NOVIKOV INEQUALITIES WITH SYMMETRY

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Abstract. We suggest Novikov type inequalities in the situation of a compact Lie groups action assuming that the given closed 1-form is invariant and basic. Our inequalities use equivariant cohomology and an appropriate equivariant generalization of the Novikov numbers. We test and apply our inequalities in the case of a finite group. As an application we obtain Novikov type inequalities for a manifold with boundary.

Inégalités de Novikov avec symétrie

Résumé. Nous suggérons des inégalités du type de Novikov dans le cas d'une action de groupes de Lie compacts, en supposant que la 1-forme différentielle donnée est invariante et basique. Nous utilisons une cohomologie équivariante et une généralisation équivariante appropriée des nombres de Novikov. Nous appliquons nos inégalités dans le cas d'un groupe fini. Comme application nous obtenons des inégalités du type de Novikov pour une variété à bord.

Version française abrégée. En 1981 S.Novikov a trouvé une généralisation des inégalités classiques de Morse pour les 1-formes fermées. Dans cet article, nous suggérons une version équivariante des inégalités de Novikov. Nous considéron une G-variété compacte M, où G est un groupe de Lie compact, et une 1-forme θ sur M fermée et invariante. Nous supposons que θ est non-dégénérée au sens de Bott et notre probléme est de trouver des estimations de la topologie de l'ensemble C des points critiques de θ en utilisant des invariants topologiques globaux de M.

Nous construisons la série équivariante de comptage de Morse, qui contient de l'information sur la cohomologie équivariante de toutes les composantes connexes de C. Supposant que la forme θ est basique (cf. ci-dessoùs) nous définissons une généralisation équivariante des nombres de Novikov et, utilisant ces nombres, nous construisons la série équivariante de Novikov. Notre théorème principal (théorèm 7) affirme que la série équivariante de Morse est plus grande (en un sens approprié) que la série équivariante de Novikov. Il contient une infinité d'inégalités imliquant les dimensions de la cohomologie équivariante des composantes connexes de C et des nombres équivariants des Novikov.

De simples examples montrent qu'en appliquant les inégalités équivariantes de Morse bien connues d'Atiyah et Bott dans le cas d'un groupe G fini, on obtient des estimations qui sont parfois moins bonnes que celles obtenues avec les inégalités standard de Morse (qui ignorent l'action du groupe!). La situation peut être améliorée, cependant, par l'utilisation de la cohomologie équivariante croisée. Si G est un groupe fini, toute représentation de G donne naissance à un fibré vectoriel équivariant plat et alors (appliquant notre construction générale)

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à une famille d'inégalités. Des examples montrent que seulement toutes ces inégalités (correspondant à toutes les représentations irréducibles) mises ensemble donnent une estimation asssez bonne de la topologie de C.

Comme application simple nous obtenons des inégalités du type de Novikov pour des variétés à bord.

1. Consider a closed 1-form θ on a closed manifold M. Our problem is to find estimates on the topology of the set C of critical points of θ by using global topological invariants of M. Suppose that G is a compact Lie group acting on M. We will assume that θ is G *invariant*, i.e. $g^*\theta = \theta$ for any $g \in G$. Moreover, we will assume that θ is *basic*; recall, that this means that θ vanishes on vectors tangent to the orbits of G.

Note that any *exact* invariant form $\theta = df$ is basic. Also, (cf., for example, [6, Lemma 3.4]), *if M is connected and if the set of fixed points of the action of G on M is not empty, then any closed G-invariant 1-form on M is basic.*

2. Equivariant flat vector bundles. Let $\mathcal{F} \to M$ be a flat vector bundle over M endowed with a smooth action of G such that the projection $\mathcal{F} \to M$ is G-equivariant. We will assume that the action $g : \mathcal{F}_x \to \mathcal{F}_{g\cdot x}$ is linear for any $g \in G$ and $x \in M$.

The group G acts naturally on the space of differentials forms $\Omega^*(M, \mathcal{F})$ on M with values in $\mathcal{F}$. For any element $X \in \mathfrak{g}$, (where $\mathfrak{g}$ denotes the Lie algebra of G) we will denote by $\mathcal{L}^{\mathcal{F}}(X) : \Omega^*(M, \mathcal{F}) \to \Omega^*(M, \mathcal{F})$ the corresponding Lie algebra action.

A flat bundle $\mathcal{F} \to M$ as above, is called *G-equivariant flat vector bundle* if

$$g \circ \nabla \;=\; \nabla \circ g, \qquad \nabla_{X_M} \;=\; \mathcal{L}^{\mathcal{F}}(X) \tag{1}$$

for any $g \in G$, and for any $X \in \mathfrak{g}$. Here ∇ denotes the covariant derivative of $\mathcal{F}$ and for $X \in \mathfrak{g}$, X_M denotes the vector field on M defined by the action of $\mathfrak{g}$. The second condition determines the covariant derivative in the directions tangent to the orbits.

If the group G is connected, then first condition in (1) follows from the second. Conversely, if G is finite, the second condition in (1) carries no information. However, these conditions are independent in general.

As an *example*, consider a closed G-invariant 1-form θ on M with complex values. Consider the flat vector bundle determined by the form θ. Namely, let $\mathcal{E}_\theta = M \times \mathbb{C}$ with the G action coming from the factor M and with the flat connection $\nabla = d + \theta \wedge \cdot$. This flat bundle always satisfy the first condition of (1) and it satisfies the second condition iff the form θ is basic.

3. The pushforward. Suppose (only in this section) that *the action of G on M is free.* Then the quotient $B = M/G$ is a smooth manifold and the map $q : M \to B$ is a locally trivial fibration. We will discuss a construction which produces a flat vector bundle over B starting from an equivariant vector bundle over M.

Namely, let $\mathcal{S}(\mathcal{F})$ denote the locally constant sheaf of flat sections of $\mathcal{F}$. Then the direct image $q_*\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$ is a locally constant sheaf over B. Let $q_*\mathcal{F}$ denote the flat bundle corresponding to this sheaf. The group G acts naturally on $q_*\mathcal{F}$ and this action is compatible with the flat structure. Moreover, the action of the connected component G_0 of the unit element $e \in G$ is trivial. Thus, the bundle $q_*\mathcal{F}$ splits into a direct sum of its flat subbundles corresponding to different irreducible representations of the finite group G/G_0. The most important for us will be the subbundle corresponding to the trivial representation; we will denote it by $q_*^G\mathcal{F}$. Note that if G is connected, then $q_*^G\mathcal{F} = q_*\mathcal{F}$. *We will say that the flat bundle $q_*^G\mathcal{F}$ is the pushforward of the bundle $\mathcal{F}$.*

4. Twisted equivariant cohomology. Next we define equivariant cohomology $H^*_G(M, \mathcal{F})$ of M with coefficients in an equivariant flat vector bundle $\mathcal{F}$. The idea of the construction is as follows. Let $EG \to BG$ be the universal principal bundle. Given an equivariant flat vector bundle $\mathcal{F}$ over M, it induced equivariant flat bundle $p^*\mathcal{F}$ over $EG \times M$, where $p : EG \times M \to M$ is the projection. Now, we want to form the pushforward $q^G_* p^*\mathcal{F}$, where $q : EG \times M \to EG \times_G M = M_G$ is the projection with respect to the diagonal action. The result is a flat vector bundle over the Borel's quotient M_G. Now we define the equivariant cohomology $H^*_G(M, \mathcal{F})$ as the cohomology of M_G twisted by the flat vector bundle $q^G_* p^*\mathcal{F}$.

We cannot literally apply the construction of the previous paragraph since our category is the category of smooth finite dimensional manifolds and the universal principal bundle $EG \to BG$ is usually infinite dimensional. The problem may be overcome by using *finite dimensional approximations of EG*. We refer to [6] (see also [2]) for details.

5. Equivariant generalization of the Novikov numbers. Given an equivariant flat bundle $\mathcal{F}$ over M and a closed basic 1-form θ on M with real values, consider the one-parameter family $\mathcal{F} \otimes \mathcal{E}_{t\theta}$ of equivariant flat bundles, where $t \in \mathbb{R}$, (*the Novikov deformation*). Here $\mathcal{E}_\theta$ denotes the equivariant flat bundle corresponding θ, cf. §2. For a fixed i consider the twisted equivariant cohomology

$$H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}), \qquad \text{where} \quad t \in \mathbb{R}, \tag{2}$$

as a function of $t \in \mathbb{R}$. Then [6, Lemma 1.3] there exists a *finite* subset $S \subset \mathbb{R}$ such that the dimension of the cohomology $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ is constant for $t \notin S$ and the dimension of the cohomology $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ jumps up for $t \in S$. The subset S, is called the *set of jump points*; the value of the dimension of $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ for $t \notin S$ is called *the background value of the dimension.*

Definition. *The i-dimensional equivariant Novikov number $\beta^G_i(\xi, \mathcal{F})$ is defined as the background value of the dimension of the cohomology $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$, $t \in \mathbb{R}$.*

Here ξ denotes the cohomology class of θ. Note that $\xi \in H^*(M, \mathbb{R})$ lies in the image of the natural map $H^*_G(M, \mathbb{R}) \to H^1(M, \mathbb{R})$ if and only if it may be represented by a basic differential form. Also the equivariant flat bundle $\mathcal{E}_\theta$ is determined (up to gauge equivalence) only by ξ. Thus, the equivariant Novikov numbers $\beta^G_i(\xi, \mathcal{F})$ are well defined for all classes in the image $\xi \in \mathrm{im}[H^1_G(M, \mathbb{R}) \to H^1(M, \mathbb{R})]$. The formal power series

$$\mathcal{N}^G_{\xi, \mathcal{F}}(\lambda) = \sum_i \lambda^i \beta^G_i(\xi, \mathcal{F}) \tag{3}$$

will be called *the equivariant Novikov series.*

6. The equivariant Morse series. Let C denote the set of critical points of θ (i.e. the set of points of M, where θ vanishes). We assume that θ is *non-degenerate in the sense of Bott*, i.e. C is a submanifold of M and the Hessian of θ is a non-degenerate on the normal bundle $\nu(C)$ to C in M. Here by the Hessian of θ we understand the Hessian of the unique function f defined in a neighborhood of C and such that $df = \theta$ and $f|_C = 0$.

Let Z be a connected component of the critical point set C and let $\nu(Z)$ denote the normal bundle to Z in M. The bundle $\nu(Z)$ splits into the Whitney sum of two subbundles $\nu(Z) = \nu^+(Z) \oplus \nu^-(Z)$, such that the Hessian is strictly positive on $\nu^+(Z)$ and strictly negative on $\nu^-(Z)$. The dimension of the bundle $\nu^-(Z)$ is called the *index* of Z (as a critical submanifold of θ) and is denoted by $\mathrm{ind}(Z)$. Let $o(Z)$ denote the *orientation bundle of* $\nu^-(Z)$, *considered as a flat line bundle.*

If the group G is connected, then Z is a G-invariant submanifold of M. In general, we denote by $G_Z = \{g \in G|\ g \cdot Z \subset Z\}$ the stabilizer of the component Z in G. Let $|G : G_Z|$ denote the index of G_Z as a subgroup of G; it is always finite.

The compact Lie group G_Z acts on the manifold Z and the flat vector bundles $\mathcal{F}|_Z$ and $o(Z)$ are G_Z-equivariant. Let $H^*_{G_Z}(Z, \mathcal{F}|_Z \otimes o(Z))$ denote the *equivariant cohomology* of the flat G_Z-equivariant vector bundle $\mathcal{F}|_Z \otimes o(Z)$. Consider the *equivariant Poincaré series*

$$\mathcal{P}^{G_Z}_{Z,\mathcal{F}}(\lambda) \ = \ \sum_i \lambda^i \dim_{\mathbb{C}} H^i_{G_Z}(Z, \mathcal{F}|_Z \otimes o(Z)) \tag{4}$$

and define using it the following *equivariant Morse counting series*

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) \ = \ \sum_Z \lambda^{\mathrm{ind}(Z)} |G : G_Z|^{-1} \mathcal{P}^{G_Z}_{Z,\mathcal{F}}(\lambda) \tag{5}$$

where the sum is taken over all connected components Z of C.

7. Theorem. *Suppose that G is a compact Lie group, acting on a closed manifold M and let $\mathcal{F}$ be an equivariant flat vector bundle over M. Then for any closed non-degenerate (in the sense of Bott) basic 1-form θ on M, there exists a formal power series $\mathcal{Q}(\lambda)$ with non-negative integer coefficients, such that*

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) - \mathcal{N}^G_{\xi,\mathcal{F}}(\lambda) = (1 + \lambda)\mathcal{Q}(\lambda), \quad \textit{where} \quad \xi = [\theta]. \tag{6}$$

The proof (cf. [6]) is based in its main part on the Novikov type inequalities for forms with non-isolated zeros, obtained in [4,5].

If G acts freely on M then the basic form θ defines a closed 1-form θ' on M/G and the inequalities of Theorem 7 (with $\mathcal{F}$ being the trivial line bundle) reduce to the usual Novikov inequalities with respect to the form θ' on the quotient manifold M/G.

In the case of a (non free) circle action $G = S^1$, the equivariant Novikov numbers $\beta^G_i(\xi, \mathcal{F})$ for large i are two-periodic and coincide with the sum of even or odd (depending on the parity of i) usual (i.e. non-equivariant) Novikov numbers of the fixed point set.

An application of Theorem 7 for symplectic torus actions is given in [6].

8. The finite group case. Let's illustrate Theorem 7 in the case when the group G *is finite*. In this situation, one can explicitly calculate the equivariant cohomology in terms of the action of G on the usual cohomology.

Let $\rho : G \to \mathrm{End}\, V_\rho$ be a representation of G on a finite dimensional complex vector space V_ρ. Consider the bundle $\mathcal{F}_\rho = M \times V_\rho$ over M with the trivial connection and with the diagonal G action. It is an equivariant flat bundle over M, which is trivial as the flat bundle but it may be not trivial as an equivariant flat bundle. We will apply Theorem 7 with the bundle $\mathcal{F} = \mathcal{F}_\rho$. The twisted equivariant cohomology $H^*_G(M, \mathcal{F}_\rho \otimes \mathcal{E}_{t\theta})$ can be calculated as

$$H^*_G(M, \mathcal{F}_\rho \otimes \mathcal{E}_{t\theta}) = \mathrm{Hom}_G \left(V^*_\rho, H^*(M, \mathcal{E}_{t\theta}) \right). \tag{7}$$

Here V^*_ρ denotes the representation dual to ρ; the cohomology $H^*(M, \mathcal{E}_{t\theta})$ is considered with its induced G-action. For an irreducible representation ρ the equivariant Novikov number

$\beta_i^G(\xi, \mathcal{F}_\rho) = \beta_i^G(\xi, \rho)$ equals to the *background multiplicity* (i.e. generic with respect to t) of V_ρ^* in the decomposition of $H^i(M, \mathcal{E}_{t\theta})$. The equivariant Poincaré series (4) can be calculated using a formula similar to (7) in terms of the action of G on the usual cohomology. We will denote $\mathcal{M}^G_{\theta,\mathcal{F}_\rho}(\lambda) = \mathcal{M}^G_\theta(\lambda;\rho)$ and $\mathcal{N}^G_{\xi,\mathcal{F}_\rho}(\lambda) = \mathcal{N}^G_\xi(\lambda;\rho)$ and view them as functions of two variables: λ (which will be formal) and ρ (which will run over the set of irreducible representations of G). Applying Theorem 7, we obtain

$$\mathcal{M}^G_\theta(\lambda;\rho) - \mathcal{N}^G_\xi(\lambda;\rho) = (1+\lambda)\mathcal{Q}(\lambda;\rho) \tag{8}$$

where ρ is an irreducible representations of G and $\mathcal{Q}(\lambda;\rho)$ denotes a polynomial in λ with non-negative integral coefficients for any ρ. We may view the last statement as establishing one family of inequalities of Novikov type for any irreducible representation ρ.

The inequalities of M.Atiyah and R.Bott [1] correspond (in the case $\xi = 0$) to the inequalities (8) with ρ the trivial representation. The usual (non-equivariant) Novikov inequalities [7,8] correspond to the regular representation in (8). Using these remarks, one may construct very simple examples (one and two dimensional!) such that the non-equivariant inequalities are better than the inequalities of [1].

9. Novikov inequalities for manifolds with boundary. Our strategy will be to reduce the problem on a manifold with boundary to a problem on the double with its natural $\mathbb{Z}_2$ action. From section 8 we know that we should expect two families of inequalities – since there are two irreducible representations of $\mathbb{Z}_2$.

Let M be a compact manifold with boundary $\Gamma = \partial M$ and let θ be a closed 1-form on M. We will denote by C the set of all critical points of θ. We will suppose that $\theta|_{\mathrm{int}(M)}$ and $\theta|_\Gamma$ are both *non-degenerate in the sense of Bott* (cf. §6) and that *any critical point of $\theta|_\Gamma$ is a critical point of θ as well.* Additionally, we will suppose that for any connected component $Z \subset \Gamma$ of the critical point set of $\theta|_\Gamma$ holds either

(1) *Z is nondegenerate as a critical manifold of θ, or*
(2) *Z is the boundary of a connected component $Z' \subset C$ such that $Z = Z' \cap \Gamma$ and the intersection $Z' \cap \Gamma$ is transversal.*

If the first possibility holds, then the Hessian $h_\theta(\cdot,\cdot)$ of θ is a nondegenerate quadratic form on the normal bundles to Z in M and in Γ and so there exists a unique nonvanishing vector field X on Z normal to Γ such that h_θ splits as a direct sum: $h_\theta(X, Y) = 0$ for any $Y \in T\Gamma$. The function $h_\theta(X, X)$ is everywhere positive or negative on Z; we will call the corresponding component $Z \subset \Gamma \cap C$ *positive* or *negative* respectively.

Represent C as the union of 4 disjoint submanifolds $C_{in} \cup C_+ \cup C_- \cup C_{bd}$, where C_{in} denotes the union of the connected components of C which do not intersect Γ, C_{bd} denotes the union of the components which are manifolds with nonempty boundary, and $C_\pm$ denotes the union of the positive (negative) components in Γ (as defined in the previous paragraph).

For simplicity we will assume that M and all submanifolds $Z \subset C$ are *orientable* .

For any connected component $Z \subset C$ denote by $\mathrm{ind}_+(Z)$ and $\mathrm{ind}_-(Z)$ the dimensions of the positive and the negative subbundles $\nu_+(Z)$ and $\nu_-(Z)$ of the normal bundle $\nu(Z)$ in M correspondingly, compare §6. Now we will define two *Morse counting polynomials*

$$\mathcal{M}^\pm_\theta(\lambda) = \sum_{Z \subset C_{in} \cup C_{bd} \cup C_\pm} \lambda^{\mathrm{ind}_\pm(Z)} \mathcal{P}_Z(\lambda), \tag{9}$$

where $\mathcal{P}_Z(\lambda) = \sum \lambda^i \dim H^i(Z, o(Z))$ is the Poincaré polynomial of Z; in this formula Z runs over the connected components contained in $C_{in} \cup C_{bd} \cup C_+$ in the case of $+$ and contained in $C_{in} \cup C_{bd} \cup C_-$ in the case of $-$.

The Novikov numbers $\beta_i(\xi)$ will be defined as *the background values of the dimension* of $H^i(M, \mathcal{E}_{t\theta})$, cf. §5. Here $\xi \in H^1(M, \mathbb{R})$ denotes the class of θ.

10. Theorem. *Under the assumption described above holds*

$$\mathcal{N}_\xi(\lambda) - \mathcal{M}_\theta^\pm(\lambda) = (1+\lambda)\mathcal{Q}^\pm(\lambda), \quad \textit{where} \quad \mathcal{N}_\xi(\lambda) = \sum \lambda^i \beta_i(\xi), \tag{10}$$

where $\mathcal{Q}^\pm(\lambda)$ *are polynomials with nonnegative integral coefficients.*

Note that in the case of a closed manifold, these two statements (for $\pm$ equal to $+$ and $-$ correspondingly) are equivalent (as follows from Poincaré duality), but for the case of manifolds with boundary they are independent.

The proof of Theorem 10 follows by applying Theorem 7 to the double $D(M)$, two copies of M glued along the boundary Γ, which has the natural $\mathbb{Z}_2$ action. Using our assumptions on θ we may construct an appropriate invariant 1-form $\tilde{\theta}$ on $D(M)$ and then use Theorem 7. The computation of the twisted equivariant cohomology of the double is based on:

Lemma. *Let* $\mathcal{E} \to D(M)$ *be a flat* $\mathbb{Z}_2$ *equivariant vector bundle. Suppose that the action of* $\mathbb{Z}_2$ *on* $\mathcal{E}|_{\partial M}$ *is trivial. Then the twisted equivariant cohomology* $H^i_{\mathbb{Z}_2}(D(M), \mathcal{E} \otimes \mathcal{F}_\rho)$ *is isomorphic to* $H^i(M, \mathcal{E}|_M)$*, if* ρ *is the trivial representation, and to* $H^i(M, \partial M, \mathcal{E}|_M)$*, if* ρ *is the not trivial irreducible representation of* $\mathbb{Z}_2$*.*

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