Lecture Series: Observables in the effective BV-formalism; Talk 5: The factorization algebra of observables

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We reach the main construction of this lecture series, which we will state as one of the central theorems of [1]

Theorem 1. [1] *Let M be a manifold. There is an assignment*

$$\mathrm{Obs}^q : \{\text{QFTs on } M\} \to \{\text{factorization algebras on } M\}$$

called the quantum observables.

There is a simpler construction at the classical level. Let us fix a classical BV-theory $(\mathcal{E}, \langle -, - \rangle, I)$. We have defined the global observables via the classical BV-complex

$$\mathrm{Obs}^{cl}(M) := \left(\mathrm{Sym}(\mathcal{E}(M)^\vee), Q + \{I, -\}\right).$$

Since $\mathcal{E}$ is a sheaf of sections of some vector bundle, it makes sense to consider, for each open U, the subcomplex

$$\mathrm{Obs}^{cl}(U) := (\mathrm{Sym}(\mathcal{E}(U)^\vee), Q + \{I, -\})$$

that we call the *classical observable supported on U*.

Proposition 1. *The assignment $U \mapsto \mathrm{Obs}^{cl}(U)$ defines a factorization algebra on M.*

In fact, this is a corollary of the $\mathbb{O}$-construction from the last talk, but we can be explicit. If $\sqcup_i U_i \to V$ is a disjoint union of open subsets inside of the open set V then we have a map

$$\mathcal{E}(V) \to \mathcal{E}(\sqcup_i U_i) = \oplus_i \mathcal{E}(U_i)$$

because $\mathcal{E}$ is a sheaf. Taking the duals and noticing that Sym is a symmetric monoidal functor we have a map

$$\otimes_i \mathrm{Sym}(\mathcal{E}(U_i)^\vee) \to \mathrm{Sym}(\mathcal{E}(V)).$$

That is, a map $\otimes_i \mathrm{Obs}^{cl}(U_i) \to \mathrm{Obs}^{cl}(V)$. One shows directly that this is a cochain map and defines the factorization structure maps.

Now, suppose we have a quantum field theory on M. This is the data of $(\mathcal{E}, Q, \langle -, - \rangle)$ together with a collection $\{I[r]\}$ of effective functionals that satisfy the RG-flow equation and the regularized quantum master equation. We have constructed the global quantum observables $\mathrm{Obs}^q(M)$. An element is a collection of functionals $\{O[r]\}$ where each $O[r] \in \mathrm{Obs}^q(M)[r]$ that are related by RG-flow.

To define the factorization algebra, we first need to define what we mean by a quantum observable $\{O[r]\}$ to be *supported* on an open set U. The naive definition used in the classical case does not work here: both the regularized BV-laplacian Δ_r and the Poisson bracket $\{-, -\}_r$ increase the support of an element $O[r]$ so that the total differential

$$\hat{Q}_r := Q + \hbar \Delta_r + \{I[r], -\}$$

also increases support. For instance, if $O[r]$ is in an element of subspace $\mathrm{Sym}(\mathcal{E}(U))[[\hbar]]$ then $\hat{Q}_r O[r]$ may not be.

Luckily, the magnitude in which $\hat{Q}_r$ does increase support is controllable. One says that a quantum observable $\{O[r]\}$ is *supported* on $U \subset M$ iff there exists a closed subset $K \subset U$ and a small enough regularization r such that

$$\mathrm{Supp}\ O[r] \subset K.$$

A main technical result of [1] is that if we have such an observable supported on U then $\hat{Q}_r$ applied to it is still supported on U. Thus we have defined the subcomplex

$$\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$$

of *observables* supported on U.

We now describe the structure maps of the factorization algebra. Focus on the case $U \sqcup U' \hookrightarrow V$ where U, U' are disjoint. We need to describe a map

$$\mathrm{Obs}^q(U) \otimes \mathrm{Obs}^q(U') \to \mathrm{Obs}^q(V).$$

Take quantum observables $\{O[r]\}$ and $\{O[r']\}$ supported on U, U' respectively. Viewing the functionals as elements of the symmetric algebra $\mathcal{O}(\mathcal{E})[[\hbar]]$ we may consider the product

$$O[r] \cdot O[r'] \in \mathcal{O}(\mathcal{E})[[\hbar]].$$

Theorem 2. *The following limit*

$$\lim_{r' \to 0} W^r_{r'}(O[r] \cdot O[r']) \in \mathcal{O}(\mathcal{E})[[\hbar]]$$

exists and will be denoted $(O \cdot O')[r]$.

We can then define the factorization product ([**?**]) by

$$\{O[r]\} \otimes \{O'[r]\} \mapsto \{(O \cdot O')[r]\}.$$

It is straightforward to check that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization map. A spectral sequence argument is needed to show that

$$\mathrm{Obs}^q : U \mapsto \mathrm{Obs}^q(U)$$

actually is a factorization algebra.

The connection with the classical observables is the following.

Theorem 1. [1] *Suppose* $\{I[r]\}$ *is a quantization of the classical theory* $I \in \mathcal{O}_{loc}(\mathcal{E})$. *Then* Obs^q *is a factorization algebra in* $\mathbb{C}[[\hbar]]$*-modules. Moreover, there is an isomorphism*

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$*, and the factorization algebra of classical observables.*

References

[1] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.