Lecture Series: Observables in the effective BV-formalism; Talk 1: Effective quantum field theory

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In this talk we describe Costello's mathematical formulation of the low-energy effective field theory approach to perturbative quantum field theory (QFT). Physically, this approach was developed by Kadanoff, Polchinski, Wilson, and others. A key theorem of Costello is a bijection between local functionals on fields and (effective) pertubative QFTs.

The setting for field theory is an action $\mathcal{S}$ which is a function on a space of fields

$$\mathcal{S} : \mathcal{E} \to \mathbb{C}.$$

Classical field theory studies the critical set of the function $\mathcal{S}$. A sample computation in quantum field theory is computing the expectation of an observable, i.e., another function $\mathcal{O} : \mathcal{E} \to \mathbb{C}$. The expectation of $\mathcal{O}$ is given (at least formally) by a functional integral

$$\langle \mathcal{O} \rangle = \int_{\mathcal{E}} \mathcal{O}(\varphi) e^{-\mathcal{S}(\varphi)/\hbar} D\varphi.$$

This integral is often ill-defined, but, in good cases, it has a well defined expansion in the limit $\hbar \to 0$. If $\mathcal{E}$ is finite dimensional and $D\varphi$ is the Lebesgue measure, then this $\hbar \to 0$ limit concentrates on a neighborhood of the critical set of $\mathcal{S}$ and this procedure is the classical stationary phase approximation.

A key element in the definition of (effective) perturbative quantum field theory is renormalization flow (called renormalization group flow in [1] and sometimes exact renormalization group flow in the physics literature).

Let V be a finite dimensional vector space over $\mathbb{R}$ and Φ a non-degenerate negative definite quadratic form Φ. Define $P \in \mathrm{Sym}^2 V$ to be the inverse to $-\Phi$. Let

$$\mathscr{O}(V) \overset{def}{=} \widetilde{\mathrm{Sym}}(V^\vee),$$

so $\mathscr{O}(V)$ is the ring of formal power series in a variable $v \in V$. Denote by $\mathscr{O}^+(V)[[\hbar]] \subset \mathscr{O}(V)[[\hbar]]$ the subspace of functionals which are at least cubic modulo $\hbar$. For a functional $I \in \mathscr{O}(V)[[\hbar]]$, we write

$$I = \sum_{i,k \geq 0} \hbar^i I_{i,k},$$

where $I_{i,k}$ is homogeneous of degree k.

Given a triple (V, P, I) as above, we define the a new functional $W(P, I) \in \mathscr{O}^+(V)[[\hbar]]$ as follows

$$W(P, I) = \sum_\gamma \hbar^{g(\gamma)} \frac{w_\gamma(P, I)}{|\mathrm{Aut}(\gamma)|},$$

where the sum is over connected (stable) graphs γ, and $g(\gamma)$ is the genus of the graph. The *graph weight* $w_\gamma(P, I) \in \mathscr{O}(V)$ is defined by contracting tensors with the components of I placed on the vertices (a vertex of valency k and internal degree i is labeled by $I_{i,k}$) and internal edges are labeled by P. The map

$$W(P,-) : \mathscr{O}^+(V)[[\hbar]] \to \mathscr{O}^+(V)[[\hbar]]$$

is called the *renormalization flow operator*. The diagrammatic expansion appearing in the definition of $W(P,I)$ can also be understand as an asymptotic series in $\hbar$ for an integral on U (assuming we've normalized the measure on U appropriately):

$$W(P,I)(a) = \hbar \log \int_{x \in U} e^{(\Phi(x,x)+2I(x+a))/2\hbar}.$$

The integral appearing above doesn't alway make sense in infinite dimensions, however contraction of tensors does. Therefore, we can still define $W(P,I)$ in the case that V is replaced by a nuclear Fréchet space $\mathcal{E}$ (e.g., $\mathcal{E}$ is the space of sections of a vector bundle E over a manifold M); we work with strong duals and use the completed projective tensor product. In particular, for any $P \in \mathrm{Sym}^2$ we have the renormalization flow operator

$$W(P,-) : \mathscr{O}^+(\mathcal{E})[[\hbar]] \to \mathscr{O}^+(\mathcal{E})[[\hbar]].$$

The Wilsonian yoga is that we have a collection of effective actions $\{S[\Lambda]\}$ and that they are related by renormalization flow.

Let us discuss this paradigm in the setting of scalar field theory on a compact Riemannian manifold M. In this case, our fields are just the smooth functions $C^\infty(M)$. Let D be the (positive) Laplacian on M and $m \in \mathbb{R}_{>0}$, we assume our effective action has the form

$$S[\Lambda](\phi) = -\frac{1}{2}\langle \phi, (D+m^2)\phi \rangle + I[\Lambda](\phi).$$

The functional $I[\Lambda]$ (which is at least cubic modulo $\hbar$) is called the *effective interaction*. In this picture Λ corresponds to "energy" and let $C^\infty(M)_{[\Lambda',\Lambda)}$ denote the span of functions whose eigenvalues lie between Λ' and Λ. The key requirement is that the effective interactions satisfy the *flow equation*:

$$I[\Lambda'](a) = \hbar \log \int_{\phi \in C^\infty(M)_{[\Lambda',\Lambda)}} e^{(-\langle \phi,(D+m^2)\phi \rangle + 2I[\Lambda](\phi+a))/2\hbar}.$$

If we define a cut off kernel $P_{[\Lambda',\Lambda)}$ (we sum only over certain eigenvalues of the operator $(D+m^2)$), then we can rewrite the flow equation as

$$I[\Lambda'](a) = W(P_{[\Lambda',\Lambda)}, I[\Lambda])(a).$$

For a number of reasons, we actually use a smooth cut-off based on the heat kernel. For $l \in \mathbb{R}_{>0}$, let K_l be the kernel for the operator $e^{-l(D+m^2)}$. Our *propagator* with infrared cut-off L and ultraviolet cut-off ϵ ($\epsilon, L \in [0,\infty]$), is given by

$$P(\epsilon, L) = \int_{l=\epsilon}^{L} K_l \, dl.$$

The operator $W(P(\epsilon,L),-)$ implements *renormalization flow from length scale ϵ to length scale L.*

Lastly, we call a functional $I \in \mathscr{O}(C^\infty(M))$ *local* if it is given by an integral of some Lagrangian density.

Definition 1. *A perturbative QFT, with fields $C^\infty(M)$ and kinetic action $-\frac{1}{2}\langle\phi,(D+m^2)\phi\rangle$, is given by a set of effective interactions $I[L] \in \mathscr{O}^+(C^\infty(M))[[\hbar]]$ for all $L \in (0,\infty]$, such that*

(1) *The flow equation is satisfied for all $\epsilon, L \in (0,\infty]$:*
$$I[L] = W(P(\epsilon, L), I[\epsilon]).$$
(2) *For each i,k, $I_{i,k}[L]$ has a small L asymptotic expansion by local functionals.*

There is an extension of this definition to vector-bundle valued theories, i.e., where the space of fields is given by the space of sections of a vector bundle over M.

Theorem 1 (Costello)**.** *Fix a renormalization scheme. There is a bijection between the set of perturbative QFTs and the set of local action functions $I \in \mathscr{O}^+_{loc}(C^\infty(M))[[\hbar]]$.*

A renormalization scheme is a way to extract the singular part of certain functions of one variable; we won't belabor this detail. The proof of the theorem above is constructive. Given a local functional I, we can construct a series of counterterms $I^{\mathrm{CT}}(\epsilon)$ which cancel certain ultraviolet divergences, so that the effective interaction is given by
$$I[L] = \lim_{\epsilon\to 0} W(P(\epsilon, L), I - I^{\mathrm{CT}}(\epsilon)).$$
Conversely, if $I[L]$ is a family of effective interactions, then a certain renormalized limit as $L \to 0$ defines a local functional (the naive limit doesn't exist and certain counter terms must be subtracted).

References

[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.