Quantum groups and factorization homology on surfaces

Time: Friday, 3:00 PM – 5:00 PM.

Place: MIT 2-135 / NEU ???

Factorization homology [BD04, AF15, AFT17, CG17] is a mechanism that takes as input an n-manifold together with some algebraic data (like an algebra or category) and produces an invariant that can be thought of as "integrating" the algebraic data over the n-manifold. The algebraic data required is that of a n-dimensional disk algebra, which are algebras over the n-disks operad. The most well-studied algebraic inputs are in familiar linear situations, where the disk algebras are valued in a symmetric monoidal category like vector spaces or chain complexes. The goal of this course is to study a particular instance of factorization homology where $n = 2$ and the input algebraic data is of a *non-linear* nature: a braided monoidal category. We will follow the papers [BZBJ18b, BZBJ18a] and potentially some additional topics from [Coo, GJS].

If $\mathcal{A}$ is a braided monoidal category, then the factorization homology along the oriented surface S:

$$\int_S \mathcal{A}$$

is defined and produces a category. To an open disk $S = D^2$, the resulting category is simply $\mathcal{A}$ itself. Moreover, since every surface S admits a disk embedding $D^2 \hookrightarrow S$, by functoriality of factorization homology the category $\int_S \mathcal{A}$ always has a distinguished element $\mathcal{O}_S$ coming from the unit of the category $\mathcal{A}$. In the case that the surface S has an S^1-boundary, the resulting category $\int_S \mathcal{A}$ can be identified with a module category of the algebra of endomorphisms of $\mathcal{O}_S$. A presentation for this algebra can be given using the excision axiom of factorization homology. In short, we see that factorization homology recovers many natural objects related to the braided monoidal category $\mathcal{A}$ that we started with.

An important example will be the braided monoidal category $\mathcal{A} = \mathrm{Rep}_q G$ of representations of the quantum group $U_q\mathfrak{g}$ associated to a reductive group G and an arbitrary $q \in \mathbb{C}^{\times}$. In the case $q = 1$, and when one works in the derived setting, it was shown in [BZFN10] that the factorization homology of Rep G along S is equivalent to the category of quasi-coherent sheaves on the character stack of S: this is the moduli stack of G-local systems on S. Once we turn on a nontrivial q, one can thus interpret the resulting category $\int_S \mathrm{Rep}_q G$ as the quantum analog of sheaves on the character stack. There are concrete characterizations of this category when one works on a particular surface. For instance, when S is the annulus factorization homology recovers the quantum reflection algebra $\mathcal{O}_q(G)$. When S is a punctured torus one obtains quantum differential operators on G.

Another application deals with skein categories. It turns out that there is a close relationship between factorization homology for braided monoidal categories on surfaces and skein categories. Skein categories [RS02, FG09, Wal] provide a mechanism to quantize SL_n-character varieties and to produce 3-manifold invariants. While they are elementary to define, they are difficult to do computations with. The advantage of factorization homology, specifically the excision axiom, gives new insight into the calculation of invariants coming from skein categories.

References

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