Factorization homology along surfaces and quantum groups
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2-categories and Barr-Beck for module categories

1. Main definitions

Main reference for this talk is [BBJ, Sections 3,4]. We start from defining some basic properties of categories which will allow us define in section 2 main four $(2,1)$ categories which we will study.

We will always denote by k some field, Vect_k is a category of k-vector spaces, $\mathrm{Vect}_{k,f.d.}$ is a category of finite dimensional vector spaces.

Definition 1.1
A category $\mathscr{C}$ is called k-linear if for any two objects $X, Y \in \mathrm{Ob}(\mathscr{C})$ a class $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is equipped with a k-linear structure. Such that for any $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ the map $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ is k-bilinear.

Functor $\mathcal{F}\colon \mathscr{C} \to \mathscr{D}$ between two k-linear categories is called k-linear if for any $a, b \in \mathrm{Ob}(\mathscr{C})$ the morphism $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is k-linear.

Remark 1.2
One can easily show that for any k-linear (additive) category $\mathscr{C}$ there exists a canonical bilinear functor $\mathrm{Vect}_{k,f.d.} \times \mathscr{C} \to \mathscr{C}$, $(V,X) \mapsto V \otimes X$ such that for $X \in \mathrm{Ob}(C)$ the object $V \otimes_k X$ represents the functor $\mathscr{C} \to \mathrm{Vect}_k$, $Y \mapsto \mathrm{Hom}_{\mathrm{Vect}_k}(V, \mathrm{Hom}_{\mathscr{C}}(X,Y))$ i.e.

$$\mathrm{Hom}_{\mathscr{C}}(V \otimes_k X, Y) \simeq \mathrm{Hom}_{\mathrm{Vect}_k}(V, \mathrm{Hom}_{\mathscr{C}}(X,Y)).$$

To construct this object we can use a basis in V and the fact that it represents some functor shows that this object is actually canonical.

Definition 1.3
A category $\mathscr{C}$ is called small if both objects and Hom*-spaces of $\mathscr{C}$ are sets.*

Definition 1.4
A category $\mathscr{C}$ is called essentially small if it is equivalent to a small category.

Let us now discuss colimits. The following lemma is very usefull.

Lemma 1.5
Consider a small diagram $J\colon \mathscr{I} \to \mathscr{A}$ (i.e. category $\mathscr{I}$ is small). We denote by $\mathrm{Arr}(\mathscr{I})$ the set of arrows of $\mathscr{I}$. For an arrow $a \in \mathrm{Mor}(\mathscr{I})$ we denote by $s(a), t(a) \in \mathrm{Ob}(\mathscr{I})$ its start and target respectively. Set $X := \coprod_{a \in \mathrm{Mor}(\mathscr{I})} J(s(a))$, $Y := \coprod_{i \in \mathrm{Ob}(\mathscr{I})} J(i)$ (we assume that they exist). We have two morphisms $\psi, \phi\colon X \to Y$ defined as follows: $\psi|_{J(s(a))} := \iota_{t(a)} \circ a$, $\phi|_{J(s(a))} := \iota_{s(a)}$, where $\iota_{s(a)}\colon J(s(a)) \to Y$, $\iota_{t(a)}\colon J(t(a)) \to Y$ are the natural maps. Then the co-limit of $\mathcal{F}$ along $\mathscr{I}$ is exactly a co-equalizer of the pair $\psi, \phi\colon X \to Y$ (when it exists).

Definition 1.6
A category $\mathscr{C}$ is called cocomplete (resp. finite cocomplete) if it contains all small (resp. finite) co-limits.

Lemma 1.7
Category $\mathscr{C}$ is cocomplete (resp. finite cocomplete) iff it contains all small (resp. finite) coproducts and coequalizers.

Proof. Follows from lemma 1.5. □

Example 1.8. An example of cocomplete category is the category **Set**. One can show that if $\mathscr{D}$ is cocomplete and $\mathscr{C}$ is any other category then category $[\mathscr{C}, \mathscr{D}]$ of functors from $\mathscr{C}$ to $\mathscr{D}$ is cocomplete (one can compute colimits pointwisely). In particular any (small) category $\mathscr{C}$ can be fully faithfully embedded (via Yoneda) in a cocomplete category $[\mathscr{C}^{opp}, \mathbf{Set}]$.

Example 1.9. By lemma 1.5 any abelian category $\mathscr{A}$ is finite cocomplete. Indeed it's enough to show that finite coproducts and coequializers exist in $\mathscr{A}$. Existence of finite coproducts is one of the axioms of abelian category, coequalizer of two arrows $\phi, \psi \colon A \to A'$, $A, A' \in \mathscr{A}$ is nothing else but $\operatorname{coker}(\phi - \psi)$.

Example 1.10. An example af a not cocomplete category but finite cocomplete category is the category $A-\mathrm{mod}_{f.g.}$ of finitely generated modules over a noetherian ring A. Being abelian it is finite cocomplete but it does not contain a colimit of the following diagram $A \hookrightarrow A^{\oplus 2} \hookrightarrow \ldots$ (which should be $A^{\oplus\infty}$).

Definition 1.11
A non-empty categoty $\mathscr{I}$ is called filtered if
(i) for every two objects $i, j \in \mathrm{Ob}(\mathscr{I})$ there exists an object l and two morphisms $i \to l$, $j \to l$,
(ii) For every two morphisms $u, v \colon i \to j$ there exists an object $l \in \mathrm{Ob}(\mathscr{I})$ and an arrow $w \colon j \to l$ such that $w \circ v = w \circ u$.

Example 1.12. Let I be a directed set i.e. a set equipped with a preoder $\leqslant$ such that any finite subset of I has an upper bound. Then we can construct a filtered category $\mathscr{I}$ as a category whose objects are elements of I and the set $\mathrm{Hom}(a, b)$ consists of one element $a \to b$ if $a \leqslant b$ and is empty otherwise.

Lemma 1.13
Category $\mathscr{C}$ is cocomplete iff it contains all finite and filtered coproducts.

Proof. Indeed, if $\mathscr{C}$ contains all finite coproducts then it also contains coequalizers and all finite coproducts. Now any small coproduct is a filtered colimit of finite coproduct so we are done by lemma 1.5. □

Definition 1.14
A category $\mathscr{C}$ is called presentable (locally-presentable) if it is cocomplete and there exists a small subset S of $\mathrm{Ob}\,\mathscr{C}$ such that any object of $\mathscr{C}$ is a filtered colimit of objects in S.

Example 1.15. Examples of presentable categories include category **Set** and more generally categories $[\mathscr{C}, \mathbf{Set}]$ where $\mathscr{C}$ is small. Other example of presentable category is a category $\mathrm{Op}(X)$ for a topological space X, objects of $\mathrm{Op}(X)$ are open subsets

$U \subset X$ and morphisms are open embeddings, coproducts correspond to unions and coequalizers are trivial. It is an exersise to show that the category of coalgebras over a field k is presentable.

Definition 1.16
A functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ *from a cocomplete category* $\mathscr{A}$ *is called cocontinuous if it preserves colimits.*

Remark 1.17
Note that in the same way as in the proof of lemma 1.13 using lemma 1.5 we see that a functor is cocontinuous iff it preserves finite and filtered colimits.

Example 1.18. Let X, Y be two toplogical spaces and $\mathrm{Ob}(X), \mathrm{Ob}(Y)$ are the corresponding categories of open subsets. Then any continous map $f\colon Y \to X$ defines a cocontinuous functor $f^*\colon \mathrm{Op}(X) \to \mathrm{Op}(Y)$ by sending $U \in \mathrm{Op}(X)$ to $f^{-1}(U) \in \mathrm{Op}(Y)$.

Definition 1.19
An object $C \in \mathscr{C}$ *of a category* $\mathscr{C}$ *which admits all filtered colimits is called compact if the functor* $\mathrm{Hom}_{\mathscr{C}}(C, \bullet)$ *commutes with filtered colimits. We denote by* $\mathscr{C}_c \subset \mathscr{C}$ *the full subcategory consisting of compact objects of* $\mathscr{C}$*.*

Example 1.20. Compact objects in the category **Set** are precisely finite sets.

Example 1.21. Let R be a noetherian ring and $\mathscr{C} = R - \mathrm{mod}$ the category of R-modules. Then an object $C \in \mathscr{C}$ is compact iff it is finitely generated.

Proof. The implication $\Leftarrow$ is a an exersise. Let us prove the implication $\Rightarrow$. We fix a compact object $M \in \mathscr{C}$. Note that we can present M as a colimit of its finitely generated submodules $M_i \subset M$. We have $\mathrm{Hom}(M, M) = \mathrm{Hom}(M, \mathrm{colim}\, M_i) = \mathrm{colim}(\mathrm{Hom}(M, M_i))$. Consider now the element $id \in \mathrm{Hom}(M, M)$. We see that there exists $M_i \subset M$ and $f\colon M \to M_i$ such that $id = \iota_i \circ f$, where $\iota_i\colon M_i \hookrightarrow M$ is the embedding. It follows that $M = M_i$, hence, M is fnitely presented. □

Example 1.22. Let X be a topological space and recall a category $\mathrm{Op}(X)$. Then an object $C \in \mathrm{Op}(X)$ is compact iff it is compact as a topological space.

Proof. Fix a compact object $C \in \mathrm{Op}(X)$. Consider a covering $C = \bigsqcup_{i \in I} U_i$ by open subsets. For any finite subset $K \subset I$ define $U_K := \bigcup_{i \in K} U_i$. The set $\{U_K\}$ together with natural open embeddings defines a filtered system. Note that $\mathrm{colim}_K U_K = C$. We see that $\mathrm{Hom}(C, C) = \mathrm{Hom}(C, \mathrm{colim}_K U_K) = \mathrm{colim}_K \mathrm{Hom}(C, U_K)$. It follows that $C = U_K$ for some K i.e. C is compact.

It is an exersise to check that any compact subspace $C \subset X$ is compact in $\mathrm{Op}(X)$. □

Remark 1.23
Let us point out that restricting ourselves to filtered colimits is crucial in the definition of compact object. For example in the category $A - \mathrm{mod}$ compact objects are precisely finitely presented modules while objects $P \in A - \mathrm{mod}$ such that $\mathrm{Hom}_{\mathscr{C}}(C, \bullet)$ commutes with all colimits are projective finitely presented modules.

Remark 1.24
Any functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ commutes with finite filtered colimits. Indeed if $\mathscr{I}$ is a finite filtered category and $J\colon \mathscr{I} \to \mathscr{A}$ is a diagram then there is always the maximal element $i \in \mathrm{Ob}(\mathscr{I})$. Now it follows from the definitions that $\operatorname{colim} J = J(i)$ i.e. $\mathcal{F}(\operatorname{colim} J) = \mathcal{F}(J(i)) = \operatorname{colim} \mathcal{F} \circ J$.

Lemma 1.25
Let $\mathscr{C}$ be a cocomplete category then a finite colimit of compact objects is compact.

Proof. Indeed assume that $C = \operatorname{colim}_i C_i$, whith C_i-compact and such that the indexing set is finite. Consider now arbitrary filtered colimit $\operatorname{colim}_j X_j$. We have

$$\begin{aligned}\mathrm{Hom}(C, \operatorname{colim}_j X_j) = \mathrm{Hom}(\operatorname{colim}_i C_i, \operatorname{colim}_j X_j) = \lim_i \mathrm{Hom}(C_i, \operatorname{colim}_j X_j) = \\ = \lim_i \operatorname{colim}_j \mathrm{Hom}(C_i, X_j) = \operatorname{colim}_j \lim_i \mathrm{Hom}(C_i, X_j) = \\ = \operatorname{colim}_j \mathrm{Hom}(\lim_i C_i, X_j) = \operatorname{colim}_j \mathrm{Hom}(C, X_j),\end{aligned}$$

here we use the following standard fact – finite limits commute with filtered colimits (exersise). □

Definition 1.26
A functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ is called compact if it sends compact objects to compact objects.

Example 1.27. A continuous map $f\colon Y \to X$ induces a compact functor $f^*\colon \mathrm{Ob}(X) \to \mathrm{Ob}(Y)$ iff f is proper.

Definition 1.28
An object $S \in \mathrm{Ob}(\mathscr{C})$ is called a generator if for every pair of morphisms $f, g\colon X \to Y$ in $\mathscr{C}$, if $f \circ l = g \circ l$ for every morphism $l\colon S \to X$ then $f = g$.

Example 1.29. Let $\mathscr{C}$ be a category of A-modules. Then $A \in \mathrm{Ob}(\mathscr{C})$ is a generator. Indeed if M, N are two A-modules and $f, g\colon M \to N$ are two morphisms then to any $m \in M$ we can associate a morphism $l_m\colon A \to M$, $a \mapsto am$ then from $f \circ l_m = g \circ l_m$ we deduce $f(m) = g(m)$.

Definition 1.30
A Grothendieck category is an abelian cocomplete category which has a generator and such that filtered colimits are exact.

Example 1.31. Let A be a k-algebra then the category $A - \mathrm{mod}$ is Grothendieck. More generally if $(X, \mathcal{O}_X)$ is a ringed space then the category of sheaves of $\mathcal{O}_X$-modules is Grothendieck.

Let us now formulate adjoint functor theorem in the setting of presentable categories (see [AR]).

Proposition 1.32
Let $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ be a functor between locally presentable categories $\mathscr{A}, \mathscr{B}$. Then $\mathcal{F}$ admits a right adjoint iff it preserves all small co-limits.

Definition 1.33
Let $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ be a functor. Suppose that all finite limits (resp. co-limits) exist in $\mathscr{A}$. We say that $\mathcal{F}$ is left (resp. right) exact if it commutes with finite limits (resp. colimits).

Lemma 1.34
Functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ is right (resp. left) exact iff it commutes with finite products (resp. coproducts) and equalizers (resp. coequalizers).

Proof. The implication $\Rightarrow$ is clear. The opposite implication follows from lemma 1.5. □

Corollary 1.35
For abelian categories $\mathscr{A}, \mathscr{B}$ the two notions of right (resp. left) exact functors coincide.

2. Main categories

Let $\mathscr{V}$ be a monoidal category. A $\mathscr{V}$-category $\mathscr{A}$ (or a category enriched over $\mathscr{V}$) is
(i) a class of objects $\mathrm{Ob}(\mathscr{A})$,
(ii) for any $X, Y \in \mathrm{Ob}(\mathscr{A})$ an object $\mathrm{Hom}(X, Y) \in \mathscr{V}$,
(iii) for each $X, Y, Z \in \mathrm{Ob}(\mathscr{A})$ a morphism

$$\circ_{XYZ}\colon \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z),$$

(iv) for each $X \in \mathrm{Ob}(\mathscr{C})$ a morphism $id_X\colon 1 \to \mathrm{Hom}(X,X)$
such that the following diagrams commute:

$$\begin{array}{ccc}
(\mathrm{Hom}(Z,H) \otimes \mathrm{Hom}(Y,Z)) \otimes \mathrm{Hom}(X,Y) & \xrightarrow{\circ_{YZH} \otimes id} & \mathrm{Hom}(Y,H) \otimes \mathrm{Hom}(X,Y) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \circ_{XYH}} \\
 & & \mathrm{Hom}(X,H) \\
 & & \uparrow{\scriptstyle \circ_{XZH}} \\
\mathrm{Hom}(Z,H) \otimes (\mathrm{Hom}(Y,Z) \otimes \mathrm{Hom}(X,Y)) & \xrightarrow{id \otimes \circ_{XYZ}} & \mathrm{Hom}(Z,H) \otimes \mathrm{Hom}(X,Z)
\end{array}$$

$$\begin{array}{ccc}
1 \otimes \mathrm{Hom}(X,Y) & \xrightarrow{id_Y \otimes id} & \mathrm{Hom}(Y,Y) \otimes \mathrm{Hom}(X,Y) \\
 & {\scriptstyle \lambda} \searrow \quad \swarrow {\scriptstyle \circ_{XYY}} & \\
 & \mathrm{Hom}(X,Y) &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}(X,Y) \otimes 1 & \xrightarrow{id \otimes id_X} & \mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(X,X) \\
 & {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \circ_{XXY}} & \\
 & \mathrm{Hom}(X,Y) &
\end{array}$$

where α, λ, ρ are the natural morphisms in the tensor category $\mathscr{V}$.

Example 2.36. A category enriched over Vect_k is nothing else but k-linear category. A category enriched over **Set** is nothing else but a small category.

Recall that if we have two categories $\mathscr{C}, \mathscr{D}$ then we can form their cartesian product whose objects are $\mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$ and $\mathrm{Hom}_{\mathscr{C}\times\mathscr{D}}((C,D),(C',D')) := \mathrm{Hom}_{\mathscr{C}}(C,C') \times \mathrm{Hom}_{\mathscr{D}}(D,D')$ for $C, C' \in \mathrm{Ob}(\mathscr{C}), D, D' \in \mathrm{Ob}(\mathscr{D})$.

Definition 2.37
A (strict) 2-category is a category enriched over **Cat** *with monoidal structure given by cartesian product.*

Definition 2.38
A $(2,1)$-category is a 2-category in which any 2-morphism is invertible.

Our main players will be the following four $(2,1)$-categories:

(1) **Rex** is a category of essentially small finitely cocomplete k-linear categories with morphisms right exact functors and 2-morphisms – k-linear natural isomorphisms.

(2) **Pr** is a category of presentable k-linear categories with morphisms cocontinuous functors and 2-morphisms – k-linear natural isomorphisms.

(3) $\mathbf{Pr}_c$ is a category of k-linear cocomplete categories $\mathscr{C}$ such that $\mathscr{C}_c$ is essentially small and any object of $\mathscr{C}$ is a filtered limit of compact objects. Morphisms between such categories are compact cocontinuous functors and 2-morphisms – k-linear natural isomorphisms.

(4) **Gr** is a category of Grothendieck categories.

Remark 2.39
Note that by proposition 1.32 morphisms in categories **Pr**, $\mathbf{Pr}_c$ have right adjoints.

Example 2.40. One very important example of an object of **Rex** is a category $C-\mathrm{comod}_{f.d.}$ of finite dimensional comodules over some k-coalgebra C.

Now our goal is to construct an equivalence $\mathbf{Rex} \simeq \mathbf{Pr}_c$. Starting from a small finitely co-complete category $\mathscr{C}$ we can construct its ind-completion as follows. Its objects are functors $\mathcal{F}\colon \mathscr{I} \to \mathscr{C}$ from a small filtered category $\mathscr{I}$ to $\mathscr{C}$. Morphisms in $\mathrm{ind}(\mathscr{C})$ are natural transformations of functors. We have a canonical fully faithfull embedding $\iota\colon \mathscr{C} \hookrightarrow \mathrm{ind}(\mathscr{C})$. The following lemma (exersise) describes a universal property of the category $\mathrm{ind}(\mathscr{C})$.

Lemma 2.41
For any category $\mathscr{D}$ which has all filtered colimits and a functor $\mathcal{F}\colon \mathscr{C} \to \mathscr{D}$ there exists a unique functor $\mathrm{ind}(\mathcal{F})\colon \mathrm{ind}(\mathscr{C}) \to \mathscr{D}$ which prerserves filtered colimits and such that $\mathcal{F} = \mathrm{ind}(\mathcal{F}) \circ \iota$.

The following is true.

Lemma 2.42
(a) For any $C \in \mathrm{Ob}(\mathscr{C})$ the object $\iota(C) \in \mathrm{ind}(\mathscr{C})$ is compact.

(b) If $\tilde{C} \in \mathrm{ind}(\mathscr{C})$ is compact then $\tilde{C}$ is a retract of some object C of $\mathscr{C}$ i.e. there exist morphisms $\tilde{C} \xrightarrow{\iota} C \xrightarrow{\pi} \tilde{C}$ such that $\pi \circ \iota = \mathrm{id}_{\tilde{C}}$.

(c) If $\mathscr{C}$ is finite cocomplete then any compact object of $\tilde{\mathscr{C}}$ is isomorphic to $\iota(C)$ for some $C \in \mathrm{Ob}(\mathscr{C})$.

Proof. Let us prove (a). Fix an object $C \in \mathscr{C}$ and let $\tilde{J}\colon \mathscr{I} \to \mathscr{C}$, $i \mapsto X_i$ be a filtered diagram. From the definitions it follows that it is enough to deal with the diagrams $\tilde{J} = \iota \circ J$, $J\colon \mathscr{I} \to \mathscr{C}$. We have to show that $\mathrm{Hom}_{\tilde{\mathscr{C}}}(\iota(C), \mathrm{colim}\, X_i) = \mathrm{colim}(\mathrm{Hom}(C, X_i))$. So we need to check a universal property of colimit for $\mathrm{Hom}_{\tilde{\mathscr{C}}}(\iota(C), \mathrm{colim}\, X_i)$. Take a set Z together with compatible homomorphisms $\psi_i\colon \mathrm{Hom}(C, X_i) \to Z$. We need to construct a morphism $\mathrm{Hom}(C, \mathrm{colim}\, X_i) \to Z$. Note that one can consider $\mathrm{colim}\, X_i$ as an object of $\mathrm{ind}(\mathscr{C})$. By the definition an element of $\mathrm{Hom}(C, \mathrm{colim}\, X_i)$ is a family of compatible morphisms $\varphi_i\colon C \to X_i$. We now construct a morphism $\mathrm{Hom}(C, \mathrm{colim}\, X_i) \to Z$ by sending (φ_i) to an element $\psi_i(\varphi_i)$ (it does not depend on i). The claim follows.

Let us now prove part (b). Fix a compact object $\tilde{C} \in \mathrm{ind}(\mathscr{C})$. We can write $\tilde{C} = \mathrm{colim}_{i \in \mathrm{Ob}(\mathscr{I})}(X_i)$ for some diagram $\mathscr{I} \to \mathscr{C}$, $i \mapsto X_i$. We see that $\mathrm{Hom}_{\mathrm{ind}(\mathscr{C})}(\tilde{C}, \tilde{C}) = \mathrm{Hom}_{\mathrm{ind}(\mathscr{C})}(\tilde{C}, \mathrm{colim}_{i \in \mathrm{Ob}(\mathscr{I})}(X_i)) = \mathrm{colim}_i(\tilde{C}, X_i)$ so, in particular, morphism $\mathrm{id}\colon \tilde{C} \to \tilde{C}$ can be decomposed as $\mathrm{id} = \pi \circ \iota$ for some $X_i \in \mathscr{C}$.

To prove (c) it remains to note that $\tilde{C}$ from (b) is isomorphic to a coequalizer of the pair $\iota \circ \pi, \mathrm{id}\colon C \to C$ (we keep notations from the proof of (b)) which lies in $\mathscr{C}$ since it is finite cocomplete. □

Proposition 2.43
We have an equivalence of $(2,1)$-categories $\mathbf{Rex} \simeq \mathbf{Pr}_c$.

Proof. Fix a category $\mathscr{C} \in \mathbf{Rex}$. We can consider its ind-completion $\mathrm{ind}(\mathscr{C})$. We claim that $\mathrm{ind}(\mathscr{C}) \in \mathbf{Pr}_c$. Indeed category $\mathrm{ind}(\mathscr{C})$ contains all finite (because $\mathscr{C}$ was finite cocomplete and colimits commute with each other) and filtered colimits so by lemma 1.13 it is cocomplete, note also that by lemma 2.42 any object $X \in \mathscr{C}$ is compact as an object of $\mathrm{ind}(\mathscr{C})$. Note also that if $\mathcal{F}\colon \mathscr{C} \to \mathscr{C}'$ is a morphism in the category $\mathbf{Rex}$ i.e. a functor which commutes with finite colimits then $\mathrm{ind}(\mathcal{F})$ is cocontinuous.

So we obtain a functor $\mathrm{ind}\colon \mathbf{Rex} \to \mathbf{Pr}_c$.

Let us construct an inverse functor $\mathbf{Pr}_c \to \mathbf{Rex}$. It sends $\mathscr{D} \in \mathbf{Pr}_c$ to a full subcategory $\mathscr{D}_c \subset \mathscr{D}$ of compact objects. Recall that by the definition $\mathscr{D}_c$ is essentially small. Note also that by lemma 1.25 the category $\mathscr{D}_c$ is finite cocomplete. It follows that $\mathscr{D}_c \in \mathbf{Rex}$. We obtain a functor $\mathrm{comp}\colon \mathbf{Pr}_c \to \mathbf{Rex}_c$.

It follows from lemma 2.42 that $\mathrm{ind}, \mathrm{comp}$ are mutually inverse equivalences. □

We are now going to investigate objects $\mathscr{C}, \mathscr{D} \in \mathbf{Pr}_c \simeq \mathbf{Rex}$ and functors $L\colon \mathscr{C} \to \mathscr{D}$. For that we recall that by proposition 1.32 functor L andmits a right adjoint $R\colon \mathscr{D} \to \mathscr{C}$. We will investigate category $\mathscr{D}$ using the functor $T := R \circ L\colon \mathscr{C} \to \mathscr{D}$ (it will have a structure of a monad) and modules over T. Let us start from the general definition.

Definition 2.44
Let $(\mathscr{C}, \otimes)$ be a monoidal category. An object $A \in \mathscr{C}$ is called a unital algebra object if we have morphisms $m\colon A \otimes A \to A$, $i\colon 1 \to A$ such that standard axioms of associative

unital algebra holds:

$$\begin{array}{ccccc} (A\otimes A)\otimes A & \xrightarrow{\alpha} & A\otimes (A\otimes A) & \xrightarrow{id\otimes m} & A\otimes A\ , \\ & \searrow{\scriptstyle m\otimes id} & & & \swarrow{\scriptstyle m} \\ & A\otimes A & \xrightarrow{\quad m \quad} & A & \end{array}$$

$$\begin{array}{ccccc} 1\otimes A & \xrightarrow{i\otimes id} & A\otimes A & \xleftarrow{id\otimes i} & A\otimes A\ . \\ & \searrow{\scriptstyle \lambda} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \rho} & \\ & & A & & \end{array}$$

Example 2.45. Let $\mathscr{C} = \mathrm{Vect}_k$ be a category of vector spaces over k. Then a unital algebra object of $\mathscr{C}$ is exactly an associative unital k-algebra.

To each k-linear category $\mathscr{C}$ we can associate the tensor category $(\mathrm{End}(\mathscr{C}), \circ)$ whose objects are functors $\mathcal{F}\colon \mathscr{C} \to \mathscr{C}$, morphisms are natural transformations and tensor structure comes from the composition of morphisms.

Definition 2.46
A monad T in a category $\mathscr{C}$ is a unital algebra object in the category $\mathrm{End}(\mathscr{C})$.

Example 2.47. Let $L\colon \mathscr{C} \to \mathscr{D}$, $R\colon \mathscr{D} \to \mathscr{C}$ be an adjoint pair of functors. Then the composition $T = R \circ L$ is a monad on $\mathscr{C}$ via the adjunction and counit

$$\eta\colon \mathrm{id}_{\mathscr{C}} \to R \circ L,\ \epsilon\colon L \circ R \to \mathrm{id}_{\mathscr{D}}\,.$$

Definition 2.48
If T is a monad in the category $\mathscr{C}$ with multiplication $m\colon T^2 \to T$ and unit $\eta\colon id_{\mathscr{C}} \to T$ then we define the category T-$\mathrm{mod}_{\mathscr{C}}$ as a category of pairs (X, f), where $X \in \mathscr{C}$ and f is a morphism $T(X) \xrightarrow{f} X$ such that the following diagrams commute:

$$\begin{array}{ccc} T^2(X) & \xrightarrow{m(X)} & T(X) \\ \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle f} \\ T(X) & \xrightarrow{\ f\ } & X \end{array}\ ,\quad \begin{array}{ccc} X & \xrightarrow{\eta(x)} & T(X) \\ & \searrow{\scriptstyle id} & \downarrow{\scriptstyle f} \\ & & X \end{array}\ .$$

Morphisms between $(X, f), (X', f') \in$ $\mathrm{Ob}(T - \mathrm{mod}_{\mathscr{C}})$ are $h\colon X \to X'$ such that $h \circ f = f' \circ T(h)$.

Example 2.49. Functor $id_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ is a monad via the natural isomorphisms $id_{\mathscr{C}} \xrightarrow{\sim} id_{\mathscr{C}}$, $id_{\mathscr{C}} \circ id_{\mathscr{C}} \xrightarrow{\sim} id_{\mathscr{C}}$. We have an equivalence $id_{\mathscr{C}} - \mathrm{mod}_{\mathscr{C}} \xrightarrow{\sim} \mathscr{C}$ given by $X \mapsto (X, id_{\mathscr{C}}(X))$.

Example 2.50. More generally assume that L, R are mutually inverse equivalences. Then the morphisms $\eta\colon \mathrm{id}_{\mathscr{C}} \to R \circ L$, $\epsilon\colon L \circ R \to \mathrm{id}_{\mathscr{D}}$ are isomorphisms and we see that $T - \mathrm{mod}_{\mathscr{C}}$ is equivalent to $\mathscr{C}$ via a morphism $\mathscr{C} \to T - \mathrm{mod}_{\mathscr{C}}$, $X \mapsto (X, \eta^{-1}(X))$, the inverse functor is a forgetting functor $\mathrm{forg}\colon T - \mathrm{mod}_{\mathscr{C}} \to \mathscr{C}$.

We always have a functor $\widetilde{R}\colon \mathscr{D} \to T-\mathrm{mod}_{\mathscr{C}}$ sending $X \in \mathscr{D}$ to $R(X) \in \mathscr{C}$ equipped with the canonical action

$$act\colon R \circ L \circ R(A) \xrightarrow{id \otimes \epsilon} R(A).$$

Recall that by proposition 1.32 if $\mathscr{C}, \mathscr{D} \in \mathbf{Pr}$ that ANY morphism $L\colon \mathscr{C} \to \mathscr{D}$ of $\mathbf{Pr}$ admits a right adjoint $R\colon \mathscr{D} \to \mathscr{D}$ so we obtain a functor $\tilde{R}\colon \mathscr{D} \to T-\mathrm{mod}_{\mathscr{C}}$. Our next goal is to formulate necessary and sufficient for this functor to be an equivalence.

3. Barr-Beck

Definition 3.51
Abelian category $\mathscr{D}$ is called locally finite if it has finite dimensional spaces of morphisms and each object has finite length.

Definition 3.52
Abelian category $\mathscr{D}$ is called finite if it is locally finite has finite number of simple objects, and has enough projectives.

Let us start from the following proposition.

Proposition 3.53
Let $\mathscr{D}$ be a finite abelian category. Then $\mathscr{D} \simeq A-\mathrm{mod}_{f.d.}$ for some finite dimensional algebra A. More precisely $A = \mathrm{End}(P)^{\mathrm{opp}}$, where P is a projective generator of $\mathscr{D}$.

Proof. Let P be a projective generator of $\mathscr{D}$. We have a functor $R = \mathrm{Hom}(P, \bullet)\colon \mathscr{D} \to \mathscr{C}$, where $\mathscr{C} := \mathrm{End}(P)^{opp} - \mathrm{mod}_{f.d.}$. Note that R is left exact. Let us prove that R has a left adjoint functor (it actually already follows from a version proposition 1.32). Note that if we already know that $\mathscr{C} \simeq A-\mathrm{mod}$ for some A than P must be an A-module and the functor L is fiven by $M \mapsto P \otimes_{\mathrm{End}(P)^{opp}} M$ for $M \in \mathscr{C}$.

We now just mimick the construction of tensor product. Fix a basis $\{f_j\}$ of $\mathrm{End}(P)^{opp}$, fix also a basis $e_1, \ldots, e_m$ of M. We consider an object $P^{\oplus m}$ and want to quotient it by the elements of the form $p \otimes f_j e_i - f_j(p) \otimes e_i$. For that consider morphisms $\phi_{ij}, \psi_{ij}\colon P \to P^{\oplus m}$: ϕ_{ij} is the map f_j composed with the i-th coordinate embedding $P \hookrightarrow P^{\oplus m}$, to define ψ_{ij} we decompose $f_j e_i = \sum a_l e_l$ and define ψ_{ij} as $a_l \,\mathrm{id}_P$ on l-th summand. Then we set

$$L(M) = P \otimes_{\mathrm{End}(P)^{opp}} M := P^{\oplus m} / \mathrm{Span}_k(\mathrm{Im}(\psi_{ij} - \phi_{ij} \,|\, i,j).$$

Let us also describe a more canonical way of constructing a functor L. Note that we have a forgetfull functor $\mathscr{C} \to \mathrm{Vect}_k$ and we denote by M the corresponding vector space. Consider an object $M \otimes_k P \in \mathscr{D}$ (see remark 1.2). We have two morphisms $\phi, \psi\colon M \otimes \mathrm{End}(P) \otimes_k P \to M \otimes_k P$ given as follows: $\phi = id_M \otimes act_P, \psi = act_M \otimes id_P$, where $act_M\colon M \otimes_k \mathrm{End}(P) \to M$, $(a, f) \mapsto af$, $act_P\colon \mathrm{End}(P) \otimes_k P \to P$, $f \otimes p \mapsto f(p)$ (more formally we use the identification $\mathrm{Hom}(\mathrm{End}(P) \otimes_k P, P) \simeq \mathrm{Hom}(\mathrm{End}(P), \mathrm{Hom}(P, P))$ and then morphism act_P just corresponds to id_P). We then can define $L(M) := \mathrm{coker}(\phi - \psi)$.

It is easy to see that L is left adjoint to R.

It remains to check that the adjunction morphisms $\eta\colon \mathrm{id}_{\mathscr{C}} \to R \circ L$, $\epsilon\colon L \circ R \to \mathrm{id}_{\mathscr{D}}$ are isomorphisms. This can be done on the generators $P, \mathrm{Hom}(P,P)$ of the categories $\mathscr{D}, \mathscr{C}$ respectively.

We have

$$L(R(P)) = L(\mathrm{Hom}(P,P)) = P \otimes_{\mathrm{End}(P)^{opp}} \mathrm{Hom}(P,P) = P,$$

$$R(L(\mathrm{Hom}(P,P))) = R(P) = \mathrm{Hom}(P,P).$$

The claim follows. □

Example 3.54. Let us give couple examples of categories which satisfy the conditions of proposition 3.53. One important class of examples are blocks of BGG-category $\mathcal{O}$ for semi-simple finite dimensional Lie algebra $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_2$ one can easily describe the corresponding algebra $\mathrm{End}(P)^{opp}$ explicitly.

The following generalization of proposition 3.53 to the case of locally finite abelian categories holds (the proof is not very hard, see [EGNO, Section 1.10]).

Proposition 3.55
Let $\mathscr{D}$ be a locally finite abelian category. Then $\mathscr{D} \simeq D - \mathrm{comod}_{f.d.}$ where the later is a category of finite dimensional comodules over a unique coalgebra D. If $\mathscr{D}$ is finite then D is finite dimensional and $\mathscr{D} \simeq D^* - \mathrm{mod}_{f.d.}$ (c.f. proposition 3.53).

Remark 3.56
Note that any locally finite category lies in **Rex**. The subclass of locally finite categorties in **Rex** is very inmportant, for example, it is closed under the Deligne-Kelly tensor product $\boxtimes$ on **Rex** (see section 4 and remark 4.67).

Let us now generalize proposition 3.53 to our setting.

Definition 3.57
A fork in a category $\mathscr{A}$ is a triple $f, g\colon A \to B$, $e\colon B \to C$ such that $ef = eg$. A split coequalizer is a folk together with morphisms $t\colon B \to A$, $s\colon C \to B$ such that $es = \mathrm{id}_C, se = gt, ft = \mathrm{id}_B$.

The following lemma is easy.

Lemma 3.58
Let (f, g, e) be a split coequalizer. Then for any functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ the morphism $\mathcal{F}(e)$ is a coequalizer of the pair $\mathcal{F}(f), \mathcal{F}(g)$.

Proof. Note that for any functor $\mathcal{F}$, $(\mathcal{F}(f), \mathcal{F}(g), \mathcal{F}(e))$ is a split coequalizer. So it remains to check that if (f, g, e) is a split coequalizer then e is a coequalizer of f, g. Indeed if $h\colon B \to D$ is some morphism such that $h \circ f = h \circ g$ then $h \circ s\colon C \to D$ gives us the desired morphism. If $p\colon C \to D$ is any other map which makes diagram commutative then we must have $p \circ e = h$ so $p = p \circ e \circ s = h \circ s$ and uniquenesse follows. □

The following theorem is a generalization of the proposition 3.53. Recall adjoint functors $L\colon \mathscr{C} \to \mathscr{D}$, $R\colon \mathscr{D} \to \mathscr{C}$.

Theorem 3.59
The functor $\widetilde{R}\colon \mathscr{D} \to T-\mathrm{mod}_{\mathscr{C}}$ *is an equivalence if and only if*

(1) *R is conservative i.e. if $f\colon X \to Y$ in $\mathscr{D}$ is such that $R(f)$ is an isomorphism then f is an isomorphism,*

(2) *$\mathscr{D}$ has coequalizers of R-split parallel pairs (those parallel pairs of morphisms in $\mathscr{D}$, which R sends to pairs having a split coequalizer in $\mathscr{D}$) and R preserves those coequalizers.*

Proof. Let us prove the implication $\Rightarrow$. We assume that $\mathscr{D}$ is equivalent to $T-\mathrm{mod}_{\mathscr{C}}$ via $\tilde{R}$. After this equivalence the functor $R\colon \mathscr{D} \to \mathscr{C}$ becomes isomorphic to a forgetfull functor $\mathrm{Forg}\colon T-\mathrm{mod}_{\mathscr{C}} \to \mathscr{C}$ which is obviously conservative.

Let us now fix two maps $h, l\colon (X, f) \to (Y, g)$ in $T-\mathrm{mod}_{\mathscr{C}}$ such that the corresponding maps $h, l\colon X \to Y$ have a split coequalizer $e\colon Y \to Z$ in $\mathscr{C}$. We need to construct a coequalizer for $h, l\colon (X, f) \to (Y, g)$. We have the following diagram

$$\begin{array}{ccccc} T(X) & \xrightarrow{T(h),T(l)} & T(Y) & \xrightarrow{T(e)} & T(Z) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow{h,l} & Y & \xrightarrow{e} & Z \end{array}$$

Note that by lemma 3.58, $T(e)$ is a coequalizer. It follows from the definitions that $egT(l) = elf = ehf = egT(h)$. We conclude that there exists $m\colon T(Z) \to Z$ such that the diagram commute. It is an exersise to check that m is compatible with a monad (algebra) structure. It remains to show that $e\colon (Y, g) \to (Z, m)$ is a coequalizer.

Consider any map $d\colon (Y, g) \to (Q, p)$ in $T-\mathrm{mod}_{\mathscr{C}}$ such that $dl = dh$. Recall that $e\colon Y \to Z$ is a coequalizer in $\mathscr{C}$ so there exists a unique $d'\colon Z \to Q$ such that $d'e = d$. Using the fact that $T(e)$ is a coequalizer (uniquenesse part) we obtain $p \circ T(d') = d' \circ m$ (maps $p \circ T(d'), d' \circ m\colon T(Z) \to Q$ satisfy $p \circ T(d') \circ T(h) = d' \circ m \circ T(h), p \circ T(d') \circ T(l) = d' \circ m \circ T(l)$ so they must coincide). So we get a desired map $d'\colon (Z, m) \to (Q, p)$.

Let us prove the implication $\Leftarrow$. Recall unit and counit morphisms

$$\eta\colon id_{\mathscr{C}} \to R \circ L,\ \epsilon\colon L \circ R \to id_{\mathscr{D}}.$$

Recall that we have a multiplication $m\colon T^2 \to T$. Let us firs off all note that for any $(x, f) \in \mathrm{Ob}(T-\mathrm{mod}_{\mathscr{C}})$ the folk $(m(x), T(f), f)$ splits by the pair $s = \eta(x), t = \eta(Tx)$.

We have the forgetfull functor $\mathrm{Forg}_{\mathscr{C}}\colon T-\mathrm{mod}_{\mathscr{C}} \to \mathscr{C}$. We also have a functor $\tilde{L}\colon \mathscr{C} \to T-\mathrm{mod}_{\mathscr{C}}$ given by $X \mapsto (TX, m(X))$ on the level of objects. It is easy to see that $\tilde{L}$ is left adjoint to $\mathrm{Forg}_{\mathscr{C}}$.

So we have two pairs of adjoint functors

$$(R, L), (\mathrm{Forg}_{\mathscr{C}}, \tilde{L}),\ R\colon \mathscr{D} \to \mathscr{C},\ \mathrm{Forg}_{\mathscr{C}}\colon T-\mathrm{mod}_{\mathscr{C}} \to \mathscr{C}.$$

Note that $R \circ L = \mathrm{Forg}_{\mathscr{C}} \circ \tilde{L}\colon \mathscr{C} \to \mathscr{C}$. so these adjoint pairs define the same monads.

Let us now prove the followig general lemma which will allow us to...

Lemma 3.60
Let (R, L), (R', L') be two adjoint pairs, $R\colon \mathscr{D} \to \mathscr{C}$, $R'\colon \mathscr{D}' \to \mathscr{C}$ such that $R \circ L = R' \circ L'$. Assume also that the condition (2) of theorem 3.59 holds for R. Then there

exists a unique functor $Q\colon \mathscr{D}' \to \mathscr{D}$ such that $RQ = R', QL' = L$ (in particular, $Q\epsilon' = \epsilon Q$).

Proof. Let us start from the uniqueness of Q. Note that for any $x \in \mathrm{Ob}(\mathscr{C})$ the triple

$$(LR(\epsilon(x)), \epsilon(LR(x)), \epsilon(x)),\ LR(\epsilon(x)), \epsilon(LR(x))\colon LRLR(x) \to LR(x),\ \epsilon(x)\colon LR(x) \to x,$$

is folk.

Let us consider a folk which corresponds to $x = Qy$ for some $y \in \mathrm{Ob}(\mathscr{D}')$. We obtain the folk $(LR(\epsilon(Qy)), \epsilon(LR(Qy)), \epsilon(Qy)) = (LR'(\epsilon'(y)), \epsilon(LR'(y)), Q\epsilon'(y))$.

Let us now apply R to this folk. We obtain a folk $(RLR'(\epsilon'(y)), R\epsilon(LR'(y)), RQ\epsilon'(y))$ which splits because it coincides with the folk $(m(R'y), T(f), f)$ for $f = \epsilon'(R'y)$. It follows that Qy should be a coequalizer of $(LR'(\epsilon'(y)), \epsilon(LR'(y)))$ i.e. it is uniquelly defined. The uniqueness of Q follows.

Let us prove the existence of Q. It follows from the above that to any $y \in \mathrm{Ob}(\mathscr{D}')$ we can associate some (uniquelly defined) object to be denoted $Q(y)$. We now should define Q on morphisms. Consider a map $y \to z$ for some $y, z \in \mathrm{Ob}(\mathscr{D}')$. We have already realised $Q(y), Q(z)$ as coequalizers of certain diagrams. It is clear that f induces a morphism of these diagrams. So we obtain a desired morphism $Q(f)$. $\square$

Let us now apply lemma 3.60 to $R' = \mathrm{Forg}_{\mathscr{C}}, L' = \tilde{L}$. We obtain a functor $Q\colon T-\mathrm{mod}_{\mathscr{C}} \to \mathscr{D}$. Note now that the functor $\widetilde{R}\colon \mathscr{D} \to T-\mathrm{mod}_{\mathscr{C}}$ coincides with functor from lemma 3.60 for a pair $(\mathrm{Forg}_{\mathscr{C}}, \tilde{L}), (R, L)$ (we can apply this lemma to this pair because it follows from the proof of the implication $\Rightarrow$ that the condition (2) of theorem 3.59 holds for $\mathrm{Forg}_{\mathscr{C}}$). It now follows from the uniqueness part of lemma 3.60 that $Q \circ \widetilde{R} = id_{\mathscr{D}}, \widetilde{R} \circ Q = id_{T-\mathrm{mod}_{\mathscr{C}}}$. $\square$

Remark 3.61
Note that if $\mathscr{D}$ is (finite) cocomplete and $\mathcal{F}$ is cocontinuous then (2) holds authomatically.

The following lemma is very usefull for checking condition (1) of Theorem 3.59.

Lemma 3.62
Suppose $\mathscr{D}$ is abelian and R is right exact. Then R is conservative iff for any X with $R(X) \simeq 0$ we have $X \simeq 0$.

Proof. Implication $\Rightarrow$ is obvious. Let us prove the implication $\Leftarrow$. Suppose that we have $f\colon X \to Y$ such that $R(f)$ is an isomorphism. Note that R being right exact and right adjoint is exact. It follows that $R(\ker f) = \ker R(f) = 0$, $R(\mathrm{coker} f) = \mathrm{coker} R(f) = 0$ so $\ker f \simeq \mathrm{coker} f \simeq 0$. We conclude that f is an isomorphism. $\square$

We are now answering to the question when $T-\mathrm{mod}_{\mathscr{A}}$ is abelian.

Proposition 3.63
If $\mathscr{A}$ is an abelian category and $T\colon \mathscr{A} \to \mathscr{A}$ is a right exact monad on $\mathscr{A}$ then $\mathscr{B} := T-\mathrm{mod}_{\mathscr{A}}$ is abelian.

Proof. Let us first of all show that the category $T-\mathrm{mod}_{\mathscr{A}}$ is pre-abelian.

To do this we need to construct kernels and co-kernels in the category $T-\mathrm{mod}_{\mathscr{A}}$. If $h\colon X\to X'$ induces a morphism $(X,f)\to(X',f')$ in $\mathscr{D}$ then the kernel of this morphism is exactly $(\ker h,l)$, where $l\colon T(\ker h)\to\ker h$ is obtained as the composition of the morphisms $T(\ker h)\to\ker T(h)\to\ker h$. To construct co-kernel we recall that T is right exact so $T(\operatorname{coker} h)=\operatorname{coker} T(h)$ and we can define cokernel of the morphism $h\colon(X,f)\to(X',f')$ to be $(\operatorname{coker}(h),p)$, where $p\colon T(\operatorname{coker} h)=\operatorname{coker} T(h)\to\operatorname{coker}(h)$ is the natural morphism.

Consider now a forgetful functor $\mathrm{forg}\colon\mathscr{B}\to\mathscr{A}$. It follows from the constructions that forg is an exact functor. Note that forg is conservative and $\mathscr{C}$ is abelian so it follows that the comparison morphism

$$coker(ker(h))\to ker(coker(h))$$

is an isomorphism in $\mathscr{D}$, hence, $\mathscr{D}$ is abelian.

□

4. Deligne-Kelly tensor product

Recall that on the previous lecture factorization homologies were constructed. They were constructed as a colimit along some diagram. So we want to work with categories which have enough colimits (cocomplete). So a good setting for us will be $\mathbf{Pr}_c$. Recall that by proposition 2.43 we have an equivalence $\mathbf{Pr}_c\simeq\mathbf{Rex}$. We want to have a tensor structure on the category $\mathbf{Rex}\simeq\mathbf{Pr}_c$.

Definition 4.64
Given two k-linear categories $\mathscr{A},\mathscr{B}$ we define their tensor product $\mathscr{A}\otimes\mathscr{B}$ as a category with objects the pairs (A,B) with $A\in\mathrm{Ob}(\mathscr{A})$, $B\in\mathrm{Ob}(\mathscr{B})$ and morphisms $\mathrm{Hom}((A,B),(A',B')):=\mathrm{Hom}(A,B)\otimes_k\mathrm{Hom}(A',B')$.

Proposition 4.65
(a) Category $\mathscr{A}\otimes\mathscr{B}$ has the following universal property – we have a natural equivalence between k-bilinear functors $\mathscr{A}\times\mathscr{B}\to\mathscr{C}$ to some k-linear category $\mathscr{C}$ and functors $\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$.

(b) For any k-linear $\mathscr{A},\mathscr{B},\mathscr{C}$ we have $[\mathscr{A}\otimes\mathscr{B},\mathscr{C}]\simeq[\mathscr{A}[\mathscr{B},\mathscr{C}]]$.

It turns out that if $\mathscr{A},\mathscr{B}$ are cocomplete then the category $\mathscr{A}\otimes\mathscr{B}$ need not to be cocomplete. There is a way to define other tensor product to be denoted by $\boxtimes$ such that tensor product of two cocomplete categories will be cocomplete. Refferences are [K, Section 6.5], [S, Section 2.3], [EGNO, Section 1.11],

Theorem 4.66
(a) For any two $\mathscr{A},\mathscr{B}\in\mathbf{Rex}$ there exists a category $\mathscr{A}\boxtimes\mathscr{B}$ uniquelly defined by the following property

$$\mathbf{Rex}[\mathscr{C}\boxtimes\mathscr{D},\mathscr{E}]\simeq\mathrm{Bilin}(\mathscr{C}\times\mathscr{D},\mathscr{E}),$$

where $\mathrm{Bilin}(\mathscr{C}\times\mathscr{D},\mathscr{E})$ is the category of k-bilinear functors preserving finite colimits in each variable.

We have an equivalence

$$\mathbf{Rex}[\mathscr{C}\boxtimes\mathscr{D},\mathscr{E}]\simeq\mathbf{Rex}[\mathscr{C},\mathbf{Rex}[\mathscr{D},\mathscr{E}]].$$

(b) The Kelly tensor product $\boxtimes$ equips **Rex** *with the structure of a symmetric closed monoidal* $(2,1)$*-category.*

(c) The tensor product $\boxtimes$ extends to a monoidal structure on $\mathbf{Pr}_c$ *and the functor* ind *extends to an equivalence* $\mathbf{Rex}^{\boxtimes} \xrightarrow{\sim} \mathbf{Pr}_c^{\boxtimes}$ *of symmetric monoidal* $(2,1)$*-categories.*

Remark 4.67
Assume that $\mathscr{C}, \mathscr{D}$ are locally finite abelian. Then by proposition 3.56 we have an equivalence $\mathscr{C} \simeq C - \text{comod}$, $\mathscr{D} \simeq D - \text{comod}_{f.d.}$ for some coalgebras C, D over k. Then we have $\mathscr{C} \boxtimes \mathscr{D} \simeq (C \otimes D) - \text{comod}$ so we have a rather explicit description of the category $\mathscr{C} \boxtimes \mathscr{D}$ in this case.

5. Monadicity for module categories

Main refference for all the notions of this section is [EGNO, Chapter 7]. The main theorem is taken from [BBJ, Theorem 4.6].

Definition 5.68
Tensor category $(\mathscr{C}, \otimes)$ is called closed if for any $X \in \mathrm{Ob}(\mathscr{C})$ the functor $X \otimes \bullet \colon \mathscr{C} \to \mathscr{C}$ admits a right adjoint. If $\mathscr{C} \in$ **Pr** *then this is equivalent to the fact that $X \otimes \bullet$ is cocomplete.*

Definition 5.69
An object $C \in \mathrm{Ob}(\mathscr{C})$ of a tensor category is called right dualizable if there exists an object $C^* \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\mathrm{ev}_C \colon C^* \otimes C \to 1$, $\mathrm{coev}_C \colon 1 \to C \otimes C^*$ such that the compositions

$$C \xrightarrow{\mathrm{coev}_C \otimes id_C} (C \otimes C^*) \otimes C \xrightarrow{\alpha_{C,C^*,C}} C \otimes (C^* \otimes C) \xrightarrow{id_C \otimes \mathrm{ev}_C} C,$$

$$C^* \xrightarrow{id_{C^*} \otimes \mathrm{coev}_C} C^* \otimes (C \otimes C^*) \xrightarrow{\alpha^{-1}_{C^*,C,C^*}} (C \otimes C^*) \otimes C \xrightarrow{\mathrm{ev}_C \otimes id_C} C$$

are identity morphisms, here α are associators for the tensor structure $\otimes$ on $\mathscr{C}$.

An object $C \in \mathrm{Ob}(\mathscr{C})$ of a tensor category is called left dualizable if there exists an object ${}^*C \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\mathrm{ev}'_C \colon C \otimes {}^*C \to 1$, $\mathrm{coev}'_C \colon 1 \to {}^*C \otimes C$ such that the compositions

$$C \xrightarrow{id_C \otimes \mathrm{coev}'_C} C \otimes ({}^*C \otimes C) \xrightarrow{\alpha_{C,{}^*C,C}} (C \otimes C^*) \otimes C \xrightarrow{id_C \otimes \mathrm{ev}_C} C,$$

$${}^*C \xrightarrow{\mathrm{coev}'_C \otimes id_{C^*}} ({}^*C \otimes C) \otimes {}^*C \xrightarrow{\alpha^{-1}_{C^*,C,C^*}} {}^*C \otimes (C \otimes {}^*C) \xrightarrow{id_C \otimes \mathrm{ev}'_C} {}^*C.$$

Definition 5.70
Tensor category $(\mathscr{C}, \otimes)$ is called rigid if all compact objects of $\mathscr{C}$ are right and left dualizable.

Definition 5.71
Let $(\mathscr{A}, \otimes)$ be a tensor category in a category $\mathbf{Pr}_c$*.*

(1) A (right) $\mathscr{A}$-module category $\mathscr{M}$ in $\mathbf{Pr}_c$ *is a category $\mathscr{M} \in \mathbf{Pr}_c$ together with an action functor*

$$act_{\mathscr{M}} \colon \mathscr{M} \boxtimes \mathscr{A} \to \mathscr{M}$$

satisfying standard associativity (pentagon) axioms (this notion categorifies the notion of a module over an algebra). We will denote $act_{\mathscr{M}}(m \boxtimes X)$ *by* $m \otimes X$, $m \in \mathrm{Ob}(\mathscr{M})$, $X \in \mathrm{Ob}(\mathscr{C})$.

(2) For any $m \in \mathscr{M}$ *the functor* $act_m \colon \mathscr{A} \to \mathscr{M}$, $a \mapsto m \otimes a$ *admits a right adjoint to be denoted* $act_m^R \colon \mathscr{M} \to \mathscr{A}$ *(by proposition 1.32).*

(3) For $m, n \in \mathscr{M}$ *we set* $\underline{\mathrm{Hom}}_{\mathscr{A}}(m,n) := act_m^R(n) \in \mathscr{A}$. *Note that the object* $\underline{\mathrm{Hom}}(m,n)$ *(contravariantly) represents the functor* $\mathscr{A} \to \mathrm{Vect}_k$, $X \mapsto \mathrm{Hom}_{\mathscr{M}}(X \otimes m, n)$ *i.e.*

$$\mathrm{Hom}_{\mathscr{A}}(X, \underline{\mathrm{Hom}}(m,n)) = \mathrm{Hom}_{\mathscr{A}}(X, act_m^R(n)) \simeq \mathrm{Hom}_{\mathscr{M}}(m \otimes X, n). \tag{5.1}$$

The object $\underline{\mathrm{Hom}}(m,n)$ *is called internal* Hom *form* m *to* n.

(4) For any triple $m, n, p \in \mathscr{M}$ *there is a well-defined composition map in* $\mathscr{A}$:

$$\underline{\mathrm{Hom}}(n,p) \otimes \underline{\mathrm{Hom}}(m,n) \to \underline{\mathrm{Hom}}(m,p). \tag{5.2}$$

To construct it let us note that for any $m, n \in \mathrm{Ob}(\mathscr{M})$ *we have a canonical evaluation morphism*

$$\mathrm{ev}_{m,n} \colon \underline{\mathrm{Hom}}_{\mathscr{A}}(m,n) \otimes m \to n$$

which corresponds to $id_{\underline{Hom}_{\mathscr{A}}(m,n)}$ *under the isomorphism (5.1). We can now form the composition*

$$(\underline{Hom}_{\mathscr{A}}(n,p) \otimes \underline{Hom}_{\mathscr{A}}(m,n)) \otimes m \xrightarrow{\alpha_{\underline{Hom}_{\mathscr{A}}(n,p), \underline{Hom}_{\mathscr{A}}(m,n), m}} \underline{Hom}_{\mathscr{A}}(n,p) \otimes (\underline{Hom}_{\mathscr{A}}(m,n) \otimes m) \xrightarrow{id \otimes \mathrm{ev}_{m,n}} \underline{Hom}_{\mathscr{A}}(n,p) \otimes n \xrightarrow{\mathrm{ev}_{n,p}} p$$

which gives us the composition map (5.2) via the isomorphism (5.1).

(5) We set $\underline{\mathrm{End}}(m) := \underline{\mathrm{Hom}}(m,m) = act_m^R(m) = act_m^R(act_m(1))$. *By* (4), $\underline{\mathrm{End}}(m)$ *carries a unital algebra structure i.e.* $\underline{\mathrm{End}}(m) \in \mathscr{A}$ *is a unital algebra object. Recall that by* $\underline{\mathrm{End}}(m) - \mathrm{mod}_{\mathscr{A}}$ *we denote the category of* $\underline{\mathrm{End}}(m)$*-modules (in* $\mathscr{A}$*).*

(6) We say that m *is an* $\mathscr{A}$*-generator if* act_m^R *is faithfull.*

(7) We say that m *is an* $\mathscr{A}$*-projective if* act_m^R *is colimit-preserving (this is equivalent to say that* act_m^R *preserves finite colimits since* act_m^R *preserves filtered colimits and now apply remark 1.17).*

(8) We say that m *is an* $\mathscr{A}$*-progenerator if it is an* $\mathscr{A}$*-projective* $\mathscr{A}$*-generator.*

Definition 5.72

Let $\mathscr{M}, \mathscr{N}$ *be* $\mathscr{A}$*-module categories. A* $\mathscr{A}$*-linear functor from* $\mathscr{M}$ *to* $\mathscr{N}$ *consists of a functor* $\mathcal{F} \colon \mathscr{M} \to \mathscr{N}$ *and a natural isomorphism* $s_{X,M} \colon \mathcal{F}(X \otimes M) \xrightarrow{\sim} X \otimes \mathcal{F}(M)$, $X \in \mathrm{Ob}(\mathscr{A}), M \in \mathrm{Ob}(\mathscr{M})$ *which is associative and compatible with tensor product by* $1 \in \mathrm{Ob}(\mathscr{C})$*:*

Theorem 5.73

(Monadicity for module categories) Let $\mathscr{A}$ *be a rigid abelian tensor category in* $\mathbf{Pr}_c$ *and let* $\mathscr{M} \in \mathbf{Pr}_c$ *be an abelian* $\mathscr{A}$*-module category with an* $\mathscr{A}$*-progenerator* $m \in \mathscr{M}$.

Then the functor $\widetilde{\mathrm{act}_m^R}$ $(\underline{\mathrm{Hom}}(m,\bullet))$ *is an equivalence of* $\mathscr{A}$*-module categories,*

$$\mathscr{M} \simeq \underline{\mathrm{End}}(m) - \mathrm{mod}_{\mathscr{A}},$$

where $\mathscr{A}$ *acts on the right by multiplication.*

Proof. (Sketch) We want to apply theorem 3.59 to the functor $act_m\colon \mathscr{A} \to \mathscr{M}$. Recall that m is an $\mathscr{A}$-generator so by lemma 3.62 the functor act_m^R is conservative. Note also that m is $\mathscr{A}$-projective i.e. it is colimit-preserving in particular it preserves coequalizers. So by theorem 3.59 we obtain an equivalence

$$\mathscr{M} \xrightarrow{\sim} act_m^R \circ act_m - \mathrm{mod}_{\mathscr{A}}\,.$$

It remains to identify $act_m^R \circ act_m - \mathrm{mod}_{\mathscr{A}} \simeq \underline{\mathrm{End}}(m) - \mathrm{mod}_{\mathscr{A}}$, functor $\widetilde{act_m^R}$ will identify with $\underline{\mathrm{Hom}}(m,\bullet)$.

One can show that $act_m^R\colon \mathscr{M} \to \mathscr{A}$ carries a canonical module structure so the composition $act_m^R \circ act_m\colon \mathscr{A} \to \mathscr{A}$ is a module functor. Any module functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{A}$ is isomorphic to the functor $\mathcal{F}(1) \otimes \bullet$ (this is a categorification of the fact that if A is a k-algebra then any (right) A-module homomorphism $f\colon A \mapsto A$ is given by $a \mapsto f(1)a$). So we see that $act_m^R \circ act_m \simeq act_m^R \circ act_m(1) \otimes \bullet = \underline{\mathrm{End}}(m) \otimes \bullet$ and the claim follows.

One can also show directly (in the spirit of proposition 3.53) that the functor $\underline{\mathrm{Hom}}(m,\bullet)$ defines an equivalence $\mathscr{M} \xrightarrow{\sim} \underline{\mathrm{End}}(m) - \mathrm{mod}_{\mathscr{A}}$ (see [EGNO, section 7.10]). □

Remark 5.74
Recall that $\underline{\mathrm{End}}(m) - \mathrm{mod}_{\mathscr{A}}$ is the category of $\underline{\mathrm{End}}(m)$-modules in the tensor category $\mathscr{A}$.

Example 5.75. One can hold in head the following example. Let $\mathscr{A} = \mathrm{Vect}_{k,f.d.}$ be the category of finite dimensional vector spaces over k and $\mathscr{M}$ is some finite category. We consider $\mathscr{A}, \mathscr{M}$ as objects of the tensor category $(\mathbf{Cat}, \times)$ (analogy of $\mathbf{Pr}_c$). Then the category $(\mathrm{Vect}_{k,f.d.}, \otimes)$ is a tensor category in $\mathbf{Cat}$ and the natural functor $\mathrm{Vect}_{k,f.d.} \times \mathscr{M} \to \mathscr{M}$ (see remark 1.2) equipps $\mathscr{M}$ with a structure of $\mathrm{Vect}_{k,f.d.}$-module. For $M \in \mathscr{M}$ the functor $act_M\colon \mathrm{Vect}_{k,f.d.}$ sends V to $V \otimes_k M$ the right adjoint $act_M^R\colon \mathscr{M} \to \mathrm{Vect}_{k,f.d.}$ sends N to $\mathrm{Hom}_{\mathscr{M}}(M,N) \in \mathrm{Ob}(\mathrm{Vect}_{k,f.d.})$.

Now the theorem exactly says that if $P \in \mathscr{M}$ is a projective generator then $\mathscr{M} \simeq \underline{\mathrm{End}}(P) - \mathrm{mod}_{\mathrm{Vect}_{f.g}} = \mathrm{End}(P) - \mathrm{mod}_{f.g}$ (c.f. proposition 3.53).

References

[AR] J. Adámek, J. Rosicky, *Locally presentable and accessible categories*
[BBJ] D. Ben-Zvi, A. Brochier, D. Jordan, *Integrating quantum groups over surfaces*
[EGNO] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*
[K] G. M. Kelly, *Basic concepts of enriched category theory*
[S] D. Schäppi, *Ind-abelian categories and quasi-coherent sheaves*