TOPOLOGICAL CALCULATION OF THE PHASE OF THE DETERMINANT OF A NON SELF-ADJOINT ELLIPTIC OPERATOR

ALEXANDER G. ABANOV AND MAXIM BRAVERMAN

Abstract. We study the zeta-regularized determinant of a non self-adjoint elliptic operator on a closed odd-dimensional manifold. We show that, if the spectrum of the operator is symmetric with respect to the imaginary axis, then the determinant is real and its sign is determined by the parity of the number of the eigenvalues of the operator, which lie on the positive part of the imaginary axis. It follows that, for many geometrically defined operators, the phase of the determinant is a topological invariant. In numerous examples, coming from geometry and physics, we calculate the phase of the determinants in purely topological terms. Some of those examples were known in physical literature, but no mathematically rigorous proofs and no general theory were available until now.

Contents

1. Introduction

In the recent years several examples appeared in physical literature when the phase of the determinant of a geometrically defined non self-adjoint Dirac-type operator is a topological invariant (see e.g., [11, 12, 2, 1]). Many of those examples appear in the study of the non-linear σ-model for Dirac fermions coupled to chiral bosonic fields [2, 1]. The topologically invariant phase is called the *θ-term*. It has a dramatic effect on the dynamics of the Goldstone bosons but also has a great interest for geometers. Unfortunately, no mathematically rigorous proofs of the topological invariance of the phase of the determinant were available until now.

This paper is an attempt to better understand the above phenomenon. In particular, we find a large class of operators whose determinants have a topologically invariant phase. We also

The first author was partially supported by the Alfred P. Sloan foundation.

The second author was partially supported by the NSF grant DMS-0204421.

develop a technique for calculation of this phase. In particular, we get a first mathematically rigorous derivation of several examples which appeared in physical literature. In many cases, we also improve and generalize those examples.

Our first result is Theorem 3.2 which states that *the determinant of an elliptic operator D with a self-adjoint leading symbol, which acts on an odd-dimensional manifold and whose spectrum is symmetric with respect to the imaginary axis, is real. Moreover, the sign of this determinant is equal to $(-1)^{m_+}$, where m_+ is the number of the eigenvalues of D (counted with multiplicities) which lie on the positive part of the imaginary axis.*

Note that this result is somewhat surprising. Indeed, if one calculates the determinant of a *finite* matrix D with the spectrum symmetric with respect to an imaginary axis, then one comes to a different result. E.g., the determinant is not necessarily real.

Suppose now that we are given a family $D(t)$ of operators as above. Assuming that the eigenvalues of $D(t)$ depend continuously on t one easily concludes (cf. Theorem 3.6) that *the sign of the determinant of $D(t)$ is independent of t.* In particular, it follows that, if the definition of the operator D depends on some geometric data (Riemannian metric on a manifold, Hermitian metric on a vector bundle, etc.), then (provided the spectrum of D is symmetric) *the sign of the determinant is independent of these data, i.e., is a topological invariant.* We present numerous examples of this phenomenon. In all those examples we calculate the signs of the determinants in terms of the standard topological invariants, such as the Betti numbers or the degree of a map.

The paper is organized as follows:

In Section 2, we briefly recall the basic facts about the ζ-regularized determinants of elliptic operators.

In Section 3, we formulate and prove our main result (Theorem 3.2) and discuss its main implications.

In Section 4, we present the simplest (but still interesting) geometric examples of applications of Theorem 3.2.

In Section 5, we consider an operator D on a circle, which appeared in the study of a quantum spin in the presence of a planar, time-dependent magnetic field. This operator depends on a map from a circle to itself. We calculate the phase of the determinant of D in terms of the winding number of this map.

In Section 6, we extend some of the examples considered by P. Wiegmann and the first author in [2]. The operator in question is a Dirac type operator D on an odd dimensional manifold M, whose potential depends on a section n of the bundle of spheres in $\mathbb{R}\oplus TM$. In particular, if the manifold M is paralelizable, n is a map from M to a $\dim M$-dimensional sphere. We show that the sign of the determinant of D is equal to $(-1)^{\deg n}$, where $\deg n$ is the topological degree of n.

Acknowledgments. The first author would like to thank the Theory Institute of Strongly Correlated and Complex Systems at Brookhaven for hospitality and support.

The second author would like to thank the Max-Planck-Institut für Mathematik, where most of this work was completed, for hospitality and providing the excellent working conditions. He would also like to thank Mikhail Shubin and Raphael Ponge for valuable discussions.

2. Preliminaries on determinants of elliptic operators

Let E be a vector bundle over a smooth compact manifold M and let $D : C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order $m \geq 1$. Let $\sigma_L(D)$ denote the leading symbol of D.

2.1. **The choice of an angle.** Our aim is to define the ζ-function and the determinant of D. For this we will need to define the complex powers of D. As usual, to define complex powers we need to choose a *spectral cut* in the complex plane. We will restrict ourselves to the simplest spectral cuts given by a ray

$$R_\theta \;=\; \big\{\, \rho e^{i\theta} : 0 \leq \rho < \infty \,\big\}, \qquad 0 \leq \theta \leq 2\pi. \tag{2.1}$$

Consequently, we have to choose an angle $\theta \in [0, 2\pi)$.

Definition 2.2. *The angle θ is a* principal angle *for an elliptic operator D if*

$$\operatorname{spec}\big(\,\sigma_L(D)(x,\xi)\,\big) \;\cap\; R_\theta \;=\; \emptyset, \qquad \textit{for all} \quad x \in M,\ \xi \in T^*_xM\backslash\{0\}.$$

If $\mathcal{I} \subset \mathbb{R}$ we denote by $L_\mathcal{I}$ the solid angle

$$L_\mathcal{I} \;=\; \big\{\, \rho e^{i\theta} : 0 < \rho < \infty,\, \theta \in \mathcal{I} \,\big\}.$$

Definition 2.3. *The angle θ is an* Agmon angle *for an elliptic operator D if it is principal angle for D and there exists $\varepsilon > 0$ such that*

$$\operatorname{spec}(D) \cap L_{[\theta-\varepsilon,\theta+\varepsilon]} \;=\; \emptyset.$$

2.4. **The ζ-function and the determinant.** Let θ be an Agmon angle for D. Assume, in addition, that D is injective. The ζ-function $\zeta_\theta(s,D)$ of D is defined as follows.

Let $\rho_0 > 0$ be a small number such that

$$\operatorname{spec}(D) \cap \big\{\, z \in \mathbb{C};\, |z| < 2\rho_0 \,\big\} \;=\; \emptyset.$$

Define the contour $\Gamma = \Gamma_{\theta,\rho_0} \subset \mathbb{C}$ consisting of three curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 \;=\; \big\{\, \rho e^{i\theta} : \rho_0 \leq \rho < \infty \,\big\}, \quad \Gamma_2 \;=\; \big\{\, \rho_0 e^{i\alpha} : \theta < \alpha < \theta + 2\pi \,\big\}, \\ \Gamma_3 \;=\; \big\{\, \rho e^{i(\theta+2\pi)} : \rho_0 \leq \rho < \infty \,\big\}. \end{aligned} \tag{2.2}$$

For $\operatorname{Re} s > \frac{\dim M}{m}$, the operator

$$D_\theta^{-s} \;=\; \frac{i}{2\pi} \int_{\Gamma_{\theta,\rho_0}} \lambda^{-s}(D-\lambda)^{-1}\, d\lambda \tag{2.3}$$

is a pseudo-differential operator with smooth kernel $D_\theta^{-s}(x,y)$, cf. [14, 15].

We define

$$\zeta_\theta(s,D) \ = \ \operatorname{Tr} D_\theta^{-s} \ = \ \int_M \operatorname{tr} D_\theta^{-s}(x,x)\,dx, \qquad \operatorname{Re} s > \frac{\dim M}{m}. \tag{2.4}$$

It was shown by Seeley [14] (see also [15]) that $\zeta_\theta(s,D)$ has a meromorphic extension to the whole complex plane and that 0 is a regular value of $\zeta_\theta(s,D)$.

More generally, let Q be a pseudo-differential operator of order q. We set

$$\zeta_\theta(s,Q,D) \ = \ \operatorname{Tr} Q\, D_\theta^{-s}, \qquad \operatorname{Re} s > (q+\dim M)/m. \tag{2.5}$$

If Q is a projection, i.e., $Q^2=Q$ then [18, §6], [19] (see also [10] for a shorter proof), the function $\zeta_\theta(s,D;Q)$ also has a meromorphic extension to the whole complex plane which is regular at 0.

Finally, we define the ζ-regularized determinant of D by the formula

$$\operatorname{Det}_\theta(D) \ = \ \exp\Big(-\frac{d}{ds}\big|_{s=0}\zeta_\theta(s,D)\Big). \tag{2.6}$$

2.5. **The case of an operator close to self-adjoint.** Let us assume now that

$$\sigma_L(D)^* \ = \ \sigma_L(D), \tag{2.7}$$

where $\sigma_L(D)^*$ denotes the dual of $\sigma_L(D)$ with respect to some fixed scalar product on the fibers on E. This assumption implies that D can be written as a sum $D=D'+A$ where D' is self-adjoint and A is a differential operator of a smaller order. In this situation we say that D is *an operator close to self-adjoint*, cf. [3, §6.2], [9, §I.10].

Though the operator D is not self-adjoint in general, the assumption (2.7) guarantees that it has nice spectral properties. More precisely, cf. [9, §I.6], the space $L^2(M,E)$ of square integrable sections of E can be decomposed into the sum of finite dimensional D-invariant subspaces

$$L^2(M,E) \ = \ \bigoplus \Lambda_k \tag{2.8}$$

such that the restriction of D to Λ_k has a unique eigenvalue λ_k and $\lim_{k\to\infty}|\lambda_k|=\infty$. The sum (2.8) is not a sum of the orthogonal subspaces. We refer to [9, Ch. I] for the precise meaning of the sign "$\bigoplus$" in (2.8).

The space Λ_k are called the *space of root vectors of D with eigenvalue λ_k*. We call the dimension of the space Λ_k the *multiplicity* of the eigenvalue λ_k and we denote it by m_k.

By Lidskii's theorem [8], [13, Ch. XI], the ζ-function (2.4) is equal to the sum (including the multiplicities) of the eigenvalues of D_θ^{-1}. Hence,

$$\zeta_\theta(s,D) \ = \ \sum_{k=1}^\infty m_k\lambda_k^{-s} \ = \ \sum_{k=1}^\infty m_k e^{-s\log_\theta\lambda_k}, \tag{2.9}$$

where $\log_\theta(\lambda_k)$ denotes the branch of the logarithm in $\mathbb{C}\backslash R_\theta$ which take the real values on the positive real axis.

2.6. **Dependence of the determinant on the angle.** Assume now that θ is only a principal angle for D. Then, cf. [14, 15], there exists $\varepsilon>0$ such that $\operatorname{spec}(D)\cap L_{[\theta-\varepsilon,\theta+\varepsilon]}$ is finite and $\operatorname{spec}(\sigma_L(D))\cap L_{[\theta-\varepsilon,\theta+\varepsilon]}=\emptyset$. Thus we can choose an Agmon angle $\theta'\in(\theta-\varepsilon,\theta+\varepsilon)$ for D. In this subsection we show that $\operatorname{Det}_{\theta'}(D)$ is independent of the choice of this angle θ'. For simplicity, we will restrict ourselves with the case when D is an operator close to self-adjoint, cf. Subsection 2.5.

Let $\theta''>\theta'$ be another Agmon angle for D in $(\theta-\varepsilon,\theta+\varepsilon)$. Then there are only finitely many eigenvalues $\lambda_{r_1},\dots,\lambda_{r_k}$ of D in the solid angle $L_{[\theta',\theta'']}$. We have

$$\log_{\theta''}\lambda_k \;=\; \begin{cases}\log_{\theta'}\lambda_k, & \text{if } \ k\notin\{r_1,\dots,r_k\};\\ \log_{\theta'}\lambda_k+2\pi i, & \text{if } \ k\in\{r_1,\dots,r_k\}.\end{cases} \tag{2.10}$$

Hence,

$$\zeta'_{\theta'}(0,D)\;-\;\zeta'_{\theta''}(0,D)\;=\;\frac{d}{ds}\Big|_{s=0}\sum_{i=1}^{k} m_k\, e^{-s\log_{\theta'}(\lambda_{r_i})}(1-e^{-2\pi i s})\;=\;2\pi i\sum_{i=1}^{k} m_{r_i}. \tag{2.11}$$

and

$$\operatorname{Det}_{\theta''}D\;=\;\operatorname{Det}_{\theta'}D. \tag{2.12}$$

Note that the equality (2.12) holds only because both angels θ' and θ'' are close to a given principal angle θ so that the intersection $\operatorname{spec}(D)\cap L_{[\theta',\theta'']}$ is finite. If there are infinitely many eigenvalues of D in the solid angle $L_{[\theta',\theta'']}$ then $\operatorname{Det}_{\theta''}(D)$ and $\operatorname{Det}_{\theta'}(D)$ might be quite different.

3. Operators whose spectrum is symmetric with respect to the imaginary axis

In this section M is an *odd-dimensional* closed manifold, $E\to M$ is a complex vector bundle over M, and D is a differential operator of order $m\geq 1$ which is close to self-adjoint (cf. Subsection 2.5) and invertible.

3.1. **The phase of the determinant and the imaginary eigenvalues.** Suppose that *the spectrum of D is symmetric with respect to the imaginary axis.* More precisely, we assume that, if $\lambda=\rho e^{i\alpha}$ is an eigenvalue of D with multiplicity m, then $\rho e^{i(\pi-\alpha)}$ is also an eigenvalue of D with the same multiplicity. Since the leading symbol of D is self-adjoint, $\pm\frac{\pi}{2}$ are principal angles of D, cf. Definition 2.2. Hence, cf. Subsection 2.6, we can choose an Agmon angle $\theta\in(\frac{\pi}{2},\pi)$ such that there are no eigenvalues of D in the solid angles $L_{(\pi/2,\theta]}$ and $L_{(-\pi/2,\theta-\pi]}$.

Let m_+ denote the number of eigenvalues of D (counted with multiplicities) on the positive part of the imaginary axis, i.e., on the ray $R_{\pi/2}$ (cf. (2.1)).

Our first result is the following

Theorem 3.2. *In the situation described above*

$$\operatorname{Im}\zeta'_\theta(0,D)\;=\;-\pi\, m_+. \tag{3.1}$$

In particular, $\operatorname{Det}_\theta D=\exp\big(-\zeta'_\theta(0,D)\big)$ *is a real number, whose sign is equal to* $(-1)^{m_+}$*.*

Remark 3.3. a. For (3.1) to hold we need the precise assumption on θ which we specified above. However, if we are only interested in the sign of the determinant of D, the result remains true for all $\theta\in(-\pi,\pi)$. This follows from (2.12).

b. Note that only the eigenvalues on the positive part of the imaginary axis contribute to the sign of the determinant. This asymmetry between the positive and the negative part of the imaginary axis is coursed by our choice of the spectral cut R_θ in the upper half plane. If we have chosen the spectral cut in the lower half plane the sign of the determinant would be determined by the eigenvalues on the negative imaginary axis.

Proof. Let

$$\rho_je^{i\alpha_j}, \qquad \theta-\pi<\alpha_j<\frac{\pi}{2}, \quad j=1,2,\dots$$

be all the eigenvalues of D which lie in the solid angle $L_{(\theta-\pi,\pi/2)}$ (here and below all the eigenvalues appear in the list the number of times equal to their multiplicities). Since the spectrum of D is symmetric with respect to the imaginary axis, $\rho_je^{i(\pi-\alpha_j)}$ $(j=1,2,\dots)$ are all the eigenvalues of D in the solid angle $L_{(\theta,3\pi/2)}$.

Finally, let

$$\mu_1^+\,e^{i\frac{\pi}{2}},\dots,\mu_{m_+}^+\,e^{i\frac{\pi}{2}}, \qquad \mu_1^-\,e^{-i\frac{\pi}{2}},\dots,\mu_{m_-}^-\,e^{-i\frac{\pi}{2}}$$

be all the imaginary eigenvalues of D (since $\pm\frac{\pi}{2}$ are principal angles for D, there are only finitely many of those, cf. Subsection 2.6). Then

$$\begin{aligned}\zeta_\theta(s,D) \;&=\; \sum_{j=1}^{\infty}\rho_j^{-s}\left(e^{-i\alpha_js}+e^{i(\alpha_j+\pi)s}\right) \;+\; \sum_{j=1}^{m_+}(\mu_j^+)^{-s}\,e^{-i\frac{\pi}{2}s} \;+\; \sum_{j=1}^{m_-}(\mu_j^-)^{-s}\,e^{i\frac{\pi}{2}s}\\ &=\; 2\Big[\sum_{j=1}^{\infty}\rho_j^{-s}\,\cos{(\alpha_j+\frac{\pi}{2})s}\Big]\,e^{i\frac{\pi}{2}s} \;+\; \sum_{j=1}^{m_+}(\mu_j^+)^{-s}\,e^{-i\frac{\pi}{2}s} \;+\; \sum_{j=1}^{m_-}(\mu_j^-)^{-s}\,e^{i\frac{\pi}{2}s}.\end{aligned}$$

Set

$$z(s) \;:=\; 2\sum_{j=1}^{\infty}\rho_j^{-s}\,\cos{(\alpha_j+\frac{\pi}{2})s}.$$

Then

$$\zeta_\theta(s,D) \;=\; z(s)\,e^{i\frac{\pi}{2}s} \;+\; \sum_{j=1}^{m_+}(\mu_j^+)^{-s}\,e^{-i\frac{\pi}{2}s} \;+\; \sum_{j=1}^{m_-}(\mu_j^-)^{-s}\,e^{i\frac{\pi}{2}s},$$

and

$$\zeta_\theta'(0,D) \;=\; z'(0) \;+\; i\frac{\pi}{2}\,z(0) \;+\; \sum\log\mu_j^\pm \;+\; i\frac{\pi}{2}\left(m_--m_+\right).$$

Note that $z(s)$ and $z'(s)$ are real for $s\in\mathbb{R}$. Hence, we obtain

$$\operatorname{Im}\,\zeta_\theta'(0,D) \;=\; \frac{\pi}{2}\left(z(0)+m_--m_+\right). \tag{3.2}$$

We will now calculate $z(0)$ by comparing it with the ζ-function of the operator D^2. The angle 2θ is a principal angle for D^2 and

$$\begin{aligned}\zeta_{2\theta}(s/2,D^2) \;&=\; \sum \rho_j^{-s}\left(e^{-i\alpha_j s}+e^{i\alpha_j s}\right) \;+\; \sum (\mu_j^{\pm})^{-s}\, e^{-i\frac{\pi}{2}s}\\ &=\; 2\sum \rho_j^{-s}\,\cos(\alpha_j s) \;+\; \sum (\mu_j^{\pm})^{-s}\, e^{-i\frac{\pi}{2}s}.\end{aligned}$$

Hence,

$$\zeta_{2\theta}(s/2,D^2) \;-\; z(s) \;=\; 4\Big[\sum \rho_j^{-s}\,\sin(\alpha_j+\frac{\pi}{4})s\Big]\,\sin\frac{\pi}{4}s \;+\; \sum (\mu_j^{\pm})^{-s}\, e^{-i\frac{\pi}{2}s}. \tag{3.3}$$

Let $\Pi_{(-\pi/2,\pi/2)},\,\Pi_{(\pi/2,3\pi/2)} : L^2(M,E)\to L^2(M,E)$ be the orthogonal projections onto the spans of the eigensections of D corresponding to the eigenvalues in $L_{(-\pi/2,\pi/2)}$ and in $L_{(\pi/2,3\pi/2)}$ respectively. Then, using the notation introduces in (2.5), we obtain

$$\begin{aligned}&\zeta_\theta(s,\Pi_{(-\pi/2,\pi/2)},D)\,e^{-i\frac{\pi}{4}s} \;-\; \zeta_\theta(s,\Pi_{(\pi/2,3\pi/2)},D)\,e^{-i\frac{3\pi}{4}s}\\ &\qquad=\; \Big[\sum \rho_j^{-s}\,e^{-i\alpha_j s}\Big]\,e^{-i\frac{\pi}{4}s} \;-\; \Big[\sum \rho_j^{-s}\,e^{i(\alpha_j+\pi)s}\Big]\,e^{-i\frac{3\pi}{4}s}\\ &\qquad=\; \sum \rho_j^{-s}\,\Big(e^{-i(\alpha_j+\frac{\pi}{4})}-e^{i(\alpha_j+\frac{\pi}{4})}\Big) \;=\; -2i\sum \rho_j^{-s}\,\sin(\alpha_j+\frac{\pi}{4})s.\end{aligned}$$

Hence, cf. the discussion in the end of Subsection 2.4, the function $\sum \rho_j^{-s}\,\sin(\alpha_j+\frac{\pi}{4})s$ has a meromorphic extension to the whole complex plane, which is regular at 0. Thus, the first term in the RHS of (3.3) vanishes when $s=0$. The equality (3.3) implies now that

$$\zeta_{2\theta}(0,D^2) \;-\; z(0) \;=\; m_+ + m_-.$$

It is well known, cf. [14], that the ζ-function of a differential operator of even order on an odd-dimensional manifold vanishes at 0. In particular, $\zeta_{2\theta}(0,D^2)=0$. Thus,

$$z(0) \;=\; -\big(m_+ + m_-\big).$$

Substituting this equality into (3.2), we obtain (3.1). □

Remark 3.4. Note that the result of Theorem 3.2 is somewhat surprising. Indeed, if one thinks about $\operatorname{Det}_\theta D$ as a formal product of the eigenvalues of D, then one can do the following formal computation (where we shall use the notation introduces in the proof of Theorem 3.2): for each $j=1,2,\dots$ the product of the eigenvalues $\rho_j e^{i\alpha_j}$ and $\rho_j e^{i(\pi-\alpha_j)}$ is a real number. Hence, one expects $\operatorname{Det}_\theta D = \pm\big|\operatorname{Det}_\theta D\big| e^{i\frac{\pi}{2}(m_+-m_-)}$, which is quite different from the correct answer given by Theorem 3.2.

This example illustrates the danger of formal manipulations with determinants[1].

[1] The fact that formal computations often lead to wrong answers is well known. In particular, $\operatorname{Det}_\theta D$ might not be real even if $D=D^*$ so that all the eigenvalues of D are real, cf., for example, [20].

3.5. **Stability of the phase of the determinant.** Suppose now that $D(t)$ is a family of close to self-adjoint operators, depending on a real parameter t. We will say that *the spectrum of $D(t)$ depends continuously on t* if, for each t, we have a decomposition

$$L^2(M,E) \;=\; \bigoplus \Lambda_k(t),$$

(cf. (2.8)) of $L^2(M,E)$ into a sum of $D(t)$-invariant finite dimensional subspaces, such that

- $\dim \Lambda_k(t)$ is independent of t;
- the restriction of $D(t)$ to Λ_k has a unique eigenvalue $\lambda_k(t)$ and $\lim_{k\to\infty}|\lambda_k(t)| = 0$;
- for every $k = 1,2,\dots$, the function $\lambda_k(t)$ is continuous in t.

Theorem 3.6. *Let now $D(t)$ be a family of operators depending on a real parameter t. We assume that for each $t \in \mathbb{R}$ the operator $D(t)$ satisfies all the assumptions of Theorem 3.2. In particular, its spectrum is symmetric with respect to the imaginary axis. Assume, in addition, that the eigenvalues of $D(t)$ depend continuously on t. For each t let us choose an Agmon angle $\theta(t) \in (\pi/2,\pi)$. Then $\mathrm{Det}_{\theta(t)} D(t)$ is real and its sign of is independent of t.*

Proof. By our assumptions, the eigenvalues of $D(t)$ are symmetric with respect to the imaginary axis and never pass through zero. It follows that when one of the eigenvalues reaches $R_{\pi/2}$ from the left the other must reach it from the right. In other words, the parity of the number of the eigenvalues on the imaginary axis is independent of t. The theorem follows now from Theorem 3.2 and Remark 3.3. □

3.7. **Topological invariance of the phase of the determinant.** The eigenvalues always depend continuously on t if $D(t) = D_0 + tB$, where D_0 is an elliptic differential operator of order $m \geq 1$ and B is a differential operator whose order is less than m, cf. [7]. They also often depend continuously on t when we have a smooth family of geometric structures (i.e, Riemannian metrics on a manifold, Hermitian metrics on a vector bundle, etc.) and $D(t)$ is a family of geometrically defined operators (Dirac operators, Laplacians, etc.) depending of these geometric structures. Suppose, in addition, the spectrum of $D(t)$ is symmetric with respect to the imaginary axis. Then, in view of Theorem 3.6, it is natural to expect that the phase of the determinant is *a topological invariant.* A natural question is how to relate this invariant to the other topological invariants. In other words, we would like to find a topological method of calculating the phase of the determinant of geometric operators, whose spectrum is symmetric with respect to the imaginary axis. In the rest of the paper we present numerous examples in which such a calculation is indeed possible.

4. First examples

In this section we present some simple examples of applications of Theorem 3.2. More sophisticated examples will be considered in the subsequent sections.

4.1. **The circle.** The simplest possible example of an operator satisfying the conditions of Theorem 3.2 is the operator

$$D_a \;=\; -i\,\frac{d}{dt}\;+\;ia, \qquad a\in\mathbb{R},$$

acting on the space of function on the circle S^1. Clearly, the only imaginary eigenvalue of D_a is ia. Hence, Theorem 3.2 implies that, for $\theta\in(0,\pi)$, we have

$$\mathrm{Det}_\theta\, D_a \;<\; 0, \;\;\text{if}\;\; a>0, \quad \text{and} \quad \mathrm{Det}_\theta\, D_a \;>\; 0 \text{ if } \; a<0. \tag{4.1}$$

Remark 4.2. The determinant of the operator D_a can be calculated explicitly. In fact [6] provides a formula for this determinant in terms of the monodromy operator associated to D_a. Using this formula, one easily gets

$$\mathrm{Det}_\theta\, D_a \;=\; e^{-a\beta}-1,$$

where β is the length of the circle. Clearly, that agrees with (4.1).

4.3. **The deformed DeRham-Dirac operator.** Suppose M is a closed manifold of odd dimension $N=2l+1$. Let $d:\Omega^*(M)\to\Omega^{*+1}(M)$ be the DeRham differential and let $d^*:\Omega^*(M)\to\Omega^{*-1}(M)$ be the adjoint of d with respect to a fixed Riemannian metric on M. Let $\beta_j(M)=\dim H^j(M)$ $(j=0,\dots,N)$ denote the Betti numbers of M.

The operator

$$D_a \;:=\; d\;+\;d^*\;+\;ia, \qquad a\in\mathbb{R},\;\; a\neq 0.$$

has exactly one imaginary eigenvalue $\lambda=ia$ and its multiplicity is equal to the sum $\sum_{j=0}^N\beta_j(M)$ of the Betti numbers of M. Because of the Poincaré duality, this sum is an even number. Thus Theorem 3.2 implies that

$$\mathrm{Det}_\theta\; D_a \;>\; 0, \qquad \text{for all} \quad a\in\mathbb{R},\; 0<\theta<\pi.$$

To construct a more interesting example let us fix non-zero real numbers $a_0,\dots,a_N$ and consider the operator $A:\Omega^*(M)\to\Omega^*(M)$ defined by the formula

$$A:\,\omega\;\mapsto a_j\,\omega, \qquad \text{if} \quad \omega\in\Omega^j(M).$$

Then one easily concludes from Theorem 3.2 that

$$\mathrm{Det}_\theta\,\big(\,d+d^*+iA\,\big) \;=\; (-1)^{\sum_{\{j:a_j>0\}}\beta_j(M)}\,\big|\,\mathrm{Det}_\theta\,\big(\,d+d^*+iA\,\big)\,\big|.$$

Another interesting example can be constructed as follows. Let $*:\Omega^*(M)\to\Omega^{N-*}(M)$ denote the Hodge-star operator. Consider the operator $\Gamma:\Omega^*(M)\to\Omega^{N-*}(M)$, defined by the formula

$$\Gamma:\,\alpha\;\mapsto\; i^{\frac{N(N+1)}{2}}\,(-1)^{\frac{j(j+1)}{2}}\,*\alpha \;=\; i^{l+1}\,(-1)^{\frac{j(j+1)}{2}}\,*\alpha, \qquad \alpha\in\Omega^j(M). \tag{4.2}$$

Since $N=2l+1$ is odd, Γ is self-adjoint, satisfies $\Gamma^2=1$, and commutes with $d+d^*$. In particular, Γ acts on $\mathrm{Ker}(d+d^*)$ and this action has exactly 2 eigenvalues ±1, which have equal multiplicities $\frac12\sum_{j=0}^N\beta_j(M)$. Hence, the operator

$$D_\Gamma \;:=\; d\;+\;d^*\;+\;i\,\Gamma,$$

has exactly 2 imaginary eigenvalues $\pm i$, and multiplicities of these eigenvalues are equal to $\frac{1}{2}\sum_{j=0}^{N}\beta_j(M)$. Theorem 3.2 implies now that

$$\mathrm{Det}_\theta\ D_\Gamma \ = \ (-1)^{\frac{1}{2}\sum_{j=0}^{N}\beta_j(M)}\,\big|\ \mathrm{Det}_\theta\ D_\Gamma\ \big|.$$

Remark 4.4. All the results of this subsection can be easily extended to operators acting on the space of differential forms with values in a flat vector bundle $F\to M$. (The DeRham differential should be replaced by the covariant differential and the Betti numbers should be replaced by the dimensions of the cohomology of M with coefficients in F). We leave the details to the interested reader.

5. A Dirac-type operator on a circle.

In this section we consider the operator D on the circle, which appears, e.g., in the study of a quantum spin in the presence of a planar, time-dependent magnetic field. The phase of the determinant of D is called *the Berry phase* [5]. The main result of this section is Theorem 5.3, which is known in physical literature (see e.g., [2]), but no mathematically rigorous proofs were available until now.

5.1. **The setting.** Let S^1 be the circle, which we view as the interval $[0,\beta]$ ($\beta>0$) with identified ends. Let

$$n:\ S^1\ \longrightarrow\ \big\{\,z\in\mathbb{C}:\ |z|=1\,\big\}$$

be a smooth map. Then there exists a smooth function $\phi:\mathbb{R}\to\mathbb{R}$, satisfying the periodicity conditions

$$\phi(t+\beta)\ =\ 2\pi k+\phi(t),\qquad k\in\mathbb{Z}, \tag{5.1}$$

such that $n=e^{i\phi}$. The number k above is called the *topological degree* (or the *winding number*) of the map n.

Set $\widehat{n}=\begin{pmatrix}0&e^{i\phi}\\ e^{-i\phi}&0\end{pmatrix}$ and consider the family of operators depending on a real parameter m

$$D\ =\ i\frac{d}{dt}+im\widehat{n}\ =\ i\frac{d}{dt}+im\begin{pmatrix}0&e^{i\phi}\\ e^{-i\phi}&0\end{pmatrix}, \tag{5.2}$$

acting on the space of vector-functions $\xi:[0,\beta]\to\mathbb{C}^2$ with boundary conditions

$$\xi(\beta)=e^{i\pi\nu}\xi(0),\quad \dot{\xi}(\beta)=e^{i\pi\nu}\dot{\xi}(0),\qquad\quad \nu=0,1. \tag{5.3}$$

We shall study the determinant of D. The following lemma shows that this determinant is non-zero for m sufficiently large.

Lemma 5.2. *For $m>\max_{t\in[0,\beta]}|\dot{\phi}(t)|$, zero is not in the spectrum of D.*

Proof. Consider the following scalar product on the vector valued functions on $[0,\beta]$:

$$(\xi,\eta)\ =\ \int_0^\beta\langle\xi(t),\eta(t)\rangle\,dt,$$

where $\langle\cdot,\cdot\rangle$ stands for the standard scalar product on $\mathbb{C}^2$. Let $\|\xi\| = (\xi,\xi)^{1/2}$ denote the norm of the vector function ξ.

Integrating by parts the expression for $\|D\xi\|^2$ we obtain, for ξ satisfying the boundary condition (5.3),

$$\begin{aligned}\|D\xi\|^2 \;&=\; (\dot\xi,\dot\xi) + m(\widehat{n}\xi,\dot\xi) + m(\dot\xi,\widehat{n}\xi) + m^2\|\xi\|^2 \\ &\geq\; -m(\dot{\widehat{n}}\xi,\xi) - m(\widehat{n}\dot\xi,\xi) + m(\dot\xi,\widehat{n}\xi) + m^2\|\xi\|^2 \\ &=\; -m(\dot{\widehat{n}}\xi,\xi) + m^2\|\xi\|^2 \;\geq\; m\,\big(m - \max_{t\in[0,\beta]}|\dot\phi(t)|\,\big)\,\|\xi\|^2.\end{aligned}$$

□

Theorem 5.3. *Let $m > \max_{t\in[0,\beta]}|\dot\phi(t)|$. For every $\theta\in(0,\pi)$ such that there are no eigenvalues of D on the ray R_θ the following equality holds*

$$\mathrm{Det}_\theta\, D \;=\; -(-1)^{k+\nu}\,\big|\,\mathrm{Det}_\theta\, D\,\big|, \tag{5.4}$$

where k is defined in (5.1) *and ν is defined in* (5.3).

Remark 5.4. Theorem 5.3 relates the sign of $\mathrm{Det}_\theta\, D$ with the topological invariant of the map $e^{i\phi}$. This realizes the program outlined in Subsection 3.7.

We precede the proof of the theorem with some discussion of the spectral properties of D.

5.5. **The spectral properties of D.** In order to study the spectrum of D it is convenient to replace it by a conjugate operator as follows. The operator

$$U_\phi \;:=\; \begin{pmatrix} e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{pmatrix}$$

maps the space of vector-functions with boundary conditions (5.3) to the space of functions $\xi:[0,\beta]\to\mathbb{C}^2$ with new boundary conditions

$$\xi(\beta) = e^{i\pi(\nu+k)}\xi(0), \quad \dot\xi(\beta) = e^{i\pi(\nu+k)}\dot\xi(0). \tag{5.5}$$

Thus the operator

$$\widetilde{D} \;:=\; U_\phi^{-1}\circ D\circ U_\phi \;=\; i\,\frac{d}{dt} + \begin{pmatrix} -\dot\phi/2 & im\\ im & \dot\phi/2\end{pmatrix} \tag{5.6}$$

acting on the space of vector functions with boundary conditions (5.5) is isospectral to D.

We now consider the following deformation of $\widetilde{D}$:

$$\widetilde{D}_a \;:=\; i\,\frac{d}{dt} + \begin{pmatrix} -a\dot\phi/2 & im\\ im & a\dot\phi/2\end{pmatrix}, \qquad a\in[0,1]. \tag{5.7}$$

The same arguments which were used in the proof of Lemma 5.2 show that

Lemma 5.6. *The operator $\widetilde{D}_a$ is invertible for all $a\in[0,1]$, and all sufficiently large $m>0$.*

Let
$$\overline{\widetilde{D}_a} \;=\; -i\,\frac{d}{dt}+\begin{pmatrix}-a\dot{\phi}/2 & -im\\ -im & a\dot{\phi}/2\end{pmatrix},$$
be the complex conjugate of the operator $\widetilde{D}_a$.

The following lemma shows that the spectrum of $\widetilde{D}_a$ (and, hence, of D) is symmetric with respect to both the real and the imaginary axis.

Lemma 5.7. *The operators $\widetilde{D}_a$, $-\overline{\widetilde{D}}_a$, and $\widetilde{D}_a^*$ are conjugated to each other. Therefore, they have the same spectral decomposition* (2.8).

In particular, the operators D, $-\overline{D}$, and D^* are conjugated to each other.

Proof. An easy calculation shows that
$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\widetilde{D}_a\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}\;=\;\widetilde{D}_a^*,\qquad \begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\widetilde{D}_a\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\;=\;-\overline{\widetilde{D}}_a.$$
□

5.8. **Proof of Theorem 5.3.** Since the operators D and $\widetilde{D}$ are conjugated to each other their determinants are equal. By Theorem 3.6, the sign of the determinant of $\widetilde{D}=\widetilde{D}_a$ is equal to the sign of the determinant of the operator
$$\widetilde{D}_0 \;=\; i\,\frac{d}{dt}+\begin{pmatrix}0&im\\ im&0\end{pmatrix}.$$
It is easy to see that all the eigenvalues of $\widetilde{D}_0$ are given by the formula
$$\lambda_n^{\pm}\;=\;\pm im\;+\;\frac{\pi}{\beta}\,(2n-k-\nu),\qquad n\in\mathbb{Z}.$$
Hence, $\widetilde{D}_0$ does not have any eigenvalues on the ray $R_{\pi/2}$ if $k+\nu$ is odd and has exactly one eigenvalue $\lambda^+_{(k+\nu)/2}=im$ on this ray if $k+\nu$ is even.

Theorem 5.3 follows now from Theorem 3.2. □

6. The phase of the determinant and the degree of the map

This section essentially generalizes the previous section to manifolds of higher dimensions. For the case of a sphere of dimension $N=4l+1$ the results of this section have been obtained in [2] as topological terms in non-linear σ-models emerging as effective models for Dirac fermions coupled to chiral bosonic fields. However, no mathematically rigorous proofs were available until now. Note also that our result is more precise, since the equality (6.6) was obtained in [2] from the gradient expansion, i.e., only asymptotically for $m\to\infty$.

The section is organized as follows: first we formulate the problem in purely geometric terms as a question about the determinant of the DeRham-Dirac operator with potential. We state our main result as Theorem 6.7. Then, in Subsection 6.8, we reformulate the result in terms of an operator acting on the tensor product of the two spaces of spinors. This formulation is closer to the one considered in physical literature. Finally, we present the proof of Theorem 6.7 based on the application of Theorems 3.2 and 3.6.

6.1. **The setting.** Let M be a closed oriented manifold of odd dimension $N = 2r+1$. We fix a Riemannian metric on M and use it to identify the tangent and the cotangent bundles, $TM \simeq T^*M$. Let $\Lambda^*TM = \bigoplus_{j=0}^N \Lambda^j TM$ denote the exterior algebra of TM viewed as a vector bundle over M. The space $\Omega^*(M)$ of complex-valued differential forms on M coincides with the space of sections of the complexification $\Lambda^*TM \otimes \mathbb{C}$ of this bundle.

The bundle $\Lambda^*TM \otimes \mathbb{C}$ (and, hence, the space $\Omega^*(M)$) carries 2 anti-commuting actions of the Clifford algebra of TM (the "left" and the "right" action) defined as follows

$$c_L(v)\,\omega \;=\; v\wedge\omega - \iota_v\,\omega, \quad c_R(v)\,\omega \;=\; v\wedge\omega + \iota_v\,\omega, \qquad v\in TM,\ \omega\in\Omega^*(M). \tag{6.1}$$

where ι_v denotes the interior multiplication by v.

The DeRham-Dirac operator $\not{\partial}$ can be written now (cf. [4, Prop. 3.53]) as

$$\not{\partial} \;=\; d+d^* \;=\; \sum_{j=1}^N c_L(e_j)\,\nabla^{\mathrm{LC}}_{e_j}, \tag{6.2}$$

where ∇^{LC} denotes the Levi-Civita covariant derivative and $e_1,\dots,e_N$ is an orthonormal frame of TM.

We view the direct sum $\mathbb{R}\oplus TM$ as a vector bundle over M. Consider the corresponding sphere bundle

$$\mathbb{S} \;:=\; \big\{\,(t,a)\in\mathbb{R}\oplus TM:\, t^2+|a|^2=1\,\big\}. \tag{6.3}$$

Let n be a smooth section of the vector bundle $\mathbb{S}$. In other words, $n=(n_0,\overline{n})$, where $n_0\in C^\infty(M)$, $\overline{n}\in C^\infty(M,TM)$ and $n_0^2+|n|^2=1$.

Remark 6.2. Suppose M is a parallelizable manifold, i.e., there given an identification between TM and the product $M\times\mathbb{R}^N$. Then n can be considered as a map

$$M \;\longrightarrow\; S^N \;:=\; \big\{\,y\in\mathbb{R}^{N+1}:\,|y|^2=1\,\big\}. \tag{6.4}$$

Also, if $M\subset\mathbb{R}^{N+1}$ is a hypersurface, then, for every $x\in M$, the space $\mathbb{R}\oplus T_xM$ is naturally identified with $\mathbb{R}^{N+1}$. Hence, n again can be considered as a map $M\to S^N$. Note, however, that, even if M is parallelizable, this map is different from (6.4).

Consider the map

$$\Phi:\,\mathbb{R}\oplus TM \;\to\; \mathrm{End}\,\Lambda^*TM\otimes\mathbb{C}, \quad \Phi:\,n=(n_0,\overline{n}) \;\mapsto\; in_0+c_R(\overline{n}),$$

and define the family of deformed DeRham-Dirac operators

$$D_{mn} \;=\; \not{\partial} \;+\; m\,\Phi(n):\,\Omega^*(M) \;\longrightarrow\; \Omega^*(M). \tag{6.5}$$

We are interested in the phase of $\mathrm{Det}_\theta\, D_{mn}$ for sufficiently large m. The following lemma shows that this determinant is well defined.

Lemma 6.3. *Fix an orthonormal frame $e_1,\dots,e_N$ of TM and set*

$$\big|\,\nabla^{\mathrm{LC}}\overline{n}(x)\,\big| \;=\; \sum_{j=1}^N \big|\,\nabla^{\mathrm{LC}}_{e_j}\overline{n}(x)\,\big|, \qquad \big|\,\nabla n_0(x)\,\big| \;=\; \sum_{j=1}^N \big|\,\nabla_{e_j}n_0(x)\,\big|.$$

For $m > \max_{x\in M}\big(|(\nabla^{\rm LC}\overline{n}(x)| + |\nabla n_0(x)|\big)$, zero is not in the spectrum of D.

The lemma is a particular case of a more general Lemma 6.11, cf. below.

6.4. **The degree of a section.** Note that the bundle $\mathbb{S}\to M$ has a natural section $\sigma: M\to\mathbb{S}$, $\sigma(x) = (1,0)$.

Definition 6.5. *The* topological degree $\deg(n)$ *of the map n is the intersection number of the manifolds $\sigma(M)$ and $n(M)$ inside $\mathbb{S}$.*

Remark 6.6. Suppose M is parallelizable and consider n and σ as maps $M\to S^N$, cf. Remark 6.2. Then σ is the constant map $\sigma(x) = (1,0)$. Hence, $\deg(n)$ is the usual topological degree of the map $n: M\to S^N$.

Theorem 6.7. *Let $m > \max_{x\in M}\big(|(\nabla^{\rm LC}\overline{n}(x)| + |\nabla n_0(x)|\big)$. For every $\theta\in(0,\pi)$ such that there are no eigenvalues of D_{mn} on the ray R_θ the following equality holds*

$$\operatorname{Det}_\theta D_{mn} \;=\; (-1)^{\deg n}\,\big|\operatorname{Det}_\theta D_{mn}\big|. \tag{6.6}$$

6.8. **Reformulation in terms of spinors.** Consider the (left) *chirality operator*

$$\Gamma_L \;:=\; i^{\frac{N+1}{2}}\,c_L(e_1)\,c_L(e_2)\cdots c_L(e_N),$$

where $e_1,\dots,e_N$ is an orthonormal frame of TM. This operator is independent of the choice of the frame [4, Lemma 3.17] (in fact, it coincides with the operator Γ defined in (4.2)). Moreover, $\Gamma_L^2 = 1$ and Γ_L commutes with $c_L(v)$ and anti-commutes with $c_R(v)$ for all $v\in TM$.

Consider the map

$$\widehat{\ }: \mathbb{R}\oplus TM \;\to\; \operatorname{End}\Lambda^*TM, \qquad n=(n_0,\overline{n}) \;\mapsto \widehat{n} \;:=\; i\,\Gamma_L\,n_0 \;+\; \Gamma_L\,c_R(\overline{n}).$$

Then

$$\widehat{n}^2 \;=\; -\big(\,n_0^2 + |\overline{n}|^2\,\big).$$

Hence, the map $n\mapsto\widehat{n}$ defines a Clifford action of $\mathbb{R}\oplus TM$ on Λ^*TM.

Assume now that M is a spin-manifold (without this assumption the construction of this subsection is true only locally, in any coordinate neighborhood). In particular, there exists a bundle $\mathfrak{S}\to M$ whose fibers are isomorphic to the space of spinors over $\mathbb{R}\oplus TM$. Then (cf. [4, Prop. 3.35]) there exists a bundle $\mathcal{S}\to M$, such that $\Lambda^*TM\otimes\mathbb{C}\to M$ can be decomposed as the tensor product $\mathcal{S}\otimes\mathfrak{S}$, and the operators $\widehat{n}$ ($n\in\mathbb{R}\oplus TM$) act only on the second factor. More precisely, if we denote by $c_{\mathfrak{S}}: \mathbb{R}\oplus TM\to\operatorname{End}\mathfrak{S}$ the Clifford action of $\mathbb{R}\oplus TM$ on $\mathfrak{S}$, then $\widehat{n} = 1\otimes c_{\mathfrak{S}}(n)$.

We introduce now a new Clifford action $\widetilde{c}: TM\to\operatorname{End}\big(\Lambda^*TM\otimes\mathbb{C}\big)$ of TM on $\Lambda^*TM\otimes\mathbb{C}$, defined by the formula

$$\widetilde{c}(v) \;=\; \Gamma_L\,c_L(v), \qquad v\in TM. \tag{6.7}$$

One readily sees that $\widehat{n}$ and $\widetilde{c}(v)$ commute for all $v\in TM$, $n\in\mathbb{R}\oplus TM$. It follows (cf. [4, Prop. 3.27]) that there is a Clifford action $c_{\mathcal{S}}: TM\to\operatorname{End}\mathcal{S}$ such that $\widetilde{c}(v) = c_{\mathcal{S}}(v)\otimes 1$. Comparing dimensions we conclude that $\mathcal{S}$ is a spinor bundle over M.

It follows from (6.2), that $\not{\partial} = \Gamma_L \not{\partial}_{\mathcal{S}} \otimes 1$, where $\not{\partial}_{\mathcal{S}}$ is the Dirac operator on $\mathcal{S}$. Hence, the operator (6.5) takes the form

$$D_{mn} \;=\; \not{\partial} \;+\; m\,\Gamma_L\,\widehat{n} \;=\; \Gamma_L \left(\not{\partial}_{\mathcal{S}} \otimes 1 \;+\; m \cdot 1 \otimes c_{\mathfrak{S}}(n) \right) : C^\infty(\mathcal{S}\otimes\mathfrak{S}) \;\longrightarrow\; C^\infty(\mathcal{S}\otimes\mathfrak{S}).$$

In this form this and similar operators appeared in physical literature. In particular, for the case when M is a $(4l+1)$-dimensional sphere this operator[2] was considered in [2]. Also a result similar to our Theorem 6.7 was obtained in [2] for the operator

$$\Gamma_L \cdot D_{mn} \;=\; \not{\partial}_{\mathcal{S}} \otimes 1 \;+\; m\left(1 \otimes c_{\mathfrak{S}}(n) \right)$$

on a $(4l+3)$-dimensional sphere.

6.9. **The idea of the proof.** The rest of this section is devoted to the proof of Theorem 6.7, which is based on an application of Theorems 3.2 and 3.6. More precisely, we will deform operator D_{mn} to an operator $\widetilde{D}$ whose determinant has the same sign (in view of Theorem 3.6). We then calculate the number of imaginary eigenvalues of $\widetilde{D}$, which, in view of Theorem 3.2, will give us the sign of the determinants of $\widetilde{D}$ and D_{mn}.

First, we need to define the class of operators in which we will perform our deformation. This is done in the next subsection.

6.10. **Extension of the class of operators.** Let $a : M \to \mathbb{R}$ and $v : M \to TM$ be a smooth function and a smooth vector field on M respectively. Set

$$D(a,v) \;:=\; \not{\partial} \;+\; ia \;+\; c_R(v).$$

Clearly, $D_{mn} = D(mn_0, m\overline{n})$. Also the following analogue of Lemma 6.3 holds

Lemma 6.11. *Suppose $a(x)^2 \;+\; |v(x)|^2 > 0$ for all $x \in M$. Fix*

$$m_0 \;>\; \max_{x\in M} \left(|(\nabla^{\mathrm{LC}} v(x)| + |\nabla a(x)| \right).$$

Then, for all $m \geq m_0$, zero is not in the spectrum of $D(m_0 a, mv)$.

Proof. Set $\not{\partial}_m = \not{\partial} + (m-m_0)c_R(v)$. Then

$$D(m_0 a, mv) \;=\; \not{\partial}_m \;+\; i\,m_0\,a \;+\; m_0\,c_R(v).$$

[2]Note, however, that there is a sign discrepancy between our notation and the notation accepted in physical literature. Our operators $c_{\mathcal{S}}(v)$, $c_{\mathfrak{S}}(n)$ are skew-adjoint and satisfy the equalities $c_{\mathcal{S}}(v)^2 = -|v|^2$, $c_{\mathfrak{S}}(n) = -|n|^2$. Consequently, the operator $\not{\partial}_{\mathcal{S}}$ is self-adjoint.

Let $\alpha \in \Omega^*(M)$. Using (6.2), we obtain,

$$\begin{aligned}
&\| D(m_0 a, mv)\,\alpha \|^2 \\
&\qquad = \| \partial\!\!\!/_m \alpha \|^2 + m_0^2 \|\alpha\|^2 + m_0 \langle [\, \partial\!\!\!/_m (c_R(v) + ia) + (c_R(v) - ia)\, \partial\!\!\!/_m \,]\, \alpha, \alpha \rangle \\
&\quad \ge m_0^2 \|\alpha\|^2 + m_0 \sum_{j=1}^N \langle c_L(e_j) \big[c_R(\nabla^{\rm LC}_{e_j} v) + \nabla_{e_j} a \big]\, \alpha, \alpha \rangle + 2\, m_0\, (m - m_0) \langle |v|^2 \alpha, \alpha \rangle \\
&\qquad\qquad \ge m_0 \Big(m_0 - \max_{x \in M} \big(|(\nabla^{\rm LC} v(x)| + |\nabla a(x)| \big) \Big) \|\alpha\|^2.
\end{aligned}$$

□

The following lemma shows that we can apply Theorem 3.2 to the study of $\mathrm{Det}_\theta(a,v)$ (and, hence, of $\mathrm{Det}_\theta D_{mn}$).

Lemma 6.12. *The operators $D(a,v)$ and $-D(a,v)^*$ are conjugate to each other. Consequently, they have the same spectral decomposition* (2.8)*.*

In particular, the operators D_{mn} and $-D_{mn}^*$ are conjugate to each other.

Proof. Let $N : \Omega^*(M) \to \Omega^*(M)$ be the *grading operator* defined by the formula

$$N\,\omega = (-1)^j\, \omega, \qquad \omega \in \Omega^j(M). \tag{6.8}$$

Then

$$N \circ D(a,v) \circ N = -\partial\!\!\!/ + ia - c_R(v) = -D(a,v)^*. \tag{6.9}$$

□

6.13. **Deformation of D_{mn}.** Let $n = (n_0, \overline{n})$ be as in Theorem 6.7. Suppose that $\deg(n) = \pm k$, where k is a non-negative integer. Then there exists a section $n' = (n'_0, \overline{n}')$ of $\mathbb{S}$, which is homotopic to n and has the following properties:

- There exist k distinct points $x_1, \dots, x_k \in M$ such that
$$n'_0(x_j) = 1, \quad \overline{n}'(x_j) = 0, \qquad j = 1, \dots, k.$$
- There exists a Morse function $f : M \to \mathbb{R}$ and a neighborhood U of the set $\{x_1, \dots, x_k\}$ such that
$$\overline{n}'(x) = \nabla f(x), \qquad \text{for all} \quad x \in U,$$
and $\overline{n}'(x) \neq 0$ for all $x \in U \backslash \{x_1, \dots, x_k\}$.
- If $\overline{n}'(x) = 0$ and $x \notin \{x_1, \dots, x_k\}$, then $n'_0(x) = -1$ and $\nabla f(x) = 0$.

Let $x_{k+1}, \dots, x_l$ be the rest of the critical points of f. Then $\overline{n}'(x) \neq 0$ for all $x \notin \{x_1, \dots, x_l\}$. Fix open neighborhoods V_j ($j = 1, \dots, l$) of x_j whose closures are mutually disjoint and such that $V_j \subset U$ for all $j = 1, \dots, k$. We will assume that V_j are small enough so that $n'_0(x) \neq 0$ and $\overline{n}'(x) \neq 0$ for all $x \in V_j \backslash \{x_j\}$.

For each $j = 1, \dots, l$ fix a neighborhood W_j of x_j, whose closure lies inside V_j.

Let $a : M \to [-1,1]$ be a smooth function such that

$$a(x) \;=\; \begin{cases} \;\;1, & \text{if } x \in \bigcup_{j=1}^{k} W_j; \\ -1, & \text{if } x \notin \bigcup_{j=1}^{k} V_j. \end{cases} \tag{6.10}$$

Consider the deformation $(n_0(t), \overline{n}(t))$ of the section $(n_0', \overline{n}') \in \mathbb{S}$ given by the formulas

$$n_0(t) \;=\; \begin{cases} ta + (1-t)n_0', & 0 \le t \le 1; \\ a, & 1 \le t \le 2. \end{cases},$$
$$\overline{n}(t) \;=\; \begin{cases} \overline{n}', & 0 \le t \le 1; \\ (t-1)\nabla f + (2-t)\overline{n}', & 1 \le t \le 2. \end{cases}$$

Clearly, $(n_0(t), \overline{n}(t)) \neq 0$ for all $t \in [0,2]$. Hence, by Lemma 6.11, for large m_0 and every $m > m_0$, $t \in [0,2]$, zero is not in the spectrum of the operator $D\big(\,m_0 n_0(t),\, m\overline{n}(t)\,\big)$. Theorem 3.6 implies now that the determinant of D_{mn} has the same sign as the determinant of

$$D\big(\,m_0 n_0(2),\, m\overline{n}(2)\,\big) \;=\; D(m_0 a, m\nabla f).$$

6.14. **The spectrum of the operator** $D(0, m\nabla f)$**.** Before investigating the operator $D(m_0 a, m\nabla f)$ we consider a simpler operator

$$D(0, m\nabla f) \;=\; \not{\partial} \;+\; m\, c_R(\nabla f) \;=\; e^{-mf}\, d\, e^{mf} \;+\; \big(e^{-mf}\, d\, e^{mf}\big)^*.$$

This is a self-adjoint operator whose spectrum was studied by Witten [17] (see, for example, [16] for a mathematically rigorous exposition of the subject). In particular, $D(0, m\nabla f)$ has the following properties:

- There exist a constant $C > 0$ and a function $r(m) > 0$ such that $\lim_{t\to 0} r(m) = 0$ and, for all sufficiently large $m > 0$, the spectrum of $D(0, m\nabla f)$ lies inside the set
$$\big(-\infty,\, -C\sqrt{m}\,\big) \cup \big(-r(m), r(m)\big) \cup \big(C\sqrt{m},\, \infty\big).$$
- Let E_m denote the span of the eigenvectors of $D(0, m\nabla f)$ with eigenvalues in the interval $\big(-r(m), r(m)\big)$. Then, for all sufficiently large m, the space E_m has a basis $\alpha_{1,m}, \dots, \alpha_{l,m}$ ($\|\alpha_{j,m}\| = 1$) such that each $\alpha_{j,m}$ *is concentrated in* W_j in the following sense:
$$\int_{W_j} \alpha_{j,m} \wedge *\alpha_{j,m} \;=\; 1 \;-\; o(1), \qquad \text{as} \quad m \to \infty. \tag{6.11}$$
(Here $o(1)$ stands for a vector whose norm tends to 0 as $m \to \infty$). In particular, $\dim E_m = l$.

Note, that (6.11) and (6.10) imply that

$$a(x)\,\alpha_{j,m}(x) \;=\; \begin{cases} \alpha_{j,m}(x) + o(1), & \text{for } j = 1, \dots, k; \\ -\alpha_{j,m}(x) + o(1), & \text{for } j = k+1, \dots, l. \end{cases} \tag{6.12}$$

6.15. **The spectrum of the operator** $D(m_0a, m\nabla f)$**.** Let m_0 be as in Subsection 6.13 and let C be as in Subsection 6.14. Choose m large enough so that

$$m \ > \ \left(\frac{4(m_0+1)}{C}\right)^2.$$

and $r(m)<1$. We view the operator

$$D(m_0a, m\nabla f) \ = \ D(0, m\nabla f) \ + \ i\, m_0\, a(x)$$

as a perturbation of $D(0, m\nabla f)$.

Lemma 6.16. *The number of eigenvalues λ (counting with multiplicities) of $D(m_0a, m\nabla f)$ which satisfy*

$$|\lambda| \ < \ 2(m_0+1), \quad \operatorname{Im}\lambda \ > \ 0,$$

is equal to $k = \deg n$.

Proof. The spectral projection of the operator $D(0, m\nabla f)$ onto the space E_m (cf. Subsection 6.14) is given by the Cauchy integral

$$P_m \ = \ \frac{1}{2\pi i}\oint_\gamma \big(\,\lambda - D(0, m\nabla f)\,\big)^{-1}\, d\lambda, \tag{6.13}$$

where γ is the boundary of the disk $B = \big\{\, z\in\mathbb{C}:\, |z| < 2(m_0+1)\,\big\}$.

Note that, for all $\lambda\in\gamma$, we have

$$\Big\|\,\big(\,\lambda - D(0, m\nabla f)\,\big)^{-1}\Big\| \ \le \ \frac{1}{\operatorname{dist}\big(\,\lambda, \operatorname{spec} D(0, m\nabla f)\,\big)} \ = \ \frac{1}{2m_0+1}. \tag{6.14}$$

Hence, for all $\lambda\in\gamma$, we obtain

$$\begin{aligned} &\Big\|\,\big(\,\lambda - D(m_0a, m\nabla f)\,\big)^{-1}\Big\| \\ &\qquad \le \ \Big\|\,\big(\,\lambda - D(0, m\nabla f)\,\big)^{-1}\Big\|\cdot\big\|\,\big(\,1 - (\lambda - D(0, m\nabla f))^{-1}m_0a\,\big)^{-1}\,\big\| \\ &\qquad\qquad \le \ \frac{1}{2m_0+1}\cdot\frac{1}{1-\frac{m_0}{2m_0+1}} \ = \ \frac{1}{m_0+1}. \end{aligned} \tag{6.15}$$

In particular, γ is contained in the resolvent set of $D(m_0a, m\nabla f)$.

Let E'_m denote the span of the root vectors of $D(m_0a, m\nabla f)$ with eigenvalues in B. The spectral projection of $D(m_0a, m\nabla f)$ onto E'_m is given by the formula

$$P'_m \ = \ \frac{1}{2\pi i}\oint_\gamma \big(\,\lambda - D(m_0a, m\nabla f)\,\big)^{-1}\, d\lambda.$$

Using (6.14) and (6.15), we obtain

$$\begin{aligned} \|\,P_m - P'_m\,\| \ &= \ \frac{1}{2\pi}\,\Big\|\oint_\gamma \big(\,\lambda - D(0, m\nabla f)\,\big)^{-1} m_0a\,\big(\,\lambda - D(m_0a, m\nabla f)\,\big)^{-1} d\lambda\Big\| \\ &\le \ 2(m_0+1)\cdot\frac{1}{2m_0+1}\cdot m_0\cdot\frac{1}{m_0+1} \ = \ \frac{2m_0}{2m_0+1} \ < \ 1. \end{aligned} \tag{6.16}$$

In particular,

$$\dim E'_m \;=\; \dim E_m \;=\; l,$$

and the projection P'_m maps E_m isomorphically onto E'_m. Recall that the basis $\alpha_{1,m}, \dots, \alpha_{l,m}$ of E_m was defined in Subsection 6.14. Then $P'_m\alpha_{1,m}, \dots, P'_m\alpha_{l,m}$ is a basis of E'_m.

From (6.12), we get

$$D(m_0 a, m\nabla f)\,\alpha_{j,m} \;=\; \begin{cases} i\,m_0\,\alpha_{j,m} + o(1), & \text{for } j = 1, \dots, k; \\ -i\,m_0\,\alpha_{j,m} + o(1), & \text{for } j = k+1, \dots, l. \end{cases}$$

Since the operators $D(m_0 a, m\nabla f)$ and P'_m commute we obtain

$$D(m_0 a, m\nabla f)\,P'_m\alpha_{j,m} \;=\; \begin{cases} i\,m_0\,P'_m\alpha_{j,m} + o(1), & \text{for } j = 1, \dots, k; \\ -i\,m_0\,P'_m\alpha_{j,m} + o(1), & \text{for } j = k+1, \dots, l. \end{cases}$$

Hence, the restriction of $D(m_0 a, m\nabla f)$ to E'_m has exactly k eigenvalues (counting with multiplicities) with positive imaginary part. □

6.17. **Proof of Theorem 6.7.** Clearly, all the eigenvalues of $D(m_0 a, m\nabla f)$ satisfy

$$\big|\,\mathrm{Im}\,\lambda\,\big| \;\le\; m_0. \tag{6.17}$$

In particular, all the eigenvalues of $D(m_0 a, m\nabla f)$ which lie on the ray $R_{\pi/2}$ belong to the disc $B = \big\{\, z \in \mathbb{C} :\, |z| < 2(m_0 + 1)\,\big\}$. Since the spectrum of $D(m_0 a, m\nabla f)$ is symmetric with respect to the imaginary axis the number of these eigenvalues (counting with multiplicities) has the same parity as the number of all eigenvalues, which lie in B and have positive imaginary part. Theorem 6.7 follows now from Theorem 3.2 and Lemma 6.16. □

References

[1] A. G. Abanov, *Hopf term induced by fermions*, Phys.Lett. **B492** (2000), 321–323.

[2] A. G. Abanov and P. B. Wiegmann, *Theta-terms in nonlinear sigma-models*, Nucl.Phys. **B570** (2000), 685–698.

[3] M. S. Agranovich, *Elliptic operators on closed manifolds*, Current problems in mathematics. Fundamental directions, Vol. 63 (Russian), Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990, pp. 5–129.

[4] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, 1992.

[5] M. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. R. Soc. Lond. A **392**, (1984), 45-57.

[6] D. Burghelea, L. Friedlander, and T. Kappeler, *On the determinant of elliptic differential and finite difference operators in vector bundles over S^1*, Comm. Math. Phys. **138** (1991), no. 1, 1–18.

[7] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1966.

[8] V. B. Lidskiĭ, *Non-selfadjoint operators with a trace*, Dokl. Akad. Nauk SSSR **125** (1959), 485–487.

[9] A. S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, Translations of Mathematical Monographs, vol. 71.

[10] R. Ponge, *Spectral asymetry, zeta function and the noncommutative residue*, Preprint.

[11] A. N. Redlich, *Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions*, Phys. Rev. Lett. **52**, (1984), 18-21.

[12] ———, *Parity violation and gauge noninvariance of the effective gauge field action in three dimensions*, Phys. Rev. D **29**, (1984), 2366-2374.

[13] J. R. Retherford, *Hilbert space: compact operators and the trace theorem*, London Mathematical Society Student Texts, vol. 27, Cambridge University Press, Cambridge, 1993.

[14] R. Seeley, *Complex powers of elliptic operators*, Proc. Symp. Pure and Appl. Math. AMS **10** (1967), 288–307.

[15] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer Verlag, Berlin, New York, 1980.

[16] ———, *Semiclassical asymptotics on covering manifolds and Morse inequalities*, Geom. Funct. Anal. **6** (1996), 370–409.

[17] E. Witten, *Supersymmetry and Morse theory*, J. of Diff. Geom. **17** (1982), 661–692.

[18] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), no. 1, 143–177.

[19] ———, *Noncommutative residue. I. Fundamentals*, K-theory, arithmetic and geometry (Moscow, 1984–1986), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 320–399.

[20] K. P. Wojciechowski, *Heat equation and spectral geometry. Introduction for beginners*, Geometric methods for quantum field theory (Villa de Leyva, 1999), World Sci. Publishing, River Edge, NJ, 2001, pp. 238–292.

Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794, USA

Department of Mathematics, Northeastern University, Boston, MA 02115, USA

E-mail address: `alexandre.abanov@sunysb.edu`

E-mail address: `maxim@neu.edu`