Lecture Series: Observables in the effective BV-formalism; Talk 1: Effective quantum field theory

RYAN GRADY

In this talk we describe Costello’s mathematical formulation of the low-energy effective field theory approach to perturbative quantum field theory (QFT). Physically, this approach was developed by Kadanoff, Polchinski, Wilson, and others. A key theorem of Costello is a bijection between local functionals on fields and (effective) pertubative QFTs.

The setting for field theory is an action \( S : \mathcal{E} \to \mathbb{C} \). Classical field theory studies the critical set of the function \( S \). A sample computation in quantum field theory is computing the expectation of an observable, i.e., another function \( O : \mathcal{E} \to \mathbb{C} \). The expectation of \( O \) is given (at least formally) by a functional integral

\[
\langle O \rangle = \int_{\mathcal{E}} O(\phi) e^{-S(\phi)/\hbar} D\phi.
\]

This integral is often ill-defined, but, in good cases, it has a well defined expansion in the limit \( \hbar \to 0 \). If \( \mathcal{E} \) is finite dimensional and \( D\phi \) is the Lebesgue measure, then this \( \hbar \to 0 \) limit concentrates on a neighborhood of the critical set of \( S \) and this procedure is the classical stationary phase approximation.

A key element in the definition of (effective) perturbative quantum field theory is renormalization flow (called renormalization group flow in [1] and sometimes exact renormalization group flow in the physics literature).

Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \) and \( \Phi \) a non-degenerate negative definite quadratic form \( \Phi \). Define \( P \in \text{Sym}^2 V \) to be the inverse to \(-\Phi\). Let

\[
\mathcal{O}(V) \overset{def}{=} \text{Sym}(V^\vee),
\]

so \( \mathcal{O}(V) \) is the ring of formal power series in a variable \( v \in V \). Denote by \( \mathcal{O}^+(V)[[\hbar]] \subset \mathcal{O}(V)[[\hbar]] \) the subspace of functionals which are at least cubic modulo \( \hbar \). For a functional \( I \in \mathcal{O}(V)[[\hbar]] \), we write

\[
I = \sum_{i,k \geq 0} \hbar^i I_{i,k},
\]

where \( I_{i,k} \) is homogeneous of degree \( k \).

Given a triple \((V, P, I)\) as above, we define the a new functional \( W(P, I) \in \mathcal{O}^+(V)[[\hbar]] \) as follows

\[
W(P, I) = \sum_{\gamma} \hbar^{g(\gamma)} w_{\gamma}(P, I) \frac{1}{|\text{Aut}(\gamma)|},
\]

where the sum is over connected (stable) graphs \( \gamma \), and \( g(\gamma) \) is the genus of the graph. The graph weight \( w_{\gamma}(P, I) \in \mathcal{O}(V) \) is defined by contracting tensors with
the components of $I$ placed on the vertices (a vertex of valency $k$ and internal degree $i$ is labeled by $I_{i,k}$) and internal edges are labeled by $P$. The map

$$W(P, -) : \mathcal{O}^+(V)[[\hbar]] \to \mathcal{O}^+(V)[[\hbar]]$$

is called the renormalization flow operator. The diagrammatic expansion appearing in the definition of $W(P, I)$ can also be understand as an asymptotic series in $\hbar$ for an integral on $U$ (assuming we’ve normalized the measure on $U$ appropriately):

$$W(P, I)(a) = \hbar \log \int_{x \in U} e^{(\phi(x,x) + 2I(x+a))/2\hbar}.$$

The integral appearing above doesn’t always make sense in infinite dimensions, however contraction of tensors does. Therefore, we can still define $W(P, I)$ in the case that $V$ is replaced by a nuclear Fréchet space $\mathcal{E}$ (e.g., $\mathcal{E}$ is the space of sections of a vector bundle $E$ over a manifold $M$); we work with strong duals and use the completed projective tensor product. In particular, for any $P \in \text{Sym}^2$ we have the renormalization flow operator

$$W(P, -) : \mathcal{O}^+(\mathcal{E})[[\hbar]] \to \mathcal{O}^+(\mathcal{E})[[\hbar]].$$

The Wilsonian yoga is that we have a collection of effective actions $\{S[\Lambda]\}$ and that they are related by renormalization flow.

Let us discuss this paradigm in the setting of scalar field theory on a compact Riemannian manifold $M$. In this case, our fields are just the smooth functions $C^\infty(M)$. Let $D$ be the (positive) Laplacian on $M$ and $m \in \mathbb{R}_{>0}$, we assume our effective action has the form

$$S[\Lambda](\phi) = -\frac{1}{2} \langle \phi, (D + m^2)\phi \rangle + I[\Lambda](\phi).$$

The functional $I[\Lambda]$ (which is at least cubic modulo $\hbar$) is called the effective interaction. In this picture $\Lambda$ corresponds to “energy” and let $C^\infty(M)_{[\Lambda', \Lambda]}$ denote the span of functions whose eigenvalues lie between $\Lambda'$ and $\Lambda$. The key requirement is that the effective interactions satisfy the flow equation:

$$I[\Lambda'](a) = \hbar \log \int_{\phi \in C^\infty(M)_{[\Lambda', \Lambda]}} e^{(-\langle \phi, (D+m^2)\phi \rangle + 2I[\Lambda](\phi+a))/2\hbar}.$$

If we define a cut-off kernel $P_{[\Lambda', \Lambda]}$ (we sum only over certain eigenvalues of the operator $(D + m^2)$), then we can rewrite the flow equation as

$$I[\Lambda'](a) = W(P_{[\Lambda', \Lambda]}, I[\Lambda])(a).$$

For a number of reasons, we actually use a smooth cut-off based on the heat kernel. For $l \in \mathbb{R}_{>0}$, let $K_l$ be the kernel for the operator $e^{-l(D+m^2)}$. Our propagator with infrared cut-off $L$ and ultraviolet cut-off $\epsilon$ ($\epsilon, L \in [0, \infty]$), is given by

$$P(\epsilon, L) = \int_{l=\epsilon}^{L} K_l dl.$$

The operator $W(P(\epsilon, L), -)$ implements renormalization flow from length scale $\epsilon$ to length scale $L$. 

2
Lastly, we call a functional $I \in \mathcal{O}(C^\infty(M))$ local if it is given by an integral of some Lagrangian density.

**Definition 1.** A perturbative QFT, with fields $C^\infty(M)$ and kinetic action $-\frac{1}{2}\langle \phi, (D + m^2)\phi \rangle$, is given by a set of effective interactions $I[L] \in \mathcal{O}^+(C^\infty(M))[[h]]$ for all $L \in (0, \infty]$, such that

1. The flow equation is satisfied for all $\epsilon, L \in (0, \infty]$:
   $$I[L] = W(P(\epsilon, L), I[\epsilon]).$$

2. For each $i, k$, $I_{i,k}[L]$ has a small $L$ asymptotic expansion by local functionals.

There is an extension of this definition to vector-bundle valued theories, i.e., where the space of fields is given by the space of sections of a vector bundle over $M$.

**Theorem 1** (Costello). Fix a renormalization scheme. There is a bijection between the set of perturbative QFTs and the set of local action functions $I \in \mathcal{O}^+_{\text{loc}}(C^\infty(M))[[h]]$.

A renormalization scheme is a way to extract the singular part of certain functions of one variable; we won’t belabor this detail. The proof of the theorem above is constructive. Given a local functional $I$, we can construct a series of counterterms $I^{CT}(\epsilon)$ which cancel certain ultraviolet divergences, so that the effective interaction is given by

$$I[L] = \lim_{\epsilon \to 0} W(P(\epsilon, L), I - I^{CT}(\epsilon)).$$

Conversely, if $I[L]$ is a family of effective interactions, then a certain renormalized limit as $L \to 0$ defines a local functional (the naive limit doesn’t exist and certain counter terms must be subtracted).

**References**