}) \longleftarrow \mathcal{D}$$

Yields an equivalence.

Defined the Deligne-Kelly tensor product:

For $\mathcal{C}, \mathcal{D} \in \underline{Rex}$ we defined a new $\mathcal{C} \boxtimes \mathcal{D} \in \underline{Rex}$ characterized by the property: $\underline{Rex}[\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}] \simeq \text{Bilin}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$

$\text{Ind} : \underline{Rex} \longrightarrow \underline{Pr_c}$ extends to an equivalence of symmetric closed monoidal categories $(\underline{Rex}, \boxtimes) \longrightarrow (\underline{Pr_c}, \boxtimes)$

Closed because

$$\underline{Rex}[\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E}] \simeq \underline{Rex}[\mathcal{C}, \underline{Rex}[\mathcal{D}, \mathcal{E}]]$$

Prop: $(\underline{Rex}, \boxtimes) \simeq (\underline{Pr_c}, \boxtimes)$ is closed under small 2-colimits & the tensor product preserves 2-colimits in each factor.

technical result which lets us to calculate factorization homology $\rightsquigarrow$ uses some ∞-categorical stuff

Lets define operads: Vaguely an operad in a symmetric monoidal category $(\mathcal{M}, \otimes)$ is a collection of objects $\{P(n)\}_{n \in \mathbb{N}}$ in $\mathcal{M}$ (think $\mathcal{M} = \underset{\sim}{Set}$)

with operations:

- $e : 1 \longrightarrow P(1)$ a unit
- $\circ : P(k_1) \otimes P(k_2) \otimes \dots \otimes P(k_n) \otimes P(n) \longrightarrow P(k_1 + k_2 + \dots + k_n)$
- $S_n \curvearrowright P(n)$

\+ compatibility $\rightsquigarrow$ e acts like the unit $\left(\underbrace{P(n) \cong 1 \otimes P(n) \xrightarrow{e \otimes id} P(1) \otimes P(n) \to P(n)} \right)$

$\rightsquigarrow$ The S_n action on $P(n) \otimes P(k_1) \otimes \dots \otimes P(k_n)$

maps to the S_n action on $P(k_1 + k_2 + \dots + k_n)$ by permuting the factors.

(other S_{k_i} actions inject)

$P(n)$'s are n-ary operators. $\rightsquigarrow$ Have associativity

$\rightsquigarrow$ in general for any morphism of finite sets $\langle n \rangle^\circ \xrightarrow{\alpha} \langle m \rangle^\circ$ we get a composition morphism

$$P(m) \otimes \bigotimes_{j \in \langle m \rangle} P(|\alpha^{-1}(j)|) \longrightarrow P(n)$$

(these two def'n is equivalent)

(associativity takes the form of $\langle p \rangle^\circ \xrightarrow{\alpha} \langle q \rangle^\circ \xrightarrow{\beta} \langle r \rangle^\circ$)

$$\begin{tikzcd}
 & P(r) \otimes \bigotimes_{j \in \langle q \rangle} P(|\alpha^{-1}(j)|) \otimes \bigotimes_{k \in \langle r \rangle} P(|\beta^{-1}(k)|) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\circlearrowright"] & \\
P(r) \otimes \bigotimes_{k \in \langle r \rangle} P(|(\beta \circ \alpha)^{-1}(k)|) \arrow[dr] & & P(q) \otimes \bigotimes_{j \in \langle q \rangle} P(|\alpha^{-1}(j)|) \arrow[dl] \\
 & P(p) &
\end{tikzcd}$$

We can let $\{*, 1, 2, \dots, n\} = \langle n \rangle$ the pointed set & let Γ be the category of pointed finite sets

For an operad P we can define the category $P^{\otimes}$ to be the category w/ Obj: $[n]$ $n \in \mathbb{N}$

Mor: $[n] \longrightarrow [m]$ is a map $\alpha : \langle n \rangle \longrightarrow \langle m \rangle$ & a elt $\phi \in \bigotimes_{j \in \langle m \rangle^{\circ}} P(|\alpha^{-1}(j)|)$

Def: $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ is inert if each $(\alpha^{-1}(\{i\}))$ is a singleton.

Have an obvious forgetful functor $\pi : P^{\otimes} \longrightarrow \Gamma$

Def: An ∞-operad is a functor $\pi : \mathcal{O}^{\otimes n} \longrightarrow N(\Gamma)$ between ∞ categories w/ the properties

1) For any (inert) $f : \langle m \rangle \longrightarrow \langle n \rangle$ and $C \in \mathcal{O}^{\otimes}(\langle m \rangle)$ there is a morphism $\bar{f} : C \to C'$ (π co-Cartesian...) lifting f

$$\left(\begin{tikzcd} C \arrow[r, "\bar{f}"] \arrow[d, "\pi"'] & C' \arrow[d, "\pi"] \\ \langle m \rangle \arrow[r, "f"] & \langle n \rangle \end{tikzcd}\right)$$

Satisfies pullback diagram

$\rightsquigarrow$ induces $f_! : \mathcal{O}^{\otimes}_{\langle m \rangle} \to \mathcal{O}^{\otimes}_{\langle n \rangle}$

2) $C \in \mathcal{O}^{\otimes}(\langle m \rangle)$, $C' \in \mathcal{O}^{\otimes}(\langle n \rangle)$ objects $f : \langle m \rangle \longrightarrow \langle n \rangle$ a map and let $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C')$ union of connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ lying over f. Choosing π-co-cartesian $C' \longrightarrow C'_i$ over (inert) $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ then $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \longrightarrow \bigotimes_{i \in \langle n \rangle^{\circ}} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$ is an equivalence.

3) For every collection $C_1, \dots, C_n \in \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ $\rightsquigarrow$ exists $C \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ & π-cocartesian morph $C \to C_i$ covering $\rho^i : \langle n \rangle \to \langle 1 \rangle$

Ex: define **Ass** to be the associative operad:

- Obj are Γ
- morph: a pair $(f, \{\leq_i\}_{i\in n})$: $f: \langle m\rangle \longrightarrow \langle n\rangle$ and a linear ordering $\leq_i$ on each $f^{-1}(\{i\})$
- composition of $(f, \{\leq_i\}_{i\in\langle n\rangle^o}) : \langle m\rangle \longrightarrow \langle n\rangle$ & $(g, \{\leq'_j\}_{j\in\langle p\rangle^o}) : \langle n\rangle \to \langle p\rangle$ is the pair $(g\circ f, \{\leq''_j\}_{j\in\langle p\rangle^o})$ w/ $\leq''_j$ the lexicographic ordering: $a, b \in \langle m\rangle^o$ w/ $g\circ f(a) = g\circ f(b) = j$ we have

$a \leq''_j b$ iff $f(a) \leq'_j f(b)$ and $a \leq_i b$ if $f(a) = f(b) = i$.

Def: 1) obj of E_n are $\langle p\rangle \in \Gamma$

2) $\langle m\rangle \longrightarrow \langle n\rangle \iff \alpha: \langle m\rangle \longrightarrow \langle n\rangle$ in Γ

a rectilinear embedding $\square^k \times \alpha^{-1}(\{j\}) \to \square^k$

3) each is endowed w/ the topology on rectilinear embeddings

4) ~~...~~ Composition is obvious.

($\underline{k=0}$: maps of finite sets $\rightsquigarrow$ all going to the identity

$\underline{k=1}$: ordering of intervals $\iff$ associative operad $\underline{Ass} \simeq E_1$)

$\underline{k=2}$: ... $\underline{k=3}$: symmetric

Def An algebra over an (not ∞-) operad P in M is an object $X \in M$ with a collection of maps $P(k) \otimes X^{\otimes k} \longrightarrow X$

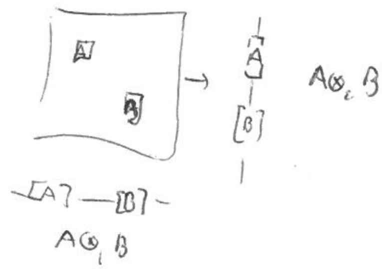
An algebra over an ∞-operad $\mathcal{O}^{\otimes n}$ is a map of ∞-operads

$$f : \mathcal{O}^{\otimes n} \longrightarrow \mathcal{C}^{\otimes n} \quad (\text{fibration}) + \text{properties}$$

E_0 algebra in $(\mathcal{C}at, \times)$: pointed categories.

E_1 algebra in $(\mathcal{C}at, \times)$: monoidal categories (use $\underline{Ass}$)

E_2 algebra in $(\mathcal{C}at, \times)$:



[in general E_{k+1}-algebra objects $=$ E_k (E_1-algebra objects)]

So we have $\otimes_2 : (\mathcal{C}, \otimes_1) \otimes^{\times} (\mathcal{C}, \otimes_1) \longrightarrow (\mathcal{C}, \otimes_1)$

$$((a \otimes_1 b), (c \otimes_1 d)) \longmapsto (a \otimes_1 b) \otimes_2 (c \otimes_1 d)$$

But it's <u>monoidal</u>

$$(a \otimes_1 b) \otimes_2 (c \otimes_1 d) \simeq (a \otimes_2 c) \otimes_1 (b \otimes_2 d)$$

let $b = c = \mathbb{1}_e \rightsquigarrow (a \otimes_1 \mathbb{1}) \otimes_2 (\mathbb{1} \otimes d) \simeq (a \otimes \mathbb{1}) \otimes_1 (\mathbb{1} \otimes d)$

so $\otimes_1$ & $\otimes_2$ are identified $\rightsquigarrow$ and

$a = d = \mathbb{1}_e \rightsquigarrow (\mathbb{1} \otimes_1 b) \otimes_2 (c \otimes_1 \mathbb{1}) \simeq (\mathbb{1} \otimes_2 c) \otimes (b \otimes \mathbb{1})$

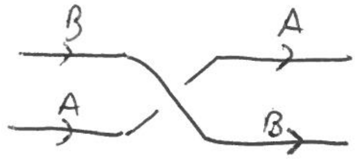
call it $\sigma_{b,c} : b \otimes c \longrightarrow c \otimes b$

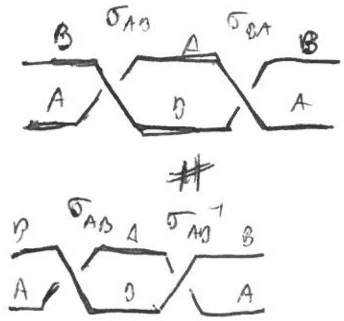
A <u>braided</u> <u>monoidal</u> <u>category</u>.

Have the usual hexagon identity

$$\begin{tikzcd} & (B\otimes A)\otimes C \arrow[r] & B\otimes(A\otimes C) \arrow[dr] & \\ (A\otimes B)\otimes C \arrow[ur] \arrow[dr] & & & B\otimes(C\otimes A) \\ & A\otimes(B\otimes C) \arrow[r] & (B\otimes C)\otimes A \arrow[ur] & \end{tikzcd}$$

Diagrammatics of braided monoidal categories

σ_{AB} 

$\sigma_{AB}^{-1} \neq \sigma_{BA}$ i.e. 

The point of DBZ, AB, DJ is to do k-linear analogues of this:

<u>Def:</u> A <u>tensor category</u> in Pr_c is an E_1-algebra $\mathcal{A}$ in Pr_c. A <u>Braided tensor category</u> in Pr_c is an E_2-algebra in Pr_c

<u>Def:</u> A tensor category is <u>rigid</u> if all compact objects are left & right dualizable

⌈ $X \in \mathcal{A}$, it has a left dual if there is an X^* & maps

$ev_X : X^* \otimes X \longrightarrow \mathbb{1}$, $coev : \mathbb{1} \longrightarrow X \otimes X^*$ s.t.

$X = \mathbb{1}\otimes X \xrightarrow{coev \otimes 1_X} X \otimes X^* \otimes X \xrightarrow{1_X \otimes ev_X} X \otimes \mathbb{1} = X$ is the identity

right dualizable if we have ${}^*X$ & maps

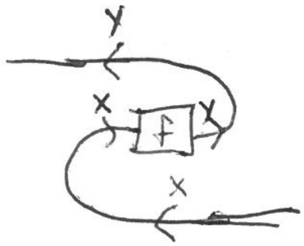
$ev'_{X} : X \otimes {}^*X \longrightarrow \mathbb{1}$, $coev'_X : \mathbb{1} \longrightarrow {}^*X \otimes X$ ⌋

<u>Diagrammatics:</u> Allow arrows in opposite direction ${}^*X$ or X^* $\longleftarrow$

coev:  , ev:  , coev':  , ev': 

For a morph $f: X \longrightarrow Y$ we get a map

$$f^*: Y^* \longrightarrow X^* \otimes X \otimes Y^* \longrightarrow X^* \otimes Y \otimes Y^* \longrightarrow X^*$$



& similarly ${}^*f$.

$\Rightarrow$ $(-)^*$ and ${}^*(-)$ become contravariant functors.

Defining the operad Disk^2_{or} as objects 2-disks w/ $SO(2)$ attached to each disk (general construction of $\mathcal{O} \rtimes G$ when $G \curvearrowright \mathcal{O}(n)$ for each $\mathcal{O}(n)$)

In order to define an Disk^2_{or}-algebra structure on $A \in \mathrm{Pr}_L$ need the additional structure of a twisting:
(twisting $\Leftrightarrow$ winding number in $SO(2)$)

<u>Def:</u> A twist on a braided monoidal category is a family of iso's

$$\theta_X : X \longrightarrow X \quad \text{with } id_{\mathbb{1}} = \theta_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1} \text{ the identity.}$$

$$\begin{tikzcd}
A\otimes B \arrow[r, "\sigma_{AB}"] \arrow[d, "\theta_{A\otimes B}"'] \arrow[dr, phantom, "\circlearrowright"] & B\otimes A \arrow[d, "\theta_B \otimes \theta_A"] \\
A\otimes B & B\otimes A \arrow[l, "\sigma_{BA}"']
\end{tikzcd}$$

<u>Balanced</u> braided monoidal category when have twists

Ribbon category:

θ_X:  ,

σ_{XY}: 

The relations are just untwisting this... $\hookrightarrow$ $\theta_{X\otimes Y}$ 

Def: If $\mathcal{A}$ is right (or left) dualizable, we call it a pivotal if we have a monoidal nat. iso:

$$c_X : X \longrightarrow {}^{**}X$$

Note that pivotal + right (or left) dualizable $\Rightarrow$ rigid.

(i.e. if we have $ev'_X : X \otimes {}^*X \longrightarrow \mathbb{1}$ and $coev'_X : \mathbb{1} \longrightarrow {}^*X \otimes X$

then ${}^*(ev'_X) : {}^*\mathbb{1} = \mathbb{1} \longrightarrow {}^*(X \otimes {}^*X) = {}^{**}X \otimes {}^*X \xrightarrow{c_X \otimes \mathbb{1}_X} X \otimes {}^*X$

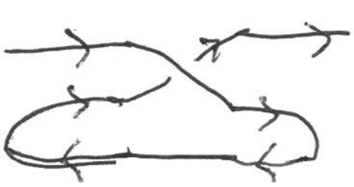
and ${}^*(coev'_X) : {}^*X \otimes X \longrightarrow \mathbb{1}$)

Actually: Braided + right dualizable $\Rightarrow$ rigid (use $\sigma_{X,{}^*X}$)

In this setting we also get

$$b_X : {}^{**}X \longrightarrow X \quad \text{via}$$

$$b_X : {}^{**}X \xrightarrow{coev \otimes \mathbb{1}_{**X}} {}^*X \otimes X \otimes {}^{**}X \xrightarrow{\mathbb{1}_{*X} \otimes \sigma_{X,{}^{**}X}} {}^*X \otimes {}^{**}X \otimes X \xrightarrow{ev_{*X}} X$$

${}^{**}X$  X $\longleftrightarrow$ $*$  X

Note b_X doesn't define a pivotal structure as it may not be <u>monoidal</u>

But we do have:

<u>Thm</u> $\mathcal{A}$ rigid, braided, monoidal category ($\Rightarrow b_X$) then $\mathcal{A}$ is pivotal $\Leftrightarrow$ balanced

<u>pf:</u> $\boxed{\Rightarrow}$ Have $i_X \rightsquigarrow$ define $\theta_X : X \xrightarrow{i_X} {}^{**}X \xrightarrow{b_X} X$

$\Leftarrow$ Have $\theta_X \rightsquigarrow$ define $i_X : X \xrightarrow{\theta_X} X \xrightarrow{b_X^{-1}} {}^{**}X$.

Lets also recall now

<u>Def:</u> • $\mathcal{A}$ a tensor cat in $\underline{Pr_c}$ a (right) $\mathcal{A}$-mod $\mathcal{M}$ is a cat & a functor $act_{\mathcal{M}} : \mathcal{M} \boxtimes \mathcal{A} \longrightarrow \mathcal{M}$ w/ associativity axioms. abbreviate $act(m \boxtimes X) = m \otimes X$

• $(\mathcal{A}, \mathcal{B})$-bimodule categories are right $\mathcal{A}^{\otimes\text{-}op} \boxtimes \mathcal{B}$ or left $\mathcal{A} \boxtimes \mathcal{B}^{\otimes\text{-}op}$-mods.

• For $m \in \mathcal{M}$ define $act_m : \mathcal{A} \longrightarrow \mathcal{M}$

$$a \longmapsto act_{\mathcal{M}}(m \boxtimes a) = m \otimes a$$

$\rightsquigarrow$ automatically has a right adjoint. [vague idea:

$act^R_m : \mathcal{M} \longrightarrow \mathcal{A}$

$$act^R_m(n) = \begin{cases} a & \text{s.t. } n = m \otimes a \ ? \\ 0 & \text{else} \end{cases}$$]

• Write $act^R_m(n) = \underline{Hom}(m,n) \in \mathcal{A}$. We also get a well-defined composition

$$\underline{Hom}(n,p) \otimes \underline{Hom}(m,n) \longrightarrow \underline{Hom}(m,p)$$

• Define $\underline{End}_{\mathcal{A}}(m) = \underline{Hom}(m,m) = act^R_m(m) = act^R_m(act_m(\mathbb{1}))$

Prop: $\mathcal{M}, \mathcal{N}$ $\mathcal{A}$-mod in Pr_c & $F: \mathcal{M} \to \mathcal{N}$ an $\mathcal{A}$-mod functor ($\mathcal{A}$ a rigid $\otimes$-cat)

(commutes w/ $act_{\mathcal{M}}$ & $act_{\mathcal{N}}$ $F(m \otimes X) = F(m) \otimes X$)

w/ right adjoint $F^R: \mathcal{N} \to \mathcal{M}$. Then F^R has an $\mathcal{A}$-mod structure.

pf: $$\mathrm{Hom}_{\mathcal{M}}(m, F^R(n \otimes X)) = \mathrm{Hom}_{\mathcal{N}}(F(m), n \otimes X) = \mathrm{Hom}_{\mathcal{N}}(F(m) \otimes {}^*X, n)$$

$$= \mathrm{Hom}_{\mathcal{N}}(F(m \otimes {}^*X), n) = \mathrm{Hom}_{\mathcal{M}}(m \otimes {}^*X, F^R(n))$$

$$= \mathrm{Hom}_{\mathcal{M}}(m, F^R(n) \otimes X)$$

Prop: A a rigid tensor cat in Pr_c, $T: A \boxtimes A \to A$ has a co-continuous right adjoint

pf: T^R comes for free, it's linear by above. Not necessarily co-continuous.

T has an A-bimod structure ($X \otimes T(Y \boxtimes Z) \otimes W$)

& since A is rigid, T^R is an $A \boxtimes A^{\otimes\text{-}op}$ functor $\Rightarrow$

$$T^R(X) = T^R((X \boxtimes \mathbb{1}_A) \otimes \mathbb{1}_A) \simeq X \boxtimes \mathbb{1}_A\, T^R(\mathbb{1}_A)$$

T^R determined by $T^R(\mathbb{1}_A)$ $\longleftarrow$ <u>note this fact later!</u>

& $\boxtimes$ is cocontinuous $\Rightarrow$ T^R is.

$\Rightarrow$ $act: \mathcal{M} \boxtimes A \to \mathcal{M}$ has a cocontinuous adjoint $\rightarrow$

$$\mathcal{M} \xrightarrow{Id_{\mathcal{M}} \boxtimes 1_{\mathcal{A}}} \mathcal{M} \boxtimes \mathcal{A} \xrightarrow{Id_{\mathcal{M}} \boxtimes T^R} \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \xrightarrow{act \boxtimes Id_{\mathcal{A}}} \mathcal{M} \boxtimes \mathcal{A}$$

Def: The relative Kelly tensor product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ is defined as the colimit: (2-sided bar complex)

$$\vdots\ \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{N} \rightrightarrows \mathcal{M} \boxtimes \mathcal{A} \boxtimes \mathcal{N} \rightrightarrows \mathcal{M} \boxtimes \mathcal{N}$$

Characterized by: $Pr_c[\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}, \mathcal{E}] \simeq Bal_{\mathcal{A}}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{E})$

$\mathcal{A}$-balanced functors:

$F: \mathcal{M} \boxtimes \mathcal{N} \longrightarrow \mathcal{E}$ is $\mathcal{A}$-balanced when we have

$$\beta_{m,X,n}: F(m \otimes X \boxtimes n) \simeq F(m \boxtimes X \otimes n)$$

nat iso's.

Lets recall the Barr-Beck theorem.

$(L, R): \mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ Left & right adjoints $T = R \circ L$ has a monad structure on $\mathcal{C}$ via unit $\eta: id_{\mathcal{C}} \longrightarrow R \circ L$ &

$m: R \circ L \circ R \circ L \xrightarrow{id \otimes \varepsilon \otimes id} R \circ L$

T-modules in $\mathcal{C}$ are pairs (X, f) $X \in \mathcal{C}$, $f: T(X) \longrightarrow X$.

Get $\widetilde{R}: \mathcal{D} \longrightarrow T\text{-mod}_{\mathcal{C}}$ w/ $X \longmapsto (R(X), act: RLR(X) \to R(X)$ via $id \otimes \varepsilon$

<u>Thm</u> (Barr–Beck): $\widetilde{R}$ is an equivalence iff

- R is conservative $\quad$ ($R(f)$ iso $\Leftrightarrow$ f iso)
- If we have a fork $\quad a \rightrightarrows b \longrightarrow c \quad$ in $\mathcal{D}$ s.t.

  $R(a) \rightrightarrows R(b) \longrightarrow R(c)$ is a split coequaliser (have maps

$$\begin{tikzcd} R(a) \arrow[r, shift left] \arrow[r, shift right] & R(b) \arrow[r] \arrow[l, bend left] & R(c) \arrow[l, bend left] \end{tikzcd}$$
)

  then $\quad c$ is a coequalizer.

<u>Def</u>: $\mathcal{A}$ an abelian tensor cat in $\underline{Pr}_c$ w/ module cat $\mathcal{M} \in \underline{Pr}_c$,

$m \in \mathcal{M}$: $\quad$ 1) m is an <u>$\mathcal{A}$-generator</u> if act_m^R is faithful

2) m is <u>$\mathcal{A}$-projective</u> if act_m^R is colim preserving

3) <u>$\mathcal{A}$-progenerator</u> if its $\mathcal{A}$-projective & $\mathcal{A}$-generator

<u>Thm</u>: Let $\mathcal{A}$ be a rigid, abelian tensor category in Pr_c, and $\mathcal{M} \in Pr_c$ abelian $\mathcal{A}$-mod w/ $\mathcal{A}$-progenerator m. Then $\widetilde{act_m^R}$ is an equivalence

$$\mathcal{M} \simeq \underline{End}(m)\text{-mod}_{\mathcal{A}}$$

<u>pf</u>: Apply above thm to $act_m : \mathcal{A} \to \mathcal{M}$. $\widetilde{act_m^R}$ is an equivalence

w/ $\quad T = act_m^R \circ act_m$ $-$ mod$_{\mathcal{A}}$.

But $\mathcal{A}$ is rigid so act_m^R is an $\mathcal{A}$-mod functor $\Rightarrow$ hence

T determined by its value on $1_{\mathcal{A}}$ & $act_m^R \circ act_m(1_{\mathcal{A}}) = \underline{End}(m)$.

Have two functorialities for this:

<u>Thm:</u> Let $\mathcal{M}$ & $\mathcal{N}$ be $\mathcal{A}$-mod categories in Pr_c, and $F: \mathcal{M} \to \mathcal{N}$ an $\mathcal{A}$-mod functor, $m \in \mathcal{M}$. Have a map $\rho_F : \underline{End}_{\mathcal{A}}(m) \to \underline{End}_{\mathcal{A}}(F(m))$

<u>pf:</u> Have comm.-diagrams

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, bend left, "act_m^R"] \arrow[dd, bend left, "F"] & \\
\mathcal{A} \arrow[ur, bend left, "act_m"] \arrow[dr, bend right, "act_{F(m)}"'] & & \\
 & \mathcal{N} \arrow[ul, "act^R_{F(m)}"'] \arrow[uu, bend right=60, "F^R"'] &
\end{tikzcd}$$

$$\begin{array}{c}
\rho_F : \underline{End}(m) = act^R_m \circ act_m(\mathbb{1}_{\mathcal{A}}) \\
\big\downarrow \eta_F \\
act_m^R \circ F^R \circ F \circ act_m(\mathbb{1}) \\
\wr\wr \\
act^R_{F(m)} \circ act_{F(m)}(\mathbb{1}) \\
\shortparallel \\
\underline{End}(F(m))
\end{array}$$

<u>Cor:</u> m an $\mathcal{A}$-progenerator for $\mathcal{M}$, $F(m)$ an $\mathcal{A}$-progenerator for $\mathcal{N}$.

Then $\mathcal{M} \simeq \underline{End}(m)\text{-mod}_{\mathcal{A}}$, $\mathcal{N} \simeq \underline{End}(F(m))\text{-mod}_{\mathcal{A}}$

& F^R is iso to the pullback functor along ρ_F.

<u>Def:</u> $F: \mathcal{A} \to \mathcal{B}$ is <u>dominant</u> if every object of $\mathcal{B}$ appears as a subobject of $F(X)$ for some $X \in \mathcal{A}$.

<u>Thm:</u> $\mathcal{A}$ rigid abelian $\otimes$-cat, $\mathcal{M}, \mathcal{N}$ $\mathcal{A}$-mod abelian cats w/ $\mathcal{A}$-progenerators m, and n. Then

$$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \simeq \underline{End}(m)\text{-mod}_{\mathcal{N}} \simeq (\underline{End}(m)\text{-}\underline{End}(n))\text{-bimod}_{\mathcal{A}}$$

<u>Cor:</u> $\mathcal{M}$ an abelian $\mathcal{A}$-mod cat, $F: \mathcal{A} \to \mathcal{B}$ dominant, $m \in \mathcal{A}$-progenerator then $m \boxtimes_{\mathcal{A}} \mathbb{1}_{\mathcal{A}}$ is a $\mathcal{B}$-progenerator of $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B}$ & we have

$\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{B} \simeq F(\underline{End}(m))\text{-mod}_{\mathcal{B}}$

If $\mathcal{A}$ is a rigid braided tensor cat in Pr_c which is an abelian category. Note that w/ the braiding we can show that

$$\mathcal{A}^{\otimes n} \longrightarrow \mathcal{A} \quad \text{is a tensor functor}$$

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_n) \xrightarrow{\sim} (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes \cdots \otimes (a_n \otimes b_n)$$

$\Rightarrow$ $\mathcal{A}$ has a left & right $\mathcal{A}^{\otimes n}$-module structure the left & right regular actions of $\mathcal{A}^{\otimes n}$ on $\mathcal{A}$

<u>Prop:</u> $\mathbb{1}_{\mathcal{A}}$ is a progenerator (remember formula for T^R)

Let $\mathcal{F}_{\mathcal{A}} = T(\underline{End}_{\mathcal{A}^{\boxtimes 2}}(\mathbb{1}_{\mathcal{A}}))$

<u>Thm:</u> $\mathcal{A} \boxtimes_{\mathcal{A} \boxtimes \mathcal{A}} \mathcal{A} \simeq \mathcal{F}_{\mathcal{A}}\text{-mod}_{\mathcal{A}}$

⋮