Lecture Series: Observables in the effective BV-formalism, Talk 4: Factorization algebras: examples and constructions

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In this talk we presented some basics of factorization algebras as defined by Costello and Gwilliam in [1]. Further, we explained some examples and constructions of factorization algebras coming from sheaves of differential graded Lie algebras.

Let M be a manifold, a *prefactorization algebra* $\mathcal{F}$ on M, taking values in vector spaces, is a rule that assigns to each open $U \subset M$ a vector space $\mathcal{F}(U)$ along with the following maps and combatibilities.

(1) For each inclusion $U \subset V$, a linear map $m_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$;
(2) For each finite collection of pairwise disjoint open sets $\{U_i\}$ with $U_i \subset V$, a linear map $m_V^{U_1,\ldots,U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$;
(3) The maps satisfy the obvious compatibility condition, i.e., if $U_{i,1} \sqcup \cdots \sqcup U_{i,n} \subset V_i$ and $V_1 \sqcup \cdots \sqcup V_k \subset W$, then the following diagram commutes.

$$\begin{array}{ccc} \otimes_{i=1}^{k} \otimes_{j=1}^{n_i} \mathcal{F}(U_j) & \longrightarrow & \otimes_{i=1}^{k} \mathcal{F}(V_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(W) & \end{array} .$$

Note that $\mathcal{F}(\emptyset)$ is necessarily a commutative algebra. A prefactorization algebra $\mathcal{F}$ is *unital* if $\mathcal{F}(\emptyset)$ is a unital commutative algebra.

A fundamental example is the factorization algebra on $\mathbb{R}$ determined by an associative algebra A. In this example, each open interval (a, b) is assigned the algebra A, the map induced by an inclusion $(a, b) \subset (c, d)$ is the identity and the map induced by including disjoint intervals is determined by the multiplication in A. That the compatibility condition (3) holds follows from the associativity of the multiplication.

The preceding example is universal for prefactorization algebras on $\mathbb{R}$ which are *locally constant*, i.e., the map induced by an inclusion of intervals is an isomorphism.

Proposition 1. *Let $\mathcal{F}$ be a locally constant, unital prefactorization algebra on $\mathbb{R}$ taking values in vector spaces. Then $\mathcal{F}(\mathbb{R})$ has the structure of an associative algebra.*

Alternatively, one can define prefactorization algebras on M valued in a multicategory C as functors $\mathcal{F} : \mathsf{Disj}_M \to \mathsf{C}$, where Disj_M is the multicategory with objects the connected open subsets of M and morphisms corresponding to inclusions of pairwise disjoint collections of opens into another open set. There is an associated symmetric monoidal category $\mathbf{S}\mathsf{Disj}_M$ and for any symmetric monoidal category $\mathsf{C}^{\otimes}$, a prefactorization algebra valued in $\mathsf{C}^{\otimes}$ is a symmetric monoidal functor $\mathcal{F} : \mathbf{S}\mathsf{Disj}_M \to \mathsf{C}$.

Prefactorization algebras have a flavor similar to precosheaves. It is often useful for objects to satisfy descent or a local-to-global property, e.g., cosheaves, and such prefactorization algebras are called *factorization algebras.*

Definition 1. *Let U be an open set. A collection of open sets $\mathfrak{U} = \{U_i\}$ is a Weiss cover of U if for any finite collection of points $\{x_1, \ldots, x_k\}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \ldots, x_k\} \subset U_i$.*

The Weiss covers define a Grothendieck topology on the category of open subsets of a space M which is called the *Weiss topology.* A Weiss cover is a cover in the traditional sense, but typically contains an enormous number of open sets. Given a manifold M of dimension n, there are several ways to construct a Weiss cover of M. For instance, the collection of all open sets in M diffeomorphic to a disjoint union of finitely many copies of the open n-disk forms a Weiss cover.

Definition 2. *A prefactorization algebra $\mathcal{F}$ on M is a factorization algebra if $\mathcal{F}$ is a cosheaf with respect to the Weiss topology.*

Generalizing the proposition of the previous section, factorization algebras (valued in cochain complexes) on $\mathbb{R}^n$ resemble E_n algebras (algebras over the operad of little n-disks). In fact, E_n algebras form a full subcategory of factorization algebras on $\mathbb{R}^n$: those that are locally constant, i.e., those for which an inclusion of open discs induces a quasi-isomorphism. The following theorem of Lurie [3] makes this claim precise (see also the work of Matsuoka [4]).

Theorem 1. *There is an equivalence of $(\infty, 1)$-categories between E_n algebras and locally constant factorization algebras on $\mathbb{R}^n$.*

Let E be a vector bundle on M and let $\mathcal{E}$ denote the sheaf of sections. Similarly, let $\mathcal{E}_c$ denote the cosheaf of compactly supported sections. It is easy to verify that the symmetric algebra of a cosheaf is a prefactorization algebra, it is more difficult to check the local-to-global (factorization) property. However, Costello and Gwilliam prove that both $\mathrm{Sym}\mathcal{E}_c$ and the completed version $\widehat{\mathrm{Sym}}\mathcal{E}_c$ form factorization algebras on M.

The preceding construction can be bootstrapped to the case of Chevalley-Eilenberg chains/cochains of a sheaf of differential graded Lie algebras. If $\mathcal{L}$ is such a sheaf, we will denote Chevalley-Eilenberg chains by $C_*(\mathcal{L})$ and cochains by $C^*(\mathcal{L})$.

Theorem 2. *Let L be a local dg Lie algebra on M. Then for $U \subset M$ an open set, the assignemnts*

$$\mathbb{U}\mathcal{L} : U \mapsto C_*(\mathcal{L}_c(U)) \quad \textit{and} \quad \mathbb{O}\mathcal{L} : U \mapsto C^*(\mathcal{L}(U))$$

define factorization algebras.

As a simple example, let $\mathfrak{g}$ be an ordinary Lie algebra and consider the sheaf of differential graded Lie algebras on $\mathbb{R}$ given by $\mathfrak{g}_{\mathbb{R}} \stackrel{def}{=} \Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$, where Ω^* denotes differential forms. By the preceding theorem $\mathbb{U}\mathfrak{g}_{\mathbb{R}}$ is a factorization algebra on $\mathbb{R}$

valued in complexes. Passing to cohomology, we obtain a locally constant factorization algebra on $\mathbb{R}$ valued in vector spaces. Hence, by the proposition above $H^*(\mathbb{U}\mathfrak{g}_{\mathbb{R}})$ corresponds to an associative algebra; one can identify this algebra as the universal enveloping algebra $U\mathfrak{g}$.

References

[1] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.

[2] P. Boavida de Brito and M. Weiss, *Manifold calculus and homotopy sheaves*, Homology Homotopy Appl., 15 (2), 361-383.

[3] J. Lurie, *Higher algebra*

[4] T. Matsuoka, *Descent and Koszul duality for factorization algebras*