Lecture Series: Observables in the effective BV-formalism; Talk 2: A rapid introduction to the BV-formalism

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The goal of classical field theory is to describe the critical locus of the action functional. The classical BV-formalism is a description of a critical locus of such an action functional in terms of homological algebra.

Suppose V is a finite dimensional vector space and that $S : V \to \mathbb{C}$ is a quadratic function. The critical locus of S is, by definition

$$\mathrm{Crit}(S) := \{v \in V \mid \mathrm{d}S(v) = 0\}$$

The exterior derivative $\mathrm{d}S$ is a linear function on the space V. That is, we can view it as a linear map

$$\mathrm{d}S : V \to V^\vee \quad v \mapsto (w \mapsto \mathrm{d}S_v(w)). \tag{1}$$

The first step is to interpret (1) as a two-term complex with V in degree zero, $V^\vee$ in degree one, and with differential $\mathrm{d}S$. I.e.

$$V \xrightarrow{\mathrm{d}S} V^\vee[-1].$$

The *classical BV-complex* is the space of algebraic functions on the differential graded vector space above. Explicitly

$$\mathcal{O}\left(V \xrightarrow{\mathrm{d}S} V^\vee[-1]\right) = \left(\mathrm{Sym}\left(V^\vee \oplus V[1]\right), Q\right)$$

where Q is the induced differential. This complex satisfies $\mathrm{H}^0 = \mathcal{O}(\mathrm{Crit}(S))$, so it is a derived replacement for the critical locus.

For a more general S (at least quadratic) we can split it up as $S = S^{\mathrm{free}} + I$ where S^{free} *is* quadratic and I is a functional with only cubic or higher terms. The BV-complex is

$$\left(\mathrm{Sym}(V^\vee \oplus V[1]), Q + \{I, -\}\right). \tag{2}$$

Again, one checks that $\mathrm{H}^0 = \mathcal{O}(\mathrm{Crit}(S))$. Note that this complex is equal to functions on the graded vector space $T^*[-1]V = V \oplus V^\vee[-1]$ with some non-trivial differential determined by S. The bracket $\{-,-\}$ of degree -1 comes from the pairing between V and $V^\vee$ and has the structure of a (shifted) Poisson bracket. This bracket is present on the space of polyvector fields on any manifold and is known as the Schouten-Nijenhuis bracket.

We consider a generalization of the above constructions to infinite dimensional vector spaces.

There are two things that we need to be careful of in this more general case:

1. All vector spaces carry a topology. Functionals will mean functions on the vector space that are continuous for this topology.
2. All vector spaces will be spaces of sections of certain sheaves on a manifold. The notion of *locality* discussed in Lecture 1 will be critical for the definition of action functionals of classical field theories.

Example 1. *Let M be a smooth manifold equipped with a Riemannian metric g and consider the space of smooth functions on M, $V = C^\infty(M)$. Define the functional S on $C^\infty(M)$ by*

$$S(\varphi) = \frac{1}{2}\int_M \varphi \mathrm{D}\varphi$$

where D denotes the Laplacian on M times the volume form. I.e. we view it as an operator

$$\mathrm{D} : C^\infty(M) \to \mathrm{Dens}(M) \ \ , \ \ \varphi \mapsto (\Delta_g \varphi)\mathrm{dvol}_g.$$

Clearly, S is a quadratic functional. Note that the functional S is local, i.e. it belongs to the subspace of local functionals $S \in \mathcal{O}_{\mathrm{loc}}(C^\infty(M)) \subset \mathcal{O}(C^\infty(M))$ defined in Lecture 1.

Note that in infinite dimensions, the bracket $\{-,-\}$ is only partially defined: the bracket between arbitrary functionals is not well defined. When at least one of the functionals is local then the bracket does make sense.

We are now ready to make a general definition of a classical field theory in our formalism. Recall some of the structure from above:

(1) We want to study the critical locus of a functional on some (infinite dimensional) vector space of fields.
(2) The fields should exists locally on the manifold in which the field theory is defined. That is, they should form a sheaf. Moreover, classical functionals should respect this locality.
(3) The collection of functions on the space of fields should have a Poisson bracket of degree 1.

With this in mind we have the following definition from [2].

Definition 1. *A free BV-theory on a manifold M consists of the following:*

(1) *A $\mathbb{Z}$-graded vector bundle $\pi : E \to M$ of finite rank;*
(2) *A map*

$$\langle -,- \rangle : E \otimes E \to \mathrm{Dens}_M$$

of degree -1 that is graded antisymmetric and fiberwise nondegenerate.

(3) *A square-zero differential operator $Q : \mathcal{E} \to \mathcal{E}$ of cohomological degree 1 that is skew self-adjoint for $\langle -,- \rangle$.*

We assume that the complex $(\mathcal{E}(M), Q)$ is elliptic.

A general BV-theory is a free BV-theory together with a local functional $I \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ of degree zero that satisfies the classical master equation

$$QI + \frac{1}{2}\{I, I\} = 0.$$

Given this data we can define the analogous BV complex as in (2). We denote

$$\mathrm{Obs}^{cl}_{\mathcal{E}}(M) := (\mathrm{Sym}(\mathcal{E}(M)^\vee), Q + \{I, -\})$$

which we will also refer to as the *global classical observables*. Note that $\{I,-\}$ is well defined as I is local, and that the operator $Q + \{I,-\}$ squares to zero by the classical master equation.

We now turn to the *quantum BV-formalism*: an approach to the path integral in QFT. More precisely, the quantum BV-formalism is a tool to make sense of expectation values of *observables* of a quantum field theory. If $S : \mathcal{E} \to \mathbb{C}$ is the action functional, an *observable* O is a function on $\mathrm{Crit}(S)$. I.e., a measurement of the physical system. It's expectation value is

$$\langle O \rangle := \frac{1}{Z_S} \int_{\varphi \in \mathcal{E}} O(\varphi) e^{-S(\varphi)/\hbar} \mathrm{D}\varphi.$$

Here $e^{-S(\varphi)/\hbar}\mathrm{D}\varphi$ is thought of a probability measure on the space of fields. The normalization Z_S is the *partition function* of the quantum field theory and equals $\langle 1 \rangle$, the expectation of the unit observable. Just as in the classical approach to the BV-formalism, there is a complex that encodes this approach to integration.

We will motivate the definition of the quantum BV-complex by means of a finite dimensional example. Let M be a closed, oriented, smooth, finite-dimensional manifold of dimension n. Let $\mu \in \Omega^n_M$ be a top form, which we think of a probability density on M. Normalize the image of μ in cohomology $[\mu] = \int_M \mu \in \mathrm{H}^n_{dR}(M)$ to be 1. Note that $\mathrm{H}^n_{dR}(M)$ is one-dimensional in our case. Contraction with μ defines an isomorphism of graded vector spaces

$$i_\mu : \mathrm{PV}^{\#}(M) \xrightarrow{\cong} \Omega^{n-\#}(M).$$

which we use to pull-back the de Rham differential to poly-vector fields which we denote div_μ. This operator on poly-vector fields is known as a *divergence* operator. The complex $(\mathrm{PV}^*(M), \mathrm{div}_\mu)$ is the simplest example of a *quantum BV-complex*. The incarnation of integration in the BV-complex is simple:

Proposition 1. *Given a function $f : M \to \mathbb{R}$, the cohomology class $[f]_{\mathrm{BV}}$ in $\mathrm{H}^0(\mathrm{PV}^*(M), \mathrm{div}_\mu)$ satisfies*

$$[f]_{\mathrm{BV}} = \langle f \rangle_\mu [1]_{\mathrm{BV}}.$$

The goal is to equip the BV-complex for a general field theory $(\mathcal{E}, Q, \langle -,- \rangle, I)$ on M

$$(\mathcal{O}(\mathcal{E}), Q + \{I, -\}) = (\mathrm{Sym}(\mathcal{E}^\vee), Q + \{I, -\})$$

with a type of divergence operator that encodes integration. This is the *BV-Laplacian*. In the case of a general field theory the naive definition of the BV-laplacian above is ill-posed. The central idea in [2] is to use the effective approach formulated in [1] to come up with a regularized version of quantum BV-complex.

References

[1] K. Costello, *Renormalization and effective field theory*, AMS, 2011.

[2] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory, Volume I & II*, Cambridge University Press (submitted), 2015.