# Lecture Series: Observables in the effective BV-formalism; Talk 3: Effective BV-quantization

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One intepretation of BV-quantization is a general approach to quantize gauge theories. As we saw in the last lecture one of the difficulties in physical/geometric applications of quantum gauge theories is the fact that the space of fields is infinite dimensional.

One incarnation of this is the so-called ultra-violet divergence which was briefly mentioned last time. Suppose $(\mathcal{E}, Q, \langle -,- \rangle)$ is a free classical BV-theory. The $(-1)$-shifted symplectic pairing $\langle -,- \rangle$ induces a partially defined Poisson bracket on $\mathcal{O}(\mathcal{E}) = \mathrm{Sym}(\mathcal{E}(M)^\vee)$. It is partially defined because the dual $\mathcal{E}(M)^\vee$ involves distributional sections and one cannot multiply such elements. Moreover, the naive definition of the BV-laplacian

$$\Delta|_{\mathrm{Sym}^{=2}} = \{-,-\}$$

is also ill-defined. In general, the naive definition of the BV-laplacian is by contraction with the element in $\bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ determined by the pairing.

The usual fix of this problem by physicists is the method of *renormalization*. In this talk, we discuss a homotopic approach to the effective renormalization of quantum gauge theories as developed by Kevin Costello in [1].

The basic idea is to use the homotopy equivalence between distributions and smooth functions to regularize the BV quantization formalism into homotopic families.

Suppose $(\mathcal{E}(M), Q)$ is an arbitrary *elliptic complex* on a manifold $M$. This means that $\mathcal{E}(M)$ is the global sections of some $\mathbb{Z}$-graded sheaf, $Q$ is a differential operator of degree $+1$ of square zero, and that the induced complex is elliptic. For instance, any free BV-theory gives such an object. One can also consider the induced complex $(\bar{\mathcal{E}}(M), Q)$ where the bar denotes distributional sections.

A famous result of Atiyah-Bott [3] states that there is a homtopy equivalence between the smooth sections and distributional sections

$$(\mathcal{E}(M), Q) \simeq (\bar{\mathcal{E}}(M), Q).$$

A lift of a distributional section to a smooth section is sometimes called a *regularization.*

The pairing of a free BV-theory determines an element $K_0 \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ of degree one. According to the above we can choose a regularization

$$K_r = K_0 + QP_r$$

where $K_r \in \mathcal{E} \otimes \mathcal{E}$ is smooth. In particular, contraction with $K_r$

$$\Delta_r := \partial_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

*is well-defined.*

Suppose $r, r'$ are two regularizations

$$K_0 = K_r + QP_r = K_{r'} + QP_{r'}.$$

Then, $K_r - K_{r'} = Q(P_r^{r'})$ for some element $P_r^{r'} \in \mathcal{E} \otimes \mathcal{E}$ of degree zero. Note that $P_r^{r'}$ is smooth.

The main idea here is that $P_r^{r'}$ is an instance of the propogator from the effective construction of local functionals. The operator $e^{\hbar \partial_{P_r^{r'}}}$ intertwines the differential:

$$e^{\hbar \partial_{P_r^{r'}}}(Q + \hbar \Delta_r) = (Q + \hbar \Delta_{r'}) e^{\hbar \partial_{P_r^{r'}}}.$$

Using this, we can "homtopy transfer" the interaction $I \in \mathcal{O}(\mathcal{E})$ via

$$I[r] = e^{\hbar P_0^r} e^{I/\hbar}.$$

This is precisely the expansion in terms of Feynman weights $I[L] = W(P_0^L, I)$ given in Lecture 1 in the case that the regularization is "length scale". This type of regularization is defined in terms of heat kernels as in [1].

**Definition 1.** *([2]) An effective BV-quantum field theory based on $(\mathcal{E}, Q, \langle -, - \rangle)$ consists of the following data:*

(1) *For each regularization $r$ we have a functional*

$$I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]].$$

*Moreover, $I[r]$ must be at least cubic.*

(2) *Given $r, r'$ then $I[r]$ must be related by RG-flow*

$$I[r] = W(P(r', r), I[r']).$$

(3) *For each $r$, $I[r]$ must satisfy the scale $r$ quantum master equation*

$$QI[r] + \hbar \Delta_L I[r] + \frac{1}{2}\{I[r], I[r]\}_r = 0.$$

(4) *Locality axiom garaunteeing that in the limit as $r \to 0$ the functionals $I[r]$ become local.*

The limit of $I[r] \mod \hbar$ exists and is local, which is denoted $I \in \mathcal{O}_{loc}(\mathcal{E})$. Moreover, it determines a classical field theory for the same underlying free BV-theory. Such a QFT is called a *quantization* of $I$.

Given a QFT we can defined the following *quantum BV-complex.* For each regularization $r$ define

$$\mathrm{Obs}^q(M)[r] := \left(\mathrm{Sym}(\mathcal{E}(M)^\vee)[[\hbar]], Q + \hbar \Delta_r + \{I[r], -\}_r\right).$$

It is called the complex of *global observables* associated to the regularization $r$. Moreover, the homotopy $P_r^{r'}$ defines a homotopy equivalence

$$\mathrm{Obs}^q(M)[r] \simeq \mathrm{Obs}^q(M)[r']$$

for any regularizations $r, r'$.

## References


[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.

[2] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.

[3] M.F. Atiyah and R. Bott, *A Lefshetz fixed point formula for elliptic complexes*, Ann. of Math. (2), **86** (1967), 374-407. MR 0212836 (35 #3701).