Koszul Duality for $E_n$ Algebras

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Assumptions We work over a field $k$ of characteristic 0 and implicitly work in the $(\infty, 1)$-setting; so by the word “category” we always mean an $(\infty, 1)$-category. The background “category of categories” is the category of stable presentable $(\infty, 1)$-categories with only colimit-preserving morphisms between them. This category is equipped with a symmetric monoidal structure given by the Lurie tensor product. The category $\text{Vect}$ is the $(\infty, 1)$-category of (unbounded) chain complexes over $k$.

1 Operads, Algebras and Modules

1.1 Operads

We shall use the following version of definition from [FG12]. Let $\mathcal{X}$ be a symmetric monoidal category. Let $\Sigma$ be the category of (nonempty) finite sets and bijections. Let $\mathcal{X}^\Sigma := \prod_{n \geq 1} \text{Rep}_\mathcal{X}(\Sigma_n)$ be the category of symmetric sequences in $\mathcal{X}$; its objects are collections $\{O(n) \in \mathcal{X}, n \geq 1\}$ such that $\Sigma_n$ acts on $O(n)$. Observe that $\mathcal{X}^\Sigma \simeq \text{Funct}(\Sigma, \mathcal{X})$. This category admits a monoidal structure $\star$ such that the following functor is monoidal:

$$
\mathcal{X}^\Sigma \to \text{Funct}(\Sigma, \mathcal{X})
$$

Namely it’s given by

$$
P \star Q = \bigoplus_{n \geq 1} (P(n) \otimes Q^\otimes n)_{\Sigma_n}
$$

where $\otimes$ is the Day convolution:

$$(P \otimes Q)(n) = \bigoplus_{i+j=n} \text{Ind}_{S_i \times S_j}^{S_n} (P(i) \otimes Q(j))$$

Note that Day convolution is symmetric monoidal because $S_i \times S_j$ and $S_j \times S_i$ are conjugate in $S_n$. The unit object of $(\mathcal{X}^\Sigma, \star)$ is $1_\star$, given by $1_\star(1) = 1_\mathcal{X}$ and $1_\star(n) = 0_\mathcal{X}$ for $n > 1$.

We define Oprd($\mathcal{X}$), the category of reduced, augmented operads over $\mathcal{X}$, to be that of augmented associative algebras in $(\mathcal{X}^\Sigma, \star)$ for which $1_\mathcal{X} \to O(1)$ is an isomorphism. The dual notion coOprd($\mathcal{X}$) of co-augmented cooperads is defined dually. This means that we have the composition maps

$$O(k) \otimes O(n_1) \otimes \ldots \otimes O(n_k) \to O(n_1 + \ldots + n_k)$$

as well as a unit element in $O(1)$, such that the unital, associative and equivariance laws are satisfied up to coherent homotopy. If we interpret the definition in the classical (non-$\infty$) setting, then we obtain the usual notion of operads.
Example 1. The associative operad \( \text{Ass} \) is given by that \( \text{Ass}(n) = k[\Sigma_n] \), the regular representation of \( \Sigma_n \); the operad maps come from substitution. Similarly, the commutative operad \( \text{Comm} \) is given by \( \text{Comm}(n) = k \), the trivial representation of \( \Sigma_n \).

Linear Dual of Operads  Given an operad \( O \) such that \( O(n) \) has finite dimensional cohomologies, we can define \( O^* \) to be \( O^*(n) = O(n)^* \), which will be a cooperad.

Shifting Operads  For an operad \( O \in \text{Oprd}(\text{Vect}) \), we use \( O[1] \) to denote the operad given on the component level by

\[
O[1](n) = O(n)[n - 1]
\]

(where the tilde indicates that the \( \Sigma_n \) action needs to be twisted accordingly), such that \( c \mapsto c[1] \) gives an equivalence \( O[1]-\text{alg}(\text{Vect}) \to O-\text{alg}(\text{Vect}) \) (see below). The dual notion of suspension of cooperads is defined analogously; namely, we also require \( O[1]-\text{coalg}(\text{Vect}) \to O-\text{coalg}(\text{Vect}) \) is given by \( c \mapsto c[1] \).

1.2  Algebras over Operads

Let \( X \) be as before, and let \( C \) be a commutative algebra object in the category of \( X \)-modules. The action

\[
(O, c) \mapsto \bigoplus_n (O(n) \otimes c^\otimes n)_{\Sigma_n}
\]

defines the \( \ast \)-action of \( X^\Sigma \) on \( C \). For any operad \( O \) and any cooperad \( O^\circ \), define

\[
O-\text{alg}(C) := O-\text{mod}(C, \ast)
\]
to be the category of \( O \)-algebras in \( C \) and

\[
O^\circ-\text{coalg}_{\text{nil}}(C) := O^\circ-\text{comod}(C, \ast).
\]
to be the category of \( O^\circ \)-coalgebras in \( C \).

Example  Algebras over \( \text{Ass} \) and \( \text{Comm} \) in a category \( C \) correspond respectively to augmented associative (that is, \( A_\infty \)) and augmented commutative (that is, \( E_\infty \)) algebras in \( C \); Similarly, coalgebras over \( \text{Ass}^* \) and \( \text{Comm}^* \) in \( C \) correspond to coaugmented cocommutative coalgebras and coaugmented coassociative coalgebras in \( C \).

Remark 1. Strictly speaking, the augmentation does not come from being a module of the operad, but rather the obvious equivalence of categories \( \text{Assoc}^{\text{non-unital}}(C) \simeq \text{Assoc}^{\text{aug}}(C) \), given by direct sum with \( 1 \)/ taking the augmentation ideal. To simplify discussion, we’ll consider associative algebras as augmented for the rest of this talk.

1.2.1 Four Types of Comodules

Notice that what we wrote was \( O^\circ-\text{coalg}_{\text{nil}}(C) \) and not \( O^\circ-\text{coalg} \); indeed the former doesn’t in general specialize to what we usual call comodules of cooperads. (Observe that, if \( A \) is a coalgebra, we ought to have maps \( A \to A^\otimes n \) and therefore a map to the direct product.) Instead, define the following \( \ast \)-action:

\[
(O, c) \mapsto \prod_n (O(n) \otimes c^\otimes n)_{\Sigma_n}
\]

and write

\[
O^\circ-\text{coalg}(C) := O^\circ-\text{comod}(C, \ast)
\]

Then this is the one that specializes to our usual notion.

In addition, define the category \( O-\text{coalg}_{\text{nil}} \) to be the one equipped with the action

\[
(O, c) \mapsto \bigoplus_n (O(n) \otimes c^\otimes n)_{\Sigma_n}
\]
and $O$-coalg$_{d.p.}$, the one equipped with the action

$$(O, c) \mapsto \prod_n (O(n) \otimes c^\otimes n)_{\Sigma_n}.$$  

For this talk we will not worry about the d.p. part, since we have the averaging functor

$$\text{avg} : O^\circ$\text{-coalg}_{d.p.} \to O^\circ$\text{-coalg}_{nil}$$

which is an isomorphism in characteristic 0. We also have the obvious functor

$$O^\circ$\text{-coalg}_{nil} \to O^\circ$\text{-coalg}_{d.p.}$$

We compose those two to get a map

$$\text{res} : O^\circ$\text{-coalg}_{d.p.} \to O^\circ$\text{-coalg}_{nil}.$$  

This functor commutes with colimits so admit a right adjoint, giving a pair

$$\text{res} : O^\circ$\text{-coalg}_{d.p.} \rightleftharpoons O^\circ$\text{-coalg} : \text{res}^R$$

**Conjecture 1** ([FG12]). \textit{res is always fully faithful.}

For categories of a specific type, this complication (and many below) disappears; namely those that are pro-nilpotent:

**Definition 1.** A category $C$ is called pro-nilpotent if we can write it as

$$C = \lim_{\mathcal{C}^\text{op}} C_i$$

in the category of stable symmetric monoidal $\mathcal{X}$-module categories, such that the following are satisfied:

1. $C_0 \simeq 0$;
2. $i \geq j \implies f_{i,j} : C_i \to C_j$ commutes with limits;
3. The monoidal map $C_i \otimes C_i \to C_i$, when restricted to $\ker f_{i,i-1} \otimes C_i$, is zero.

**Example 2.** The category $\mathcal{X}^\Sigma$ is pro-nilpotent. Namely, $C_i$ is the full subcategory of those sequences whose value on $n \geq i$ is 0.

**Remark 2.** For the results in [Cos13], the base category is that of chain complexes over a complete filtered vector space, such that each graded piece is a bounded complex. By truncating on the filtration, we can see that this category is pro-nilpotent, so all “nice” results below apply.

**Remark 3.** One of the bootstrapping observations of [FG12] is that $D(\text{Ran} \mathcal{X})$, equipped with the chiral tensor structure, is pro-nilpotent. Namely, the strata come from considering $\text{Ran} \mathcal{X}^\Sigma_n$, which is given by the same construction as $\text{Ran}$ space, but only gluing along $\Delta : X^I \to X^J$ when $|J| \leq n$. Note that $D(\text{Ran} \mathcal{X})$ equipped with the $*$-tensor structure is not pro-nilpotent.

**Proposition 1** ([FG12]). When $C$ is pro-nilpotent, $\text{res}$ is an isomorphism.

### 1.3 Modules

Let $A \in \mathcal{C}$ be an $O$-algebra, and let $\mathcal{M}$ be a module category over $\mathcal{C}$ in the category of $\mathcal{X}$-modules. Note that there is a symmetric monoidal category Sqz($\mathcal{C}, \mathcal{M}$), the “square zero extension” of $\mathcal{C}$ by $\mathcal{M}$, obtained from $\mathcal{C} \times \mathcal{M}$ by collapsing the $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ morphisms. We then define the category of $A$-modules in $\mathcal{M}$, denoted $\text{Mod}_A(\mathcal{M})$, to be the ($\infty,1$)-categoric fiber of $A$ under $\pi_1 : O$-$\text{alg}(\text{Sqz}(\mathcal{C}, \mathcal{M})) \to O$-$\text{alg}(\mathcal{C})$, which is induced by the projection $\pi_1 : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$. The dually defined comodule category is denoted $\text{Comod}^\text{nil}_{A,d.p.}(\mathcal{M})$.

Concretely speaking, an $A$-module structure amounts to an object $M \in \mathcal{M}$ and operation maps

$$O(n) \otimes A^{k-1} \otimes M \otimes A^{n-k} \to M$$

for each $1 \leq k \leq n$, such that all necessary conditions hold.
**Left/Right Module** For the operad Ass, the notion above recovers the notion of *bimodules* over an associative algebra $A$. Using colored operads it is also possible to recover the notion of left/right modules, as is detailed in [Lur]. We shall not define those concepts, but for the sake of stating results let us introduce the notation $LMod_A(M)$ and $RMod_A(M)$ to denote those two categories.

### 2 $E_n$ operads

For this talk we shall focus on the case of $E_n$ operads. Namely, for each $n \geq 1$, there is an element $E_n \in \text{Oprd}(\text{Spc})$ that is realized by the little $n$-disk or the little $n$-cube operads. The operad in $\text{Oprd}(\text{Vect})$ induced by the singular chain functor $C^\ast : \text{Spc} \to \text{Vect}$ is then called the $E_n$ operad in chain complexes; we will refer to it simply by $E_n$.

By definition we have $E_1 \simeq \text{Ass}$, so an $E_1$-algebra is nothing more than an augmented associative algebra. The other extreme is when $n = \infty$, for which we’ll write $E_\infty := \text{colim}_n E_n$. It turns out $E_\infty \simeq \text{Comm}$ (having to do with $S^\infty$ being contractible), i.e. $E_\infty$-algebras are augmented commutative algebras. The other $E_n$ cases are interpolations between those two, so can be seen as describing algebras that are “partially commutative”. More precisely, there is a sequence of maps between operads

$$E_1 \to E_2 \to \ldots \to E_n \to E_{n+1} \to \ldots \to E_\infty$$

induced from the topological counterpart

$$E_1 \to E_2 \to \ldots \to E_n \to E_{n+1} \to \ldots \to E_\infty$$

(where $E_\infty(n) = \ast$ for each $n$) by the standard embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$.

From now on, Vect will denote the homotopy category of chain complexes. When the category $\mathcal{C}$ is not specified, by $E_n$-algebras we mean elements of $E_n\text{-alg}(\text{Vect})$.

### 3 Koszul Duality

#### 3.1 Bar Construction for Associative Algebras

Let $\mathcal{A}$ be a monoidal $\mathcal{X}$-module category with limits and colimits, then we have a standard construction of a pair of adjoint functors

$$\text{Bar} : \text{AssocAlg}^{\text{aug}}(\mathcal{A}) \rightleftarrows \text{CoassocCoalg}^{\text{coaug}}(\mathcal{A}) : \text{coBar}$$

where $\text{Bar}$ maps $R$ to $1 \otimes_R 1$ ($1$ is considered as both a left and a right $R$-module, by means of the augmentation), and $\text{coBar}$ defined dually. The comultiplication on $\text{Bar}(R)$ is given by the following:

$$1 \otimes_R 1 \simeq 1 \otimes_R R \otimes_R 1 \to 1 \otimes_R 1 \otimes_R 1 \simeq 1 \otimes_R 1 \otimes_1 1 \otimes_R 1 \to (1 \otimes_R 1) \otimes (1 \otimes_R 1)$$

The counit is given by

$$1 \simeq 1 \otimes 1 \to 1 \otimes_R 1$$

It is checked in e.g. [Lur, Theorem 5.2.2.17] that this indeed lands in coassociative algebras.

Now let $\mathcal{A}$ be as above and let $\mathcal{C}$ be an $\mathcal{A}$-module category. Fix some augmented associative algebra $A \in \mathcal{A}$. By general construction we have an adjoint pair

$$\text{Bar}_A : A\text{-mod}(\mathcal{C}) \rightleftarrows \mathcal{C} : \text{triv}_A$$

Namely we have $\text{Bar}_A(M) = M \otimes_A 1$ where $A \to 1$ is the augmentation; by this notation we mean the colimit of the following diagram:

$$\cdots A \otimes M \rightrightarrows M$$

Similarly if $A^\circ$ is an coaugmented coassociative coalgebra and $\mathcal{C}$ an $A^\circ$-comodule, then we have an adjoint pair

$$\text{triv}_{A^\circ} : \mathcal{C} \rightleftarrows A^\circ\text{-mod}(\mathcal{C}) : \text{coBar}_{A^\circ}$$

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3.2 Koszul Duality for Operads

When $A$ is $\Sigma^\infty$ as above, these functors trivially lift to another adjoint pair

$$\text{Bar} : \text{Oprd}(\mathcal{A}) \rightleftarrows \text{coOprd}(\mathcal{A}) : \text{coBar}$$

that are compatible with the obvious forgetful functors. This pair we call the operadic Koszul duality.

**Proposition 2.** These are mutual equivalences.

**Proof.** Apply the algebraic Koszul duality (defined below) on $\mathcal{X} = \text{Vect}$ and $\mathcal{C} = \text{Vect}^\Sigma$ this reduces to the computation that $\text{Ass}^! = \text{Ass}^*[1]$, which is done manually. 

We shall refer to $\text{Bar}(x)$ as the Koszul dual of $x$ and write it as $x!$.

**Example 3.** The fundamental example in representation theory is $\text{Lie}^! = \text{Comm}^*[1]$, corresponding to the relationship between a Lie algebra and its Chevalley complex.

3.2.1 Koszul Dual for the $E_n$ Operads

**Proposition 3.** $E_n^! = E_n^*[1]$. More generally, we have $E_n^! \simeq E_n^*[n]$, and the map is compatible with $E_n \to E_{n+1}$.

The $n = 1$ case is a straightforward computation. For our setting (characteristic 0) the general $n$ case would follow from a corresponding computation in homology operad in [GJ94] plus the formality theorem for $E_n$ proved in loc. cit; over $\mathbb{Z}$ this is proven in [Fre11].

3.3 Koszul Duality for Algebras

Now let $\mathcal{C}$ be the same as in section 1.2. For any Koszul pair $(O, O^!)$, bar construction for modules gives an adjoint pair:

$$\text{Bar}^\text{naive}_O : O\text{-alg}(\mathcal{C}) \rightleftarrows O^!\text{-coalg}_{\text{nil.d.p.}}(\mathcal{C}) : \text{coBar}^\text{naive}_O$$

Again compatible with the forgetful functors. Now combine with the restriction adjoint pair to get

$$\text{Bar}^\text{naive}_O \circ \text{res} : O\text{-alg}(\mathcal{C}) \rightleftarrows O^!\text{-coalg}(\mathcal{C}) : \text{coBar}^\text{naive}_O \circ \text{res}^R = \text{cobar}_O$$

This is what we call the algebraic Koszul duality, and we’ll write $A^!$ for $\text{Bar}_O(A)$ as well. We shall say $A$ is Koszul if $A \to (A^!)^!$ is an isomorphism. Recall that when $\mathcal{C}$ is pro-nilpotent, the two functors agree.

**Proposition 4 ([FG12, Prop 4.1.2]).** When $\mathcal{C}$ is pro-nilpotent, both functors are equivalences.

The Two Bar Constructions Agree

In the case $O = \text{Ass}$, the Koszul duality above gives a pair of adjunction

$$[1] \circ \text{Bar}_{\text{Ass}} : \text{Assoc}^\text{aug}(\mathcal{C}) \rightleftarrows \text{Coassoc}^\text{coaug}(\mathcal{C}) : \text{coBar}_{\text{Ass}^*[1]} \circ [-1]$$

This agrees with the bar construction given at the beginning of section 3.

**Example 4.** Taking Koszul dual along $\text{Ass}^! = \text{Ass}^*[1]$ gives Hochschild complex; along $\text{Lie}^! = \text{Comm}^*[1]$ gives Chevalley complex; and along $\text{Comm}^! = \text{Lie}^*[1]$ gives Harrison complex.

3.3.1 Building an Equivalence

Unlike the operadic case, in general we have no reason to expect algebraic Koszul duality to be an equivalence.

**Example 5.** In the case of $\text{Lie}^! = \text{Comm}^*[1]$, the Bar functor sends a Lie algebra to its Chevalley complex, and this functor is clearly not fully faithful: take say $\mathfrak{sl}_2$, then its Chevalley complex is concentrated on degree ($-3$), but the trivial Lie algebra $k[3]$ would have the same Chevalley complex.
Nevertheless, [FG12] proposes a conjecture about how to make this an equivalence. We say an $O$-algebra $A$ is nilpotent if there exists an $N$ such that $n > N$ implies $O(n) \otimes A^n \rightarrow A$ is zero (nullhomotopic), and we define $O$-$\text{alg}_{\text{nil}}(C)$ to be the subcategory spanned by objects that are limits of nilpotent algebras (we call such objects pro-nilpotent).

Observe that the coBar functor lands in this subcategory: write $O^i = \text{colim}_k O^{i \leq k}$, where $O^{i \leq k}$ is obtained by erasing $O^i(s)$ terms for all $s > k$. For $B \in O^{i}$-$\text{coalg}_{\text{nil}}(C)$ and $A = \text{coBar}_O(B)$, define $A^{\leq k} := \text{coBar}_{O^{i \leq k}}(B)$, then one can check that $A = \text{lim}_{O}$-$\text{alg}(C_{\leq k})$ and $O(s)$ acts on $A^{\leq k}$ by zero for $s > k$. So by adjunction, the functor $\text{Bar}_O$ factors as $\text{Bar}_O \circ \text{compl}_O$, where the completion functor $\text{compl}_O$ is the left adjoint to the limit-preserving embedding $O$-$\text{alg}_{\text{nil}}(C) \rightarrow O$-$\text{alg}(C)$.

Conjecture 2 ([FG12]).

$\text{Bar}_O^{\text{naive}} : O$-$\text{alg}_{\text{nil}}(C) \rightleftarrows O^{i}$-$\text{coalg}_{\text{d.p.}}(C) : \text{coBar}_O$

is an equivalence of categories.

Remark 4. The $0$-connected case for modules over a commutative ring spectrum is proven in [CH15].

This can be understood as a generalization of the classical results in [BGS96] of the auto-equivalence of left finite Koszul algebras.

3.3.2 Koszul Duality for $E_1$ Algebras

Let us look at $E_1$-algebras, i.e. the case of associative algebras in Vect.

Theorem 3.1 ([Lur11, Corollary 3.1.15]). Let $A$ be an $E_1$-algebra. If $A$ is coconnective and locally finite, then $A$ is Koszul.

Note that coconnective means $\pi_0(A) = k$, $\pi_i(A) = 0$ for $i > 0$, and locally finite means $\text{dim} \pi_i(A) < \infty$ for each $i$. In the classical setting this simply means our $A$ is Artinian; in the dg setting, it means that our algebra is connective and has finite dimensional cohomologies.

**Sample Computation** Let’s do a concrete example with chain complexes. Consider $k[x]$ for $x$ in degree $-1$, so it is the complex $0 \rightarrow k \rightarrow k \rightarrow 0$ concentrated in degree $0$ and $-1$. Let’s compute what the (associative) Koszul dual $k \otimes_{k[x]}^L k$ is. The complex $k$ (concentrated on degree $0$) admits the following resolution

$$\ldots \rightarrow k[x][2] \rightarrow k[x][1] \rightarrow k[x] \rightarrow k,$$

where the maps between complexes are given by

\[
\begin{array}{ccc}
0 & \rightarrow & k \\
\downarrow & & \downarrow id \\
0 & \rightarrow & k \\
& & \downarrow \\
& & k \\
& & \rightarrow 0
\end{array}
\]

Thus we can compute the derived tensor product as

$$\text{Tot}(\ldots \rightarrow k[2] \rightarrow k[1] \rightarrow k)$$

which is given by

$$\ldots \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow k$$

i.e. $k[y]$ for $y$ placed in degree $-2$. Now we compute $\text{coBar}(k[y]) = \text{Hom}_{k[y]}(k, k) = \text{Hom}_{k[y^*]}(k, k)$ where $y^*$ is on degree $2$. We use the following resolution:

$$0 \rightarrow k[y^*][-2] \rightarrow k[y^*] \rightarrow k \rightarrow 0$$
So the derived hom is given by 
\[ \text{Tot}(k \to k[2] \to 0) \]
which is \( k[x] \) again. More generally, if we place a vector space \( V \) on degree -1, then the trivial \( \text{Sym}(V[1]) \)-module admits the following resolution:
\[
\ldots \wedge^2 (V[1]) \otimes \text{Sym}(V[1]) \to V[1] \otimes \text{Sym}(V[1]) \to \text{Sym}(V[1]) \to k \to 0
\]
From which we can derive that \( \text{Sym}(V[1])^! = \text{Sym}(V[2]) \), considered as a coalgebra.

**The Case of Lie Algebras**  
The computation above is the abelian case of the general computation for Lie algebras. Namely, given a (dg) Lie algebra \( \mathfrak{g} \), the Bar construction computes its Chevalley complex, which could be obtained from the following resolution of the trivial module:
\[
\ldots \wedge^2 (\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathfrak{g} \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \to k \to 0
\]
Let us briefly explain the case of Lie algebra Koszul duality. Because there is an operad morphism \( \text{Lie} \to \text{Ass} \), we have a natural morphism \( \text{res} : \text{Ass}(C) \to \text{Lie}(C) \), which admits a left adjoint \( U : \text{Lie}(C) \to \text{Ass}(C) \), and we have
\[
[1] \circ \text{Bar}_{\text{Ass}} \circ U \simeq \text{oblv}_{\text{Cocomm}} \circ \text{Coass} \circ [1] \circ \text{Bar}_{\text{Lie}}
\]
as functors \( \text{Lie}(C) \to \text{Coassoc}(C) \). This is a lossy functor, however: whereas \( U(\mathfrak{g}) \) is a cocommutative Hopf algebra, now we have a coassociative coalgebra.
To make this precise, define \( \text{CocommBialg}(C) := E_1\text{-alg}((\text{Cocomm-alg}(C))) \simeq \text{Cocomm-alg}(E_1\text{-alg}(C)) \) — note that this equivalence is not automatic and is checked in [GR, IV.2], and further define \( \text{CocommHopf}(C) := \text{Grp}(\text{Cocomm-alg})(C) \). To upgrade to an equivalence of cocommutative Hopf algebras one has to loop our Lie algebra; namely we can upgrade \( U \) to \( U^{\text{Hopf}} : \text{Lie}(C) \to \text{CocommHopf}(C) \), and we have
\[
[1] \circ \text{Grp}(\text{Bar}_{\text{Lie}}) \circ \Omega_{\text{Lie}} \simeq U^{\text{Hopf}}
\]
as functors \( \text{Lie}(C) \to \text{CocommHopf}(C) \). (One might be worried that the Lie structure becomes trivial after the looping, but the Lie bracket can be recovered from the homotopy data.) The same story holds for \( E_n \)-algebras in place of \( E_1 \), the only difference being that we have to loop \( n \) times instead.

**3.4 Koszul Duality for Modules**

Now let \( \mathcal{M} \) be the same as in section 1.3. By taking left adjoint to the trivial module functor \( \mathcal{M} \to \text{Mod}_A(\mathcal{M}) \) we obtain another Bar functor, and similarly a cobar functor. By same reasoning as in the algebra case, this pair factors through another pair:
\[
\text{Bar}_A : \text{Mod}^{\text{nil}}_A(\mathcal{M}) \rightleftarrows \text{Comod}^{\text{nil}}_{A^!}(\mathcal{M}) : \text{coBar}_A^!
\]
which we call the **modular Koszul duality**. (Warning: this is slightly different from the one in [FG12] Section 7, where they used what we write as \( \text{Bar}_A^{\text{naive}} \). When \( C \) is pro-nilpotent, however, those two notions will agree.) Even if \( A \) is Koszul, there is no guarantee that its modular Koszul duality is an equivalence. However, **Proposition 5.** When \( \mathcal{M} \) is pro-unipotent, these are equivalences.

For the case of one-sided modules we have the following result:

**Theorem 3.2 ([Lur11, 3.5.2]).** For \( A \) a small \( E_1 \)-algebra (defined in [Lur11, 1.1.11]), there is an equivalence between the category of ind-coherent left/right modules (ind-object over small modules, i.e. those whose homotopy groups are finite dimensional) over \( A \) and that of left/right comodules over \( A^! \).
4 More on $E_n$ Operads

The following, known as Dunn Additivity, is the key fact that makes things work:

**Theorem 4.1** ([Dun88], [Lur, 5.1.2.2]). For any $n, m$, we have $E_{n+m}\text{-alg}(C) = E_n\text{-alg}(E_m\text{-alg}(C))$.

We will not try to prove this theorem, but let us mention that this has a generalization to factorization algebras. Namely it would follow from Lurie’s result (locally constant factorization algebras on $\mathbb{R}^n$ are the same as $E_n$ algebras) and the following statement:

**Theorem 4.2** ([Roz]). For any manifolds $M, N$, the factorization algebras on $M$ valued in factorization algebras on $N$ are the same as factorization algebras on $M \times N$.

In terms of left-right modules, $E_k$ algebra also behave well (everything below would also hold for RMod):

**Corollary 1** ([Lur, 4.8.5.20]). For $A$ an $E_n$-algebra and $M$ as in section 1.3, $LMod_A(M)$ (where $A$ is viewed as an $E_1$-algebra) are $E_{n-1}$-categories.

In fact something stronger is true:

**Corollary 2.** If $M$ is such that for every $A \in E_n\text{-alg}(C)$, there exists $M_A \in LMod_A(M)$ such that $A \simeq \text{End}_A(M_A)$, then the functor $LMod_A(M)$ is a fully faithful functor from $E_n\text{-alg}(C)$ to $E_{n-1}\text{-alg}(C\text{-ModCat})$.

**Example 6.** If $A$ is an $E_3$ algebra, i.e. quasi-triangular Hopf algebra, then its module category is a braided monoidal ($E_2$) category.

Now if we have a $E_1$-algebra $A$, its left module category would have no monoidal structure; however, its bimodule category would again have an $E_1$ structure. The general statement is the following:

**Theorem 4.3** ([Lur, 3.4.4.6]). For $M = C$ and $A \in E_n\text{-alg}(C)$, we have $\text{Mod}_A(C) \in E_{n-1}\text{-alg}(A\text{-ModCat})$.

**Remark 5.** The theorem is true more generally for $O$ a coherent operad, as defined in [Lur, 3.3.1]. Also it should be straightforward to separate the exact condition on $M$ for this to hold.

4.1 (Co)Hochschild (Co)homology

Notice that when we take $M = C$, we have in particular $A \in \text{Mod}_A(C)$, so it makes sense to discuss

$$HH^*(A) := \text{Hom}_{\text{Mod}_A(C)}(A, A)$$

and

$$HH_*(A) := A \otimes_{\text{Mod}_A(C)} A.$$ 

We shall refer to them as the Hochschild cohomology/homology of $A$ respectively. Dually we can define $\text{CHH}^*(A)$ and $\text{CHH}_*(A)$, the coHochschild cohomology/homology of a coalgebra. The following statement is usually referred to as (higher) Deligne Conjecture:

**Proposition 6** ([Lur09, 2.5.13], [KS00], [Tam03]). Hochschild cohomology of an $E_n$-algebra is an $E_{n+1}$-algebra.

**Example 7.** For $C$ a monoidal category, its Hochschild cohomology would be $E_2$; this is the Drinfeld center.
5 Koszul Duality for $E_2$ Algebras and Modules

Define Bialg($\mathcal{C}$), the category of bialgebras in $\mathcal{C}$, to be

$$E_1\text{-alg}(E_1^*\text{-coalg}(\mathcal{C})) \simeq (E_1^*\text{-coalg}(E_1\text{-alg}(\mathcal{C})))$$

(That these two definitions are equivalent is again not obvious.) Let Hopf($\mathcal{C}$) denote the full subcategory of Hopf algebra objects.

**Remark 6.** Let us admit that we do not yet have a workable $\infty$-definition for Hopf($\mathcal{C}$), so the following can only be understood at the dg level. (In an earlier version of this note an incorrect definition was given.)

Using additivity, we can write $E_2$-$\text{alg}(C)$ as $E_1$-$\text{alg}(E_1$-$\text{alg}(\mathcal{C}))$; applying Koszul duality on the inner level, we end up producing an element of Bialg($\mathcal{C}$). This observation (that the $E_1$ Koszul dual of an $E_2$-algebra is a bialgebra) was due to Tamarkin.

We give two proofs for the case $\mathcal{C} = \text{Vect}$.

**Proof by Tannakian Formalism.** For any $E_2$-algebra $A$, recall that $A$-$\text{mod}(\text{Vect})$ is an $E_1$-algebra in DGCat, i.e. a monoidal DG category. Now apply modular Koszul to $A$-$\text{mod}$; in nice cases, this gives us $A^1$-$\text{comod}$ for $A^1 \in E_1^*[1]$-$\text{coalg}$, and by our remark above, the $E_1$ (monoidal) structure on $A$-$\text{mod}$ gives a monoidal structure on $A^1$-$\text{comod}$. Furthermore, by definition, shift by 1 gives an isomorphism $A^1$-$\text{comod} \simeq (A^1[1])$-$\text{comod}$, equipped with an $E_1$ structure. Since it also comes with a monoidal forgetful map to the underlying Vect, by general Tannakian formalism we can reconstruct the bialgebra $A^1[1]$. 

**Original Proof by Tamarkin.** For any given operad $O \in \text{Oprd(Vect)}$, the homology of $O$ (with trivial differential) is again an operad, which we call the homology operad $HE$. The key fact is the following, which is usually referred to as Kontsevich formality:

**Theorem 5.1** ([Tam03], [Kon97]). $E_n \simeq HE_n$.

The operad $HE_n$ is $P_n$, the operad of Poisson $n$-algebras, that is, Poisson algebras whose brackets have degree $(1 - n)$. Next, there is a combinatorially defined operad $B_\infty$, that of the brace algebras.

**Proposition 7** ([KS00]). $B_\infty \simeq HB_\infty \simeq P_2$.

This means that any $E_2$-algebra is automatically equipped with a $B_\infty$-algebra structure. Finally, an explicit check (e.g. [Foi17]) shows that Bar construction maps $B_\infty$-algebras to Hopf algebras.

Let us mention in the passing that ideas here also give another proof of the Etingof-Kazhdan quantization theorem, as noted by [Tam07]. Namely, if $g$ is a Lie bialgebra, then $\text{Sym}(g[-1])$ has, by definition, the structure of an $P_2$-algebra; then the procedure here would yield a (dg) Hopf algebra. One then checks that the resulting Hopf algebra is concentrated on degree 0, and the degree 0 piece is a bona fide Hopf algebra, which we denote by $Q(g)$. Then the Etingof-Kazhdan quantization $U_q(g)$, as a Hopf algebra (see below), is given as $\lim_n g \otimes k[t]/t^n$.

**Remark 7.** The equivalence $B_\infty \simeq P_2$ implicitly involves the choice of an associator.

**Remark 8.** Under additional conditions, this procedure can in fact produce a Hopf algebra (i.e. we get the antipode map). For instance, Tannakian formalism recovers the Hopf algebra structure if the module category turns out to be rigid; likewise, if the Lie bialgebra $g$ is conilpotent (i.e. $x \mapsto \delta(x) - (1 \otimes x + x \otimes 1)$ is a nilpotent operator), then the resulting bialgebra is also conilpotent, thus equipped with an antipode structure. In particular, this is satisfied by $g \otimes k[t]/t^n$ mentioned above.

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$^1g \otimes k[t]/t^n$ is the Lie bialgebra over $k[t]/t^n$, equipped with the same Lie bracket and the cobracket $\delta(x \otimes a) = t\delta(x)$, $\delta(x)$ being the Lie cobracket on $g$. 

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9
6 The General Case for $E_n$

Finally we list some facts about general $E_n$ algebras and modules.

**Proposition 8.** Under the identification $E_n\text{-alg} \simeq E_1\text{-alg}(E_1\text{-alg}(\ldots))$, applying the $E_n$ Koszul duality is the same thing as applying the $E_1$ Koszul duality on each of the $E_1$-structures.

**Proposition 9** ([Lur11 4.4.5]). Let $A$ be an $E_n$-algebra that is $n$-coconnective (meaning $\pi_i = 0$ for $i \geq n$) and locally finite. Then $A$ is Koszul.

**Proposition 10** ([AF14]). $HH_*(A) \simeq CHH_*(A^\vee)$ for $A \in E_n\text{-alg}$ that is $(-n)$-coconnective.
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