Joint Northeastern–MIT Graduate Research Seminar  
Fall 2018  
The Yangian and four–dimensional gauge theory

The Yangian $Y_{\mathfrak{g}}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is a Hopf algebra which deforms the enveloping algebra of the current algebra $\mathfrak{g}[z]$ of $\mathfrak{g}$–valued polynomials. It was introduced by Drinfeld in the mid–80s as one of the algebraic structures underpinning the study of integrable 1 and 2–dimensional models in Statistical Mechanics [Dri85].

The purpose of this course is to explore a far more recent connection between the Yangian and a certain class of 4–dimensional gauge theories. Mathematically, a gauge theory involves studying connections on bundles over a given smooth manifold. These theories are motivated from physics, but often have neat and concise mathematical descriptions.

The gauge theory we study is defined on manifolds of the form $\Sigma \times S$ where $\Sigma$ is a Riemann surface and $S$ is a real two-dimensional manifold. The class of connections defining the gauge theory are, in a precise sense, holomorphic in the direction of $\Sigma$ and flat in the direction of $S$. The connection between gauge theory and quantum groups we will study is similar in spirit to perhaps a more well-studied relationship between Chern-Simons theory, the study of flat connections on three-manifolds, and quantum groups.

We follow the seminal work of Costello [Cosa, Cosb] showing how Yangians arise from the algebra of operators of the four-dimensional gauge theory. The key to this result relies on the formalism of Costello-Gwilliam [CG17, CG] that the algebra of operators of a quantum field theory form a factorization algebra. This is a vast generalization of the description of algebras of operators in conformal field theory as vertex algebras. Factorization algebras simultaneously generalize the notion of a vertex algebra and algebras over more familiar operads, such as the operad of little disks. The primary goal of this seminar is to study Costello’s construction which starts from the factorization algebra description of the operators of a four dimensional gauge theory and recovers the Yangian quantum group of the gauge Lie algebra. Time permitting, we can focus on more concrete and computational sides of the program began in the works [CWYa, CWYb].

Some keywords: Factorization algebras, Koszul duality, quantum groups, Yangians, complex geometry.

Informal prerequisites: The seminar will be aimed at mathematicians. In particular, no knowledge of physics will be assumed. Knowledge of the following topics will be assumed: basic category theory, homological algebra, rudiments of Lie algebras, and some basic differential geometry including vector bundles, connections, and differential forms.

Organizers: Chris Beasley, Valerio Toledano Laredo, Brian Williams (Northeastern) and Pavel Etingof (MIT).

Time/Place The weekly seminar will take place on Tuesday afternoon, staring on September 11, and alternate between Northeastern (4:30–7:30PM beginning on Sept. 11, room TBA) and MIT (4:10–7PM, room 2–139).
Detailed (approximate) syllabus

Week 1, Sep 11  Introduction and overview of the seminar. Speaker: Brian Williams.

Week 2, Sep 18  An introduction to factorization algebras. Speaker: Ryan Mickler.

The definition of a factorization algebra with values in a symmetric monoidal category. Lurie’s result that locally constant factorization algebras on $\mathbb{R}^n$ are equivalent to $E_n$-algebras. Hochschild homology as a special case of factorization homology. [CG17, Lur, AF15].

Week 3, Sep 25  Koszul duality for $E_n$-algebras. Categories of (co)modules for $E_n$-algebras. Koszul duality for associative algebras and its generalization for augmented $E_n$-algebras (with special attention to the case $n = 2$). Outline result of Tamarkin that the Koszul dual of an $E_2$-algebra is a Hopf algebra. Interplay between Hochschild homology and Koszul duality. [Cosb, Tam03].

Week 4, Oct 2  “Four-dimensional Chern-Simons theory”. The moduli space of holomorphic, partially flat, connections on a complex surface. The moduli space of multiplicative Higgs bundles on a Riemann surface.


Week 6, Oct 16  Batalin-Vilkovisky quantization and renormalization (cont.).

Week 7, Oct 23  Observables of the four-dimensional gauge theory. Deformation of functions on the classical moduli space defined by BV quantization to small orders in $\hbar$. Line operators and a generalized version of “conformal blocks” from CFT. [Cosb11, CG, Cosb]

Week 8, Oct 30  The Yangian and Costello’s main result.

Theorem 0.1. The Koszul dual of the $E_2$-algebra of quantum observables of four-dimensional Chern-Simons theory on $\mathbb{C}_z \times \mathbb{R}^2_{\theta}$ (restricted to a factorization algebra on $\{z = 0\} \times \mathbb{R}^2_{\theta}$) is Koszul dual to the Yangian.

[Dri85, Cosb11, CG, Cosb, CWYa, CWYb]

Week 9, Nov 6  Quantum groups and the Yangian. Drinfeld’s universal $R$-matrix of $Yg$. Relation to integrable systems and lattice models [Dri85, ES02, CP94].

Week 10, Nov 13  Holomorphic factorization and the universal $R$-matrix Background on vertex algebras. Describe functor from the category of holomorphic factorization algebras to vertex algebras. Relationship to chiral algebras of Beilinson-Drinfeld. [CG17, BD04]. The quantum OPE as a map of $E_2$-algebras. Hochschild homology for categories [Cosb].

Week 11, Nov 27  The quantum Yang-Baxter equation How the quantum master equation for BV quantization implies the quantum Yang-Baxter equation. [CWYa, CWYb].

Week 12, Dec 4  Enhancements and variations of the construction. Realizing various spin systems by tweaking the input data. Coupling to surface operators.

Week 13, Dec 11  TBD.
References


