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Dedicated to M. I. Vishik on the occasion of his 80th *birthday*

Essential self-adjointness of Schrödinger-type operators on manifolds

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Abstract. Several conditions are obtained ensuring the essential self-adjointness of a Schrödinger-type operator $H_V = D^*D + V$, where D is a first-order elliptic differential operator acting on the space of sections of a Hermitian vector bundle E over a manifold M with positive smooth measure $d\mu$ and V is a Hermitian bundle endomorphism. These conditions are expressed in terms of completeness of certain metrics on M naturally associated with H_V. The results generalize theorems of Titchmarsh, Sears, Rofe-Beketov, Oleinik, Shubin, and Lesch. It is not assumed *a priori* that M is endowed with a complete Riemannian metric. This enables one to treat, for instance, operators acting on bounded domains in $\mathbb{R}^n$ with Lebesgue measure. Singular potentials V are also admitted. In particular, a new self-adjointness condition is obtained for a Schrödinger operator on $\mathbb{R}^n$ whose potential has a Coulomb-type singularity and can tend to $-\infty$ at infinity. For the special case in which the principal symbol of D^*D is scalar, more precise results are established for operators with singular potentials. The proofs of these facts are based on a refined Kato-type inequality modifying and improving a result of Hess, Schrader, and Uhlenbrock.

Contents

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§ 1. Introduction

The spectral theory of differential operators is deeply rooted in non-relativistic quantum mechanics, although the first results on eigenvalues and eigenfunctions are much older. Let us recall that, according to von Neumann, a state of a quantum system is a vector ψ in a complex Hilbert space $\mathcal{H}$ (ψ is defined up to a non-zero complex multiple), whereas a physical quantity is a self-adjoint operator A on $\mathcal{H}$. A special physical quantity H, the so-called *energy* or *Hamiltonian*, is responsible for the evolution of any given state, and this evolution is defined by the Schrödinger equation

$$\frac{1}{i}\,\frac{\partial\psi}{\partial t} = H\psi, \tag{1.1}$$

where the system of units is chosen in such a way that the Planck constant is equal to 1. For a given initial condition

$$\psi(0) = \psi_0 \tag{1.2}$$

we can write $\psi(t) = \exp(itH)\,\psi_0$, where the exponential can be defined by the spectral theorem; the equation (1.1) is satisfied in the strong sense if $\psi_0 \in \operatorname{Dom}(H)$.

Concrete Hamiltonians H come from a vaguely defined procedure which is applied to a classical system and is called *quantization*. This procedure leads to Hamiltonians H which are usually second-order differential operators, so-called *Schrödinger operators*. However, it is rare that one can immediately see that such an operator H is self-adjoint in a natural L^2 space. It is usually clear that H is *symmetric* (or *Hermitian*, in a slightly different terminology). The distinction between symmetric and self-adjoint operators is often ignored by physicists, even by very good ones, but this distinction is very important. Indeed, a solution of the initial-value problem (1.1)–(1.2) can fail to exist for a symmetric operator H even if $\psi_0 \in \operatorname{Dom}(H)$. One can try to extend H to obtain a maximal operator (that is, an operator defined on all $\psi \in L^2$ such that $H\psi \in L^2$, where $H\psi$ is understood in the distribution sense). However, in this case a solution of (1.1)–(1.2) can fail to be unique.

The existence and uniqueness of a solution of (1.1)–(1.2) can be guaranteed (for $\psi_0 \in \operatorname{Dom}(H)$) only if H is self-adjoint. However, the natural domain of H is often a problem. Then one starts from smooth compactly supported functions and tries to find out whether the operator H can be extended to a self-adjoint operator *in a unique way*. If this is the case, then H is said to be *essentially self-adjoint*. Under weak restrictions on the regularity of the coefficients, this property is equivalent to the condition that the closure of the operator H defined on the compactly supported smooth functions coincides with the maximal operator. In this case there is only one natural self-adjoint operator associated with the given differential operator, and the evolution of the quantum system is naturally well defined. For this reason, essential self-adjointness is often referred to as *quantum completeness*.

It is clear from the above arguments that the study of the essential self-adjointness of a Schrödinger-type operator must be a starting point of any further investigation of the corresponding quantum system in which this operator is the Hamiltonian. Therefore, finding exact and effective conditions for essential self-adjointness is a fundamental problem of mathematical physics.

The problem of essential self-adjointness for Schrödinger-type operators has been mainly studied in $L^2(\mathbb{R}^n)$. There are thousands of papers devoted to this study, and the first of them (though in a different terminology) is due to Weyl (1909). However, it is quite reasonable to consider the problem in a curved space or, more generally, on manifolds, especially on Riemannian manifolds. Here one usually meets an interesting interaction of analysis and geometry, because the geometry of the manifold can play an important role. The first paper in this direction is due to Gaffney (1954), and in it the essential self-adjointness of the Laplacian on any complete Riemannian manifold was proved.

In this paper we try to develop an approach which unifies and extends almost all earlier results on the essential self-adjointness of Schrödinger-type operators on vector bundles on manifolds. The main results are of two types. An important feature of results of the first type is that a Riemannian metric is not assumed to be given *a priori* but is constructed from the operator. Another important feature is that we trace the trajectories of the corresponding classical Hamiltonian, as was originated by Oleinik (1993, 1994). Our work was mainly inspired by his results and also by a recent paper of Lesch (2000), who extended Oleinik's results and approaches to a much more general context of Schrödinger-type operators on sections of vector bundles. Adding anisotropy with respect to the momentum variables, we eventually came to the first results on essential self-adjointness for Lesch-type operators (with singular potential, on sections of vector bundles), with anisotropy with respect to both spatial and momentum variables.

The second type of results is obtained by extending Kato's inequality to vector bundles, first done by Hess, Schrader, and Uhlenbrock (1980). However, they established their inequalities for smooth sections only, and this is insufficient for applications to essential self-adjointness if a non-smooth potential is added. We correct and improve the Hess–Schrader–Uhlenbrock result and then use it to extend to operators with scalar principal symbol the results on essential self-adjointness that were previously known only for scalar operators.

One of the main tools which enables us to apply Kato's inequality is that the Schwartz kernel of the resolvent of the Laplacian is positive. In fact, a slightly stronger result is needed. This result is obtained in Appendix B, where we also discuss other related questions and state a conjecture, which could simplify our proof if it were established. This conjecture is also of independent interest.

The main results of the paper are actually new not only on manifolds but also on $\mathbb{R}^n$.

We tried to make this paper as self-contained as possible. To this end, we sometimes repeat arguments of older papers, even if only minor modifications are necessary to obtain the results needed for our purposes. Sometimes this is done for some well-known results if there are no easily available references. However, in most cases we improve the exposition of older papers and translate them into more modern language. The most standard material is presented in the appendices.

§ 2. Main results

2.1. Main assumptions. Let M be a C^∞-manifold without boundary and let $\dim M = n$. We always assume that M is connected. We also assume that a positive smooth measure $d\mu$ is given, that is, in any local coordinates $x^1, x^2, \dots, x^n$ there is a strictly positive C^∞-density $\rho(x)$ such that $d\mu = \rho(x)\,dx^1\,dx^2\cdots dx^n$. Let E be a Hermitian vector bundle over M. We denote by $L^2(E)$ the Hilbert space of sections of E that are square-integrable with respect to the inner product

$$(u,v) = \int_M \langle u(x), v(x)\rangle_{E_x}\,d\mu(x).$$

Here $\langle\cdot,\cdot\rangle_{E_x}$ stands for the fibrewise inner product.

Let F be another Hermitian vector bundle on M. Consider a first-order differential operator $D\colon C_c^\infty(E)\to C_c^\infty(F)$ (here C_c^∞ stands for the space of smooth compactly supported sections). We assume that the principal symbol of D is injective. In other words, D is elliptic (possibly overdetermined).

Let D^* be the formal adjoint of D, that is, $D^*: C_c^\infty(F)\to C_c^\infty(E)$ is the differential operator such that $(Du,v) = (u, D^*v)$ for any $u\in C_c^\infty(E)$ and $v\in C_c^\infty(F)$.

The main purpose of this paper is to give sufficient conditions for the essential self-adjointness of the operator

$$H_V = D^*D + V, \tag{2.1}$$

where $V\in L^2_{\mathrm{loc}}(\operatorname{End}E)$ is a linear self-adjoint bundle endomorphism,[1] that is, the operator $V(x)\colon E_x\to E_x$ is self-adjoint for any $x\in M$.

Let us recall the definitions of the minimal and maximal operators associated with a differential expression H_V. The minimal operator $H_{V,\min}$ is the closure of the operator H_V with initial domain $C_c^\infty(E)$, whereas the maximal operator can be defined by the formula $H_{V,\max} = (H_{V,\min})^*$. In particular, the domain $\operatorname{Dom}(H_{V,\max})$ of $H_{V,\max}$ coincides with the set of all sections $u\in L^2(E)$ such that $H_Vu\in L^2(E)$ (here the expression H_Vu is understood in the distribution sense). Then the proof of the essential self-adjointness of H_V can be reduced to proving that $H_{V,\max}$ is a symmetric operator.

We make the following assumption about V.

Assumption A. $V = V_+ + V_-$, where

(i) $V_+(x)\geqslant 0$ and $V_-(x)\leqslant 0$ as linear operators $E_x\to E_x$ for every $x\in M$;

(ii) for every compact set $K\subset M$ there are positive constants $a_K<1$ and C_K such that

$$\left(\int_K |V_-|^2|u|^2\,d\mu\right)^{1/2} \leqslant a_K\|\Delta_M u\| + C_K\|u\| \ \text{ for any } \ u\in C_c^\infty(M), \tag{2.2}$$

[1]The assumption that V is locally square-integrable is needed for the validity of the relation $H_Vu\in L^2_{\mathrm{loc}}(E)$ for $u\in C_c^\infty(E)$.

where $|V_-(x)|$ stands for the norm of the linear map $V_-(x)\colon E_x \to E_x$, $\Delta_M := d^*d$ for the scalar Laplacian of an arbitrary Riemannian metric on M (see Definition 5.4), and $\|\cdot\|$ for the norm in $L^2(E)$.

Remark 2.2. The condition (2.2) is automatically satisfied if $V_- \in L^p_{\mathrm{loc}}(\operatorname{End} E)$ with $p \geqslant n/2$ for $n \geqslant 5$, $p > 2$ for $n = 4$, and $p = 2$ for $n \leqslant 3$. (If K is contained in a coordinate neighbourhood, then this follows from Theorem IX.28, arguments in the proof of Theorem X.15, and Theorems X.20 and X.21 in [64]. The general case can be proved by using the localization technique as explained in [79], §5.2.) Another possibility is to assume that $V_- \in S_{n,\mathrm{loc}}$, where $S_{n,\mathrm{loc}}$ is a local Stummel class (see Appendix C).

Assumption A is closely related to the regularity of sections belonging to the maximal domain $\operatorname{Dom}(H_{V,\max})$ of the operator H_V. More precisely, the following results hold.

Theorem 2.3. (i) *Suppose that* $V \in L^p_{\mathrm{loc}}(\operatorname{End} E)$, *where* $p > n/2$ *for* $n \geqslant 4$ *and* $p = 2$ *for* $n \leqslant 3$. *Then* $\operatorname{Dom}(H_{V,\max}) \subset W^{2,2}_{\mathrm{loc}}(E)$.

(ii) *Suppose that* V *satisfies Assumption* A *and the operator* D^*D *has scalar principal symbol. Then* $\operatorname{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$.

We note that, by Remark 2.2, Assumption A holds for V if the conditions on V in Theorem 2.3 (i) are satisfied.

The proof of Theorem 2.3 (i) is rather simple and is given in § 4. The proof of Theorem 2.3 (ii) is presented in § 7 and is based on a generalization of Kato's inequality obtained in § 5.

With Theorem 2.3 as a motivation, we propose the following conjecture (which is non-trivial even for $M = \mathbb{R}^n$).

Conjecture 2.4. *If* V *satisfies Assumption* A, *then* $\operatorname{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$.

In particular, Theorem 2.3 (i) implies the validity of the conjecture for $n \leqslant 3$. Moreover, by Theorem 2.3 (ii) the conjecture holds if H_V acts on scalar functions (for instance, if H_V is the magnetic Schrödinger operator).

Since we could not prove the conjecture in full generality, we work in this paper under the following additional condition.

Assumption B. $\operatorname{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$ and $\operatorname{Dom}(H_{V_+,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$.

Before stating the main result, we introduce additional notation.

2.5. A metric associated with D**.** Let $\widehat{D}\colon T^*M \otimes E \to F$ be a morphism of vector bundles defined by

$$D(\phi u) = \widehat{D}(d\phi)u + \phi D u, \tag{2.3}$$

where $u \in C^\infty(E)$, $\phi \in C^\infty(M)$, and $\widehat{D}(d\phi)(u)$ is identified with $\widehat{D}(d\phi \otimes u)$. Thus, $\widehat{D} = -i\sigma(D)$, where $\sigma(D)$ is the principal symbol of D. We note that $\widehat{D^*}(\xi) = -(\widehat{D}(\xi))^*$, where $\xi \in T^*_x M$.

If $\xi \in T^*_x M$, then $\widehat{D}(\xi)$ defines a linear operator $E_x \to F_x$. For $\xi, \eta \in T^*_x M$ we write

$$\langle \xi, \eta \rangle = \frac{1}{m} \operatorname{Re} \operatorname{Tr}\big((\widehat{D}(\xi))^* \widehat{D}(\eta)\big), \qquad m = \dim E_x, \tag{2.4}$$

where Tr denotes the usual trace of a linear operator. Since D is an elliptic first-order differential operator and $\widehat{D}(\xi)$ is linear in ξ, we can readily see that the formula (2.4) defines an inner product on T_x^*M. The dual inner product defines a Riemannian metric on M, which we denote by g^{TM}.

Remark 2.6. If we write $|\xi|_0 = |\widehat{D}(\xi)|$, where $|\widehat{D}(\xi)|$ stands for the usual norm of the linear operator $\widehat{D}(\xi)\colon E_x \to F_x$, then the function $|\xi|_0$ induces a norm on T_x^*M. The dual norm on T_xM induces a Finsler metric on M. It follows from elementary linear algebra that

$$m^{-1/2}|\xi|_0 \leqslant |\xi| \leqslant |\xi|_0 \tag{2.5}$$

for every $\xi \in T_x^*M$, where $|\cdot|$ is the norm determined by (2.4).

We say that a curve $\gamma\colon [a,\infty) \to M$ *goes to infinity* if for any compact set $K \subset M$ there is a $t_K > 0$ such that $\gamma(t) \notin K$ for any $t \geqslant t_K$. The metric g^{TM} is said to be *complete* if $\int_\gamma ds = \infty$ for every curve γ going to infinity. Here ds stands for the element of arc length corresponding to the metric g^{TM}. The completeness of g^{TM} is equivalent to the geodesic completeness of M. In particular, the completeness conditions for the metrics corresponding to the norms $|\cdot|_0$ and $|\cdot|$ in Remark 2.6 are equivalent.

We say that $\int^\infty \frac{ds}{\sqrt{q}} = \infty$ for a function $q\colon M \to \mathbb{R}$ if $\int_\gamma \frac{ds}{\sqrt{q}} = \infty$ for every curve γ on M going to infinity.

The main result of the present paper is the following theorem.

Theorem 2.7. *Suppose that the potential V satisfies Assumptions* A *and* B. *Let q be a map $q\colon M \to \mathbb{R}$, and let the following conditions hold*:

(i) *$q \geqslant 1$ and $q^{-1/2}$ is globally Lipschitz, or, equivalently, there is a constant $L > 0$ such that*

$$|q^{-1/2}(x_1) - q^{-1/2}(x_2)| \leqslant Ld(x_1, x_2)$$

for every $x_1, x_2 \in M$, where d stands for the distance induced by the metric g^{TM};

(ii) *there is a $\delta \in [0,1)$ such that $\delta D^*D + V \geqslant -q$ (in the sense of quadratic forms on $C_c^\infty(E)$)*;

(iii) *$\int^\infty \frac{ds}{\sqrt{q}} = \infty$, where ds is the element of arc length for the metric g^{TM}.*

Then H_V is essentially self-adjoint on $C_c^\infty(E)$.

The proof is given in § 9.3.

Remark 2.8. Since we assume the inequality $q \geqslant 1$, it follows from the condition (iii) of the theorem that g^{TM} is complete. Moreover, the condition (iii) itself is equivalent to the statement that the new metric $g := q^{-1}g^{TM}$ is complete.

Theorem 2.7 immediately implies the following corollaries.

Corollary 2.9. *Suppose that $V \geqslant -q$, where $q\colon M \to \mathbb{R}$ satisfies the assumptions* (i) *and* (iii) *of the above theorem. Then H_V is essentially self-adjoint.*

Corollary 2.10. *If the metric g^{TM} is complete, then the operator D^*D is essentially self-adjoint.*

Remark 2.11. We note that both the space $L^2(E)$ and the operator H_V depend on the measure $d\mu$ on M. However, this measure does not appear in the conditions of Corollary 2.9. Hence, if the conditions of this corollary are satisfied, then the operator $D^*D + V$ is essentially self-adjoint for any choice of the measure. In particular, M can be of finite volume with respect to this measure.

Remark 2.12. Theorem 2.7 covers the case of some interesting operators arising in differential geometry. In particular, if $D = d\colon C^\infty(M) \to \Omega^1(M)$ is the de Rham differential, then $D^*D = \Delta_M$ is the scalar Laplacian on M; see Definition 5.4. More generally, we can consider the de Rham differential $d\colon \Omega^j(M) \to \Omega^{j+1}(M)$ on forms of arbitrary degree. Let d^* be the formal adjoint of d and let $D = d + d^*$. Then D^*D is the Laplace–Beltrami operator on differential forms; see [6], §3.6. Moreover, if $D = \nabla\colon C^\infty(E) \to \Omega^1(M, E)$ is the covariant derivative corresponding to a metric connection on a Hermitian vector bundle E (§ 5.2), then D^*D is the Bochner Laplacian on E; see Definition 5.4. However, the above operators generally differ from the classical ones, because they are defined by means of an arbitrary positive smooth measure $d\mu$ which does not necessarily coincide with the standard Riemannian measure.

The restriction $\delta < 1$ in the condition (ii) of Theorem 2.7 is essential. In § 3.4 we present an example of an operator that is not essentially self-adjoint and satisfies the conditions of Theorem 2.7 for any $\delta > 1$. Unfortunately, we do not know whether Theorem 2.7 remains valid if we admit $\delta = 1$ in condition (ii). However, in § 10 we prove the following theorem.

Theorem 2.13. *Suppose that V satisfies Assumptions* A *and* B. *We assume that the metric g^{TM} is complete and that the operator $H_V = D^*D + V$ is semibounded below on $C_c^\infty(E)$. Then H_V is essentially self-adjoint.*

This assertion was formulated in [79] for the case when H_V is a magnetic Schrödinger operator acting on scalar functions. Unfortunately, the paper [79] does not contain the details of the proof for singular potentials. However, these details can be found in the present paper.

Theorem 2.7 immediately implies the following corollary.

Corollary 2.14. *Assume that $V \in L^2_{\mathrm{loc}}(\operatorname{End} E)$ satisfies Assumption* A. *Suppose that $V = V_1 + V_2$, where $V_1, V_2 \in L^2_{\mathrm{loc}}(\operatorname{End} E)$, the operator $\delta D^*D + V_1$ is semibounded below on $C_c^\infty(E)$ for some $\delta < 1$, and $V_2 \geqslant -q$, where q satisfies the conditions* (i) *and* (iii) *of Theorem* 2.7. *Then the operator $H_V = D^*D + V$ is essentially self-adjoint.*

Remark 2.15. If M is $\mathbb{R}^n$ with the standard metric and measure and H_V is the sum $H_V = -\Delta + V$, where Δ is the standard Laplacian and V is a scalar (real-valued) function, then the semiboundedness condition in Corollary 2.14 is a consequence of the following more explicit condition on V_1: $V_1 \in L^p(\mathbb{R}^n)$, where $p \geqslant n/2$ for $n \geqslant 5$ and $p > 2$ for $n = 4$, and $V_1 \in L^2(\mathbb{R}^n)$ for $n \leqslant 3$. (See the references in Remark 2.2.)

It also suffices to assume that $V_1 \in S_n$ (the Stummel class) or $V_1 \in K_n$ (the Kato class); for the definitions and properties of these classes, see Appendix C.

2.16. Most recent history. Here we provide references to the most recent papers that were our source of inspiration. For additional historical comments, see Appendix D.

Theorem 2.7 generalizes recent papers of Oleinik [58] and Shubin [78]. The latter author considered the scalar magnetic Schrödinger operator $H_V = -\Delta_A + V$ on a complete Riemannian manifold, where $\Delta_A = d_A^* d_A$, A is a real sufficiently regular 1-form, and $d_A u = du + iuA$ is a deformed differential, while $V \in L^\infty_{\mathrm{loc}}(M)$. Keeping the assumption about L^∞_{loc}, Braverman [7] generalized the result of Oleinik to the case of Schrödinger operators on differential forms. Lesch [53] later noticed that one need not restrict oneself to Laplacian-type operators with isotropic symbols. He considered a generalized Schrödinger operator $H_V = D^*D + V$ on a complete Riemannian manifold, where D is as in the present paper, and established the following sufficient condition for the essential self-adjointness of H_V.

Theorem 2.17 (Lesch). *Let M be a complete Riemannian manifold. Assume that $V \in L^\infty_{\mathrm{loc}}(\operatorname{End} E)$ and $V \geqslant -q$, where $q \geqslant 1$ is a locally Lipschitz function, and let*

$$c(x) := \max\bigl\{1, \sup\{|\widehat{D}(x,\xi)| : |\xi|_{T^*_x M} \leqslant 1\}\bigr\}.$$

If

(i) *there is a constant $C > 0$ such that $c(x)|dq^{-1/2}(x)| \leqslant C$ for any $x \in M$,*
(ii) $\int^\infty \frac{1}{c\sqrt{q}}\, ds = \infty$,

then H_V is essentially self-adjoint.

We note that Theorem 2.7 does not assume *a priori* that M is a Riemannian manifold. It is more important that, whereas Lesch uses the function $c(x)$ constructed in an isotropic way (by taking the supremum over all vectors $\xi \in T^*M$ with $|\xi| = 1$), we use the symbol $\widehat{D}$ more effectively by considering the metric g^{TM}. In § 3.1 we construct an explicit operator for which Theorem 2.7 ensures essential self-adjointness, while the Lesch theorem gives no conclusion.

Remark 2.18. Lesch [53] considered a more general case in which the operator D need not be elliptic but satisfies the following condition: if $u \in \operatorname{Dom}(H_{V,\max})$, then $Du \in L^2_{\mathrm{loc}}(F)$. (It follows from Theorem 2.7 that this condition automatically holds if D is elliptic and V is sufficiently regular, for instance, if $V \in L^p_{\mathrm{loc}}(\operatorname{End} E)$, where p is as in Theorem 2.3 (i).) Theorem 2.7 can also be extended to the non-elliptic case.

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§ 3. Some examples

3.1. Example: an operator with anisotropic symbol. Let us give an example of an operator with anisotropic symbol (a function on T^*M depending not only on the norm of a covector but also on its direction). We use Corollary 2.10 to prove that this operator is essentially self-adjoint. This result cannot be obtained from the 'isotropic estimates' given in [58], [78], [53].

Let $M = \mathbb{R}^2$ (with the standard metric and measure) and let $V = 0$. We consider the operator

$$D = \begin{pmatrix} a(x,y)\dfrac{\partial}{\partial x} \\ b(x,y)\dfrac{\partial}{\partial y} \end{pmatrix},$$

where a and b are smooth, real-valued, and nowhere vanishing functions on $\mathbb{R}^2$. This operator is elliptic (overdetermined). We are interested in the operator

$$H := D^*D = -\frac{\partial}{\partial x}\left(a^2\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left(b^2\frac{\partial}{\partial y}\right).$$

The matrix of the inner product on T^*M determined by D via the formula (2.4) is $\operatorname{diag}(a^2/2, b^2/2)$. The matrix of the corresponding Riemannian metric g^{TM} on M is $\operatorname{diag}(2a^{-2}, 2b^{-2})$, that is, the metric itself is given by $ds^2 = 2a^{-2}dx^2 + 2b^{-2}dy^2$. By Corollary 2.10, to prove that the operator H is essentially self-adjoint, it suffices to show that the metric g^{TM} is complete, or, equivalently, that

$$\int^\infty ds = \infty, \quad \text{where } ds \text{ is the element of arc length associated with } g^{TM}. \tag{3.1}$$

Let $\gamma(t) = (x(t), y(t))$, $t \in [0,\infty)$, be a curve in $\mathbb{R}^2$ going to infinity; see § 2.5. Then the completeness condition (3.1) can be represented as

$$\int_0^\infty \frac{|x'(t)|}{a}\,dt + \int_0^\infty \frac{|y'(t)|}{b}\,dt = \infty.$$

The object of this example is to show that the last condition can hold even if the integral on the left-hand side of the condition (ii) in Theorem 2.17 converges. Roughly speaking, we intend to construct an example in which at least one of the integrals of the functions $1/a$ and $1/b$ is large, whereas the integral of $1/\sqrt{a^2+b^2}$ is small. This can be achieved, for instance, by setting

$$a(x,y) = (1-\cos(2\pi e^x))x^2 + 1,$$
$$b(x,y) = (1-\sin(2\pi e^y))y^2 + 1.$$

Let us show that the operator H is essentially self-adjoint under this choice of a and b. Let $\gamma(t)$ be as above. Then there is a number sequence $t_n \in [0,\infty)$ such that either $x(t_n) \to \infty$ or $y(t_n) \to \infty$ (as $n \to \infty$). Suppose that $x(t_n) \to \infty$ (the other case can be treated in a similar way). In this case

$$\int_0^{t_n} \frac{|x'(t)|}{a}\,dt \geqslant \int_0^{x(t_n)} \frac{dx}{a}.$$

Hence, letting $n \to \infty$, we get that

$$\int_0^\infty \frac{|x'(t)|}{a}\,dt \geqslant \int_0^\infty \frac{dx}{a}.$$

Thus, for (3.1) to be satisfied, it suffices to have

$$\int_0^\infty \frac{dx}{a} = \infty.$$

We denote by r_k the solutions of $\cos(2\pi e^x) = 1$, that is, $r_k = \log k$, $k = 1, 2, \dots$. Then there is an open interval I_k containing r_k for which $1 - \cos(2\pi e^x) \leqslant x^{-2}$ (that is, for which $|\sin(\pi e^x)| \leqslant (\sqrt{2}x)^{-1}$) for any $x \in I_k$. We note that

$$\frac{1}{a(x,y)} \geqslant \frac{1}{2} \quad \text{for any} \quad x \in I_k. \tag{3.2}$$

To find an upper bound for the length of I_k, we note that

$$\left|\sin(\pi e^{r_k + h})\right| = \left|\sin(\pi e^{r_k + h} - \pi e^{r_k})\right| \leqslant \pi e^{r_k}(e^h - 1)$$

for every $h > 0$. Thus, I_k includes, in particular, all numbers of the form $r_k + h$ with $0 < h < 1/2$ for which $\pi e^{r_k}(e^h - 1) < \dfrac{1}{\sqrt{2}(r_k + h)}$. Clearly, $h < r_k$ for $k \geqslant 2$. Therefore, if $k \geqslant 2$, then I_k contains all numbers $r_k + h$ such that $0 < h < 1/2$ and $\pi e^{r_k}(e^h - 1) < \dfrac{1}{2\sqrt{2}r_k}$. We also note that $h < e^h - 1 < 2h$ for $0 < h < 1/2$. It follows that I_k contains all numbers of the form $r_k + h$ for $0 < h \leqslant \dfrac{1}{4\sqrt{2}\pi(\log k)k}$. Hence, the length of I_k is greater than $\dfrac{1}{4\sqrt{2}\pi(\log k)k}$. By (3.2), this implies that for any $x_0 > 0$ there is a number k_0 for which

$$\int_{x_0}^\infty \frac{dx}{(1 - \cos(2\pi e^x))x^2 + 1} \geqslant \frac{1}{2} \sum_{k=k_0}^\infty \frac{1}{4\sqrt{2}\pi(\log k)k} = \infty.$$

Thus, $H = D^*D$ is essentially self-adjoint by Corollary 2.10.

It should be noted that the Lesch theorem gives no conclusion here. Indeed, the Lesch function c (see Theorem 2.17) is given by

$$c = \max\left\{1, \sup\{|a\xi_1 + b\xi_2| : \xi_1^2 + \xi_2^2 \leqslant 1\}\right\} = \max\left\{1, \sqrt{a^2 + b^2}\,\right\}.$$

Let us consider the curve $\gamma(t) = (t, t)$. Then $a^2 + b^2 \geqslant (3 - 2\sqrt{2})t^4$ at the point $\gamma(t)$, and condition (ii) of the Lesch theorem (Theorem 2.17) is clearly violated.

3.2. Example: an operator in a bounded domain. We consider the square $S = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}$. Let

$$D = \begin{pmatrix} (1 - x^2)\dfrac{\partial}{\partial x} \\ (1 - y^2)\dfrac{\partial}{\partial y} \end{pmatrix}.$$

It follows immediately from Corollary 2.10 that the operator

$$H := D^*D = -\frac{\partial}{\partial x}\left((1 - x^2)^2 \frac{\partial}{\partial x}\right) - \frac{\partial}{\partial y}\left((1 - y^2)^2 \frac{\partial}{\partial y}\right)$$

is essentially self-adjoint.

3.3. Example: a Coulomb-type potential. Let $M=\mathbb{R}^{3N}$ and let $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $\mathbb{R}^3$ be orthogonal coordinates for $\mathbb{R}^{3N}$. We set

$$H=-\sum_{i=1}^{N}\Delta_i-\sum_{i=1}^{N}\frac{1}{|\mathbf{x}_i|}+\sum_{i<j}^{N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|}+V(x), \tag{3.3}$$

where Δ_i is the Laplacian corresponding to the coordinates $\mathbf{x}_i$ and $V(x)$ is a locally bounded potential satisfying the following condition: there is a function $q(x)\geqslant 1$ such that the function $q^{-1/2}(x)$ is globally Lipschitz and

$$V(x)\geqslant -q(x),\qquad \int^{\infty}\frac{ds}{\sqrt{q}}=\infty. \tag{3.4}$$

Then H is essentially self-adjoint by Corollary 2.14 and Remark 2.15. Indeed, the terms $|\mathbf{x}_i|^{-1}$ and $|\mathbf{x}_i-\mathbf{x}_j|^{-1}$ are dominated by the Laplacian on $\mathbb{R}^{3N}$ both in the operator sense and in the sense of quadratic forms. This can be proved either by separation of variables as in the classical paper of Kato [46] or by using the Stummel and Kato classes (see Example C2 in Appendix C for the Stummel classes; the corresponding arguments can be repeated literally for the Kato classes). The space $L^{3N/2}(\mathbb{R}^{3N})$ contains none of these terms except for the case $N=1$.

To give a more specific example, we can set $V(x)=-|x|^2$ in (3.3) and claim that the operator

$$-\sum_{i=1}^{N}\Delta_i-\sum_{i=1}^{N}\frac{1}{|\mathbf{x}_i|}+\sum_{i<j}^{N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|}-\sum_{i=1}^{N}|\mathbf{x}_i|^2$$

is essentially self-adjoint (this also follows from a result of Kalf [39]). More generally, the operator (3.3) is essentially self-adjoint if there is a locally bounded function $r\colon[0,\infty)\to[1,\infty)$ such that $r^{-1/2}$ is globally Lipschitz and

$$V(x)\geqslant -r(|x|),\qquad \int_0^{\infty}\frac{dt}{\sqrt{r(t)}}=\infty. \tag{3.5}$$

For example, we can take $V(x)=-|x|^2\log(1+|x|)$.

We can also consider potentials which satisfy (3.4) but not (3.5). Constructions of rich families of such potentials can be found in [68] and [58]. The self-adjointness of operators of the form (3.3) cannot be established here by the method in [39].

3.4. Example: an operator that is not essentially self-adjoint. In this subsection we present an example of an operator that is not essentially self-adjoint but satisfies the conditions of Theorem 2.7 for every $\delta>1$.

Let us consider the operator $D\colon C_c^\infty(\mathbb{R})\to C_c^\infty(\mathbb{R})$ given by $D=\dfrac{d}{dx}+|x|^\alpha$, with $\alpha>3$. Then $D^*=-\dfrac{d}{dx}+|x|^\alpha$, and hence

$$D^*D=-\frac{d^2}{dx^2}-\alpha|x|^{\alpha-1}\operatorname{sign}x+|x|^{2\alpha}.$$

Let $V=-|x|^{2\alpha}$ and $H_V=D^*D+V$. Then

$$H_V=-\frac{d^2}{dx^2}-\alpha|x|^{\alpha-1}\operatorname{sign}x.$$

This operator *is not essentially self-adjoint*, because $\alpha-1>2$ (see Example 1.1 in § 3.1 of [5]). However,

$$\delta D^*D+V=-\delta\frac{d^2}{dx^2}-\delta\alpha|x|^{\alpha-1}\operatorname{sign}x+(\delta-1)|x|^{2\alpha}$$

for any $\delta>1$. Since $2\alpha>\alpha-1$ and $\delta-1>0$, there is a number $C_\delta>0$ such that $-\delta\alpha|x|^{\alpha-1}\operatorname{sign}x+(\delta-1)|x|^{2\alpha}>-C_\delta$ for any $x\in\mathbb{R}$. Hence, $\delta D^*D+V>-C_\delta$.

We conclude that δD^*D+V is semibounded below for every $\delta>1$. Therefore, Theorem 2.7 fails for $\delta>1$.

§ 4. Proof of the first part of Theorem 2.3

In this section we assume that the potential satisfies the conditions imposed in Theorem 2.3 (i).

Lemma 4.1. $Vu\in L^2_{\mathrm{loc}}(E)$ *for any* $u\in\operatorname{Dom}(H_{V,\max})$.

Proof. Assume that $u\in\operatorname{Dom}(H_{V,\max})$, that is, $u\in L^2(E)$ and

$$D^*Du+Vu=f\in L^2(E). \tag{4.1}$$

We set $t_1:=2$. Since $V\in L^p_{\mathrm{loc}}(\operatorname{End}E)$, it follows from the Hölder inequality that $Vu\in L^{s_1}_{\mathrm{loc}}(E)$, where

$$\frac{1}{s_1}=\frac{1}{2}+\frac{1}{p}=\frac{1}{t_1}+\frac{1}{p}\,.$$

Clearly, $1\leqslant s_1<2$. Moreover, if $n>3$, then $p>2$ and $s_1>1$.

We can now improve the exponent s_1 as follows. By (4.1), $D^*Du=f-Vu=h\in L^{s_1}_{\mathrm{loc}}(E)$. If $s_1>1$, then $u\in W^{2,s_1}_{\mathrm{loc}}(E)$ by standard results on elliptic regularity (see. [87], § 6.5). Therefore, by the Sobolev embedding theorem (cf. [1], Theorem 5.4 or [50], Theorem 4.5.8) we can conclude that $u\in L^{t_2}_{\mathrm{loc}}(E)$, where

$$\frac{1}{t_2}=\frac{1}{s_1}-\frac{2}{n}=\frac{1}{t_1}+\frac{1}{p}-\frac{2}{n}\,,$$

and hence $Vu\in L^{s_2}_{\mathrm{loc}}$ by the Hölder inequality, where

$$\frac{1}{s_2}=\frac{1}{t_2}+\frac{1}{p}=\frac{1}{s_1}-\frac{2}{n}+\frac{1}{p}=\frac{1}{s_1}-\frac{2}{n}\left(1-\frac{n}{2p}\right).$$

We can proceed in this way to obtain the series of relations $u\in L^{t_k}_{\mathrm{loc}}(E)$, $Vu\in L^{s_k}_{\mathrm{loc}}(E)$, $k=1,2,\dots$, with

$$\frac{1}{t_{k+1}}=\frac{1}{s_k}-\frac{2}{n}\,,\qquad \frac{1}{s_{k+1}}=\frac{1}{s_k}-\frac{2}{n}\left(1-\frac{n}{2p}\right),$$

until we obtain the inequality $s_k > 2$ at some step, which makes the next step impossible due to the term $f \in L^2(E)$ in (4.1). Then we can conclude that $Vu \in L^2_{\mathrm{loc}}(E)$ as desired.

We now assume that $n \leqslant 3$, and hence $p = 2$, and at the initial step we have only the relation $Vu \in L^1_{\mathrm{loc}}(E)$. Then $W^{1,q}_{\mathrm{loc}}(E) \subset C(E)$ for $q > n$ by another Sobolev embedding theorem (see [1], Case C of Theorem 5.4; $C(E)$ stands here for the set of continuous sections of the bundle E). Using standard duality arguments (see [1], §§ 3.6–3.13), we get that $L^1_{\mathrm{loc}}(E) \subset W^{-1,q'}_{\mathrm{loc}}(E)$ for $1 < q' < n/(n-1)$. Therefore, $D^*Du \in W^{-1,\tilde{q}}_{\mathrm{loc}}(E)$, where $\tilde{q} = \min\{q', 2\}$. It follows from the standard results on the elliptic regularity (see [87], Chap. 6.5) that $u \in W^{1,\tilde{q}}_{\mathrm{loc}}(E)$. In particular, $u \in L^{\tilde{q}}_{\mathrm{loc}}(E)$ for such $\tilde{q}$. Since $\tilde{q} > 1$, we can argue as in the first part of the proof and again show that $Vu \in L^2_{\mathrm{loc}}(E)$.[2]

4.2. Proof of Theorem 2.3 (i). If $u \in D(H_{V,\max})$, then $D^*Du = H_V u - Vu \in {L_{\mathrm{loc}}}^2(E)$. By elliptic regularity we have $u \in W^{2,2}_{\mathrm{loc}}(E)$.

§ 5. Kato's inequality for the Bochner Laplacian

In this section we prove a generalization of Kato's inequality to the case of the Bochner Laplacian on a manifold M endowed with a Riemannian metric $g = g^{TM}$ and a measure $d\mu = \rho\,dx$. Here dx stands for the Riemannian volume form on M and $\rho\colon M \to (0,\infty)$ for a smooth function.

We first establish a 'smooth version' of Kato's inequality; see Proposition 5.9. A similar inequality was proved by Hess, Schrader, and Uhlenbrock [35] for the measure $d\mu$ defined by the Riemannian volume form on M. Later we prove in § 5.15 an L^1_{loc} version of Kato's inequality.

5.1. A pairing on the space of forms with values in bundles. Let E be a Hermitian vector bundle over M and let $\overline{E}$ be the complex conjugate of E. It is identical to E, but multiplication by a $\lambda \in \mathbb{C}$ is defined as the original multiplication by $\overline{\lambda}$. The identity map $E \to \overline{E}$ is called *complex conjugation*, and it defines an antilinear isomorphism $E \xrightarrow{\sim} \overline{E}$. The image of a vector $v \in E_x$ under this isomorphism is denoted by $\overline{v}$. The Hermitian structure on E defines a complex linear map

$$\langle\ \rangle\colon E \otimes \overline{E} \to \mathbb{C}, \qquad a \otimes \overline{b} \mapsto \langle a \otimes \overline{b}\rangle := \langle a, b\rangle. \tag{5.1}$$

Let $\Lambda^i = T^*M \wedge \dots \wedge T^*M$ be the ith exterior power of the cotangent bundle of M. The space of smooth sections of the tensor product $\Lambda^i \otimes E$ is called the *space of E-valued differential i-forms on* M and is denoted by $\Omega^i(M,E)$. The map (5.1) extends naturally to maps

$$\langle\ \rangle\colon \Lambda^k \otimes E \otimes \overline{E} \to \Lambda^k, \qquad \langle\ \rangle\colon \Omega^k(M, E \otimes \overline{E}) \to \Omega^k(M,\mathbb{C}).$$

If $\alpha \in \Omega^i(M,E)$ and $\beta \in \Omega^j(M,E)$, then $\overline{\beta} \in \Omega^j(M,\overline{E})$, $\alpha \wedge \overline{\beta} \in \Omega^{i+j}(M, E \otimes \overline{E})$, and $\langle \alpha \wedge \overline{\beta}\rangle \in \Omega^{i+j}(M,\mathbb{C})$. If one of the forms α and β belongs to $\Omega^0(M,E) \equiv C^\infty(E)$, then we omit the symbol $\wedge$ in our notation and simply write $\langle \alpha\overline{\beta}\rangle$.

---

[2] We note that, to apply the considerations on elliptic regularity to the equation $D^*Du = f \in W^{-1,t}_{\mathrm{loc}}(E)$, we must have $t > 1$. If $t = 1$, then the equation does not imply that $u \in W^{1,t}_{\mathrm{loc}}(E)$; see [36], Theorem 7.9.8. For this reason we must start the proof of the lemma from the proof of the relation $Vu \in L^t_{\mathrm{loc}}(E)$ for $t > 1$.

Let $*\colon \Lambda^i \to \Lambda^{n-i}$ be the Hodge $*$-operator [89]. This operator extends naturally to the spaces $\Lambda^i \otimes E$ and $\Omega^i(M,E)$.

The formula

$$\langle \alpha, \beta \rangle_{\Lambda^i \otimes E} := *^{-1} \langle \alpha \wedge *\overline{\beta} \rangle \in \Lambda^0 \cong \mathbb{C}, \qquad \alpha, \beta \in \Lambda^i \otimes E, \quad i = 0, \dots, n,$$

defines a non-degenerate Hermitian inner product on $\Lambda^i \otimes E$, and we write

$$|\alpha| := \langle \alpha, \alpha \rangle^{1/2}_{\Lambda^i \otimes E}.$$

We note that, if $\alpha, \beta \in \Lambda^0 \otimes E \cong E$, then $\langle \alpha, \beta \rangle_{\Lambda^i \otimes E}$ coincides with the original Hermitian inner product $\langle \alpha, \beta \rangle$ on E.

Similarly, if $\alpha, \beta \in \Omega^i(M,E)$, then the quantities $\langle \alpha, \beta \rangle_{\Lambda^i \otimes E} \in C^\infty(M)$ and $|\alpha| \in C(M)$ are well defined.

5.2. The covariant derivative and its dual operator. Let ∇ be a connection on E. It defines a linear map $\nabla\colon \Omega^*(M,E) \to \Omega^{*+1}(M,E)$ such that

$$\nabla(\omega \wedge \beta) = d\omega \wedge \beta + (-1)^i \omega \wedge \nabla\beta, \qquad \omega \in \Omega^i(M), \quad \beta \in \Omega^j(M,E).$$

In this paper we always assume that ∇ is a Hermitian connection, which means that the following equality holds:

$$d\langle \alpha \wedge \overline{\beta} \rangle = \langle \nabla\alpha \wedge \overline{\beta} \rangle + (-1)^i \langle \alpha \wedge \overline{\nabla\beta} \rangle, \qquad \alpha \in \Omega^i(M,E), \quad \beta \in \Omega^j(M,E), \tag{5.2}$$

where $d : \Omega^*(M,\mathbb{C}) \to \Omega^{*+1}(M,\mathbb{C})$ is the de Rham differential.

Let $\Omega^i_{\mathrm{comp}}(M,E)$, $i = 0, 1, \dots, n$, be the space of compactly supported differential forms. We consider the following inner product on the space $\Omega^i_{\mathrm{comp}}(M,E)$:

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_{\Lambda^i \otimes E} \, d\mu = \int_M \langle \alpha \wedge *\overline{\beta} \rangle \rho, \qquad \alpha, \beta \in \Omega^i_{\mathrm{comp}}(M,E).$$

Let $\nabla^* : \Omega^*(M,E) \to \Omega^{*-1}(M,E)$ be the formal adjoint to ∇, that is, $(\nabla\alpha, \beta) = (\alpha, \nabla^*\beta)$ for all $\alpha, \beta \in \Omega^*_{\mathrm{comp}}(M,E)$.

Lemma 5.3. *Suppose that $\beta \in \Omega^j(M,E)$. Then*

$$\nabla^*\beta = (-1)^j *^{-1} \nabla * \beta + (-1)^j *^{-1} \frac{d\rho}{\rho} \wedge *\beta.$$

Proof. Let us fix some $\alpha \in \Omega^{j-1}_{\mathrm{comp}}(M,E)$. Using (5.2), we get that

$$\begin{aligned}
(\nabla\alpha, \beta) &= \int_M \langle \nabla\alpha \wedge *\overline{\beta} \rangle \rho \\
&= \int_M d\langle \alpha\rho \wedge *\overline{\beta} \rangle - (-1)^{j-1} \int_M \langle \alpha \wedge d\rho \wedge *\overline{\beta} \rangle - (-1)^{j-1} \int_M \langle \alpha\rho \wedge \overline{\nabla * \beta} \rangle \\
&= (-1)^j \int_M \left\langle \alpha \wedge * *^{-1} \left(\frac{d\rho}{\rho} \wedge *\overline{\beta} \right) \right\rangle \rho + (-1)^j \int_M \langle \alpha \wedge * *^{-1} \, \overline{\nabla * \beta} \rangle \rho \\
&= (-1)^j \left(\alpha, *^{-1} \nabla * \beta + *^{-1} \frac{d\rho}{\rho} \wedge *\beta \right),
\end{aligned}$$

where the third equality holds because $\int_M d\langle \alpha\rho * \overline{\beta}\rangle = 0$ by the Stokes theorem. The lemma is proved.

To simplify the notation, let $A_\rho\beta = *^{-1}\dfrac{d\rho}{\rho} \wedge *\beta$. This implies that

$$\nabla^* = (-1)^j *^{-1} \nabla * + (-1)^j A_\rho \colon \Omega^j(M,E) \to \Omega^{j-1}(M,E). \tag{5.3}$$

If $E = M \times \mathbb{C}$ is the trivial line bundle, then

$$d^* = (-1)^j *^{-1} d * + (-1)^j A_\rho \colon \Omega^j(M) \to \Omega^{j-1}(M). \tag{5.4}$$

In this paper we use relations (5.3) and (5.4) only for $j = 1$. Clearly, if $\alpha \in \Omega^0(M,E)$, $\beta \in \Omega^j(M,E)$, and $\phi \in C^\infty(M)$, then

$$A_\rho\langle \alpha\overline{\beta}\rangle = \langle \alpha A_\rho\overline{\beta}\rangle, \qquad A_\rho(\phi\beta) = \phi A_\rho\beta. \tag{5.5}$$

Definition 5.4. By the *Bochner Laplacian* we mean the operator $\nabla^*\nabla \colon C^\infty(E) \to C^\infty(E)$. Similarly, we define the *Laplacian* $\Delta_M = d^*d \colon C^\infty(M) \to C^\infty(M)$.

We note that, if $d\mu$ is the Riemannian volume form on M, then $\Delta_M = -\Delta_g$, where Δ_g is the *metric* Laplacian, $\Delta_g u = \operatorname{div}(\operatorname{grad} u)$.

Remark 5.5. In § 2.5 we introduced a metric g^{TM} associated with a first-order elliptic differential operator D. It can readily be seen that for $D = \nabla \colon C^\infty(E) \to \Omega^1(M,E)$ the metric g^{TM} coincides with the metric g (it is now assumed that the latter metric is given *a priori*).

5.6. Kato's inequality. We recall that a distribution ν on M is said to be *positive* (and this is denoted by $\nu \geqslant 0$) if $(\nu,\phi) \geqslant 0$ for every non-negative function $\phi \in C_c^\infty(M)$. Hence, ν is in fact a positive Radon measure (see, for instance, [31], Chap. II, Theorem 1 of § 2). We write $\nu_1 \geqslant \nu_2$ if $\nu_1 - \nu_2 \geqslant 0$.

The main result of this section is the following theorem.

Theorem 5.7. *Suppose that $u \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla u \in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M |u| \leqslant \operatorname{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle, \tag{5.6}$$

where

$$\operatorname{sign} u(x) = \begin{cases} \dfrac{u(x)}{|u(x)|} & \text{for } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us stress that our Laplacians are *positive* operators. This explains the disagreement between the formula (5.6) and the standard form of Kato's inequality for scalar functions on $\mathbb{R}^n$; see, for example, Theorem X.27 in [64].

The rest of the section is devoted to the proof of Theorem 5.7.

5.8. Kato's inequality in the C^∞ case. Let $u \in C^\infty(E)$. For $\epsilon > 0$ we set

$$|u|_\epsilon = (|u|^2 + \epsilon^2)^{1/2}. \tag{5.7}$$

Proposition 5.9. *Suppose that $u \in C^\infty(E)$. Then*

$$|u|_\epsilon \Delta_M |u|_\epsilon \leqslant \operatorname{Re}\langle \nabla^* \nabla u, u\rangle. \tag{5.8}$$

Proof. Let us fix some $u \in C^\infty(E)$. Using (5.2), we get that

$$\begin{aligned} 2|u|_\epsilon d|u|_\epsilon = d|u|_\epsilon^2 = d|u|^2 = d\langle u, u\rangle &= \langle (\nabla u)\overline{u}\rangle + \langle u(\overline{\nabla u}\,)\rangle \\ &= 2\operatorname{Re}\langle (\nabla u)\overline{u}\rangle = 2\operatorname{Re}\langle u\overline{\nabla u}\rangle, \end{aligned} \tag{5.9}$$

and hence

$$|u|_\epsilon\, |d|u|_\epsilon| \leqslant |u|\, |\nabla u|.$$

Since $|u|_\epsilon \geqslant |u|$, the above inequality implies that

$$|d|u|_\epsilon| \leqslant |\nabla u|. \tag{5.10}$$

Furthermore, using (5.5), we see that

$$\begin{aligned} d^*(|u|_\epsilon d|u|_\epsilon) &= - *^{-1} d * (|u|_\epsilon d|u|_\epsilon) - |u|_\epsilon A_\rho d|u|_\epsilon \\ &= - *^{-1} d(|u|_\epsilon * d|u|_\epsilon) - |u|_\epsilon A_\rho d|u|_\epsilon \\ &= - *^{-1} (d|u|_\epsilon \wedge *d|u|_\epsilon) - |u|_\epsilon *^{-1} d * d|u|_\epsilon - |u|_\epsilon A_\rho d|u|_\epsilon \\ &= -|d|u|_\epsilon|^2 - |u|_\epsilon (*^{-1} d * + A_\rho) d|u|_\epsilon \\ &= -|d|u|_\epsilon|^2 + |u|_\epsilon \Delta_M |u|_\epsilon. \end{aligned} \tag{5.11}$$

Similarly,

$$\begin{aligned} d^*\langle u\overline{\nabla u}\rangle &= - *^{-1} d * \langle u\overline{\nabla u}\rangle - A_\rho \langle u\overline{\nabla u}\rangle \\ &= - *^{-1} d\langle u * \overline{\nabla u}\rangle - \langle u A_\rho \overline{\nabla u}\rangle \\ &= - *^{-1} \langle \nabla u \wedge *\overline{\nabla u}\rangle - *^{-1}\langle u\overline{\nabla * \nabla u}\rangle - \langle u A_\rho \overline{\nabla u}\rangle \\ &= -|\nabla u|^2 + \langle u\overline{\nabla^* \nabla u}\rangle. \end{aligned} \tag{5.12}$$

It follows from (5.9), (5.11), and (5.12) that

$$-|d|u|_\epsilon|^2 + |u|_\epsilon \Delta_M |u|_\epsilon = -|\nabla u|^2 + \operatorname{Re}\langle u\overline{\nabla^* \nabla u}\rangle = -|\nabla u|^2 + \operatorname{Re}\langle (\nabla^* \nabla u)\overline{u}\rangle,$$

and by (5.10) this implies (5.8) and completes the proof of the proposition.

Corollary 5.10. *Suppose that $u \in C^\infty(E)$. Then the following inequality holds for distributions*:

$$\Delta_M |u| \leqslant \operatorname{Re}\langle \nabla^* \nabla u, \operatorname{sign} u\rangle.$$

Remark 5.11. Analogues of Proposition 5.9 and Corollary 5.10 were obtained by Hess, Schrader, and Uhlenbrock [35] for the case in which $d\mu$ is the Riemannian volume form on M (and hence $\rho \equiv 1$). These authors used a slightly more complicated definition of $\operatorname{sign} u$, which is in fact unnecessary because it differs from our definition only on a set of measure zero.

5.12. Friedrichs mollifiers. Let us now derive Theorem 5.7 from Proposition 5.9 by using the technique of Friedrichs mollifiers. For the convenience of the reader we briefly review the basic definitions and results of Friedrichs' paper [29].

Suppose that $j \in C_c^\infty(\mathbb{R}^n)$, $j(z) \geqslant 0$ for any $z \in \mathbb{R}^n$, $j(z) = 0$ for $|z| \geqslant 1$, and $\int_{\mathbb{R}^n} j(z)\,dz = 1$.

For $\rho > 0$ and $x \in \mathbb{R}^n$ we set $j_\rho(x) = \rho^{-n} j(\rho^{-1}x)$. Then $j_\rho \in C_c^\infty(\mathbb{R}^n)$, $j_\rho \geqslant 0$, $j_\rho(x) = 0$ for $|x| \geqslant \rho$, and $\int_{\mathbb{R}^n} j_\rho(x)\,dx = 1$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $f \in L^1_{\mathrm{loc}}(\Omega)$. We define

$$(J^\rho f)(x) = f^\rho(x) = \int f(x-y) j_\rho(y)\,dy = \int j_\rho(x-y) f(y)\,dy, \tag{5.13}$$

where the integration is over $\mathbb{R}^n$. Then $J^\rho f(x)$ is defined for any $x \in \Omega_\rho$, where

$$\Omega_\rho = \bigl\{x : x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \rho\bigr\}.$$

We recall the following standard lemma (see [36], Theorem 1.3.2).

Lemma 5.13. *Let $1 \leqslant p < \infty$. In the notation of* (5.13):

(i) *if $f \in L^p_{\mathrm{loc}}(\Omega)$, then $J^\rho f \in C^\infty(\Omega_\rho)$;*

(ii) *if $f \in L^p_{\mathrm{loc}}(\Omega)$, then $J^\rho f \to f$ as $\rho \to 0$ in the norm of $L^p(K)$, where K is any compact set in Ω;*

(iii) *if $f \in C(\Omega)$, then $J^\rho f \to f$ uniformly on any compact set $K \subset \Omega$ as $\rho \to 0$.*

To apply the technique of Friedrichs mollifiers to the operator $\nabla^*\nabla$, we choose a coordinate neighbourhood U of M with coordinates $x^1, x^2, \dots, x^n$. Choosing a local frame in E over U, we identify the space of sections of E over U with the space of vector functions on U. Under this identification, the Bochner Laplacian $\nabla^*\nabla$ becomes a second-order elliptic differential operator which can be represented in the form

$$\sum_{i,k} a_{ik}(x)\partial_i\partial_k + \sum_i b_i(x)\partial_i + c(x),$$

where a_{ik}, b_i, and c are $m \times m$ matrices whose entries are smooth functions ($m = \dim E$). Alternatively, we can represent the operator in the form

$$\sum_{i,k} \partial_i\, a_{ik}(x)\partial_k + \sum_i b_i(x)\partial_i + c(x) \tag{5.14}$$

with possibly different functions a_{ik}, b_i, and c. The latter form is sometimes more convenient if the coefficients are not smooth.

We can also consider $u^\rho = \mathcal{J}^\rho u$, where $\mathcal{J}^\rho$ is the integral operator whose integral kernel is $j_\rho(x-y)\,\mathrm{Id}$, where Id is the $m \times m$ identity matrix.

We now state a crucial proposition whose proof is given in Appendix A.

Proposition 5.14. *Suppose that $u \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla u \in L^1_{\mathrm{loc}}(E)$. In this case $\nabla^*\nabla u^\rho \to \nabla^*\nabla u$ in $L^1_{\mathrm{loc}}(E)$ over U as $\rho \to 0+$.*

5.15. Proof of Theorem 5.7. We first note that the statement is in fact local. Namely, by using a partition of unity, we can see that it suffices to prove the inequality over every coordinate neighbourhood in some covering of M. Let us fix such a neighbourhood U and apply Proposition 5.14. Clearly, $u^\rho \in C^\infty(E|_{U_\rho})$. By Proposition 5.9,

$$\Delta_M |u^\rho|_\epsilon \leqslant \mathrm{Re}\left\langle \nabla^* \nabla u^\rho, \frac{u^\rho}{|u^\rho|_\epsilon} \right\rangle \tag{5.15}$$

for any $\epsilon > 0$. In this inequality we first fix $\epsilon > 0$ and then pass to the limit as $\rho \to 0+$.

Let us consider the left-hand side of the inequality (5.15). Clearly,

$$\big| |u^\rho|_\epsilon - |u|_\epsilon \big| \leqslant \big| |u^\rho| - |u| \big| \leqslant |u^\rho - u|. \tag{5.16}$$

Since $u^\rho \to u$ in $L^1_{\mathrm{loc}}(E|_U)$, it follows from (5.16) that $|u^\rho|_\epsilon \to |u|_\epsilon$ in $L^1_{\mathrm{loc}}(U)$ as $\rho \to 0+$. This immediately gives $\Delta_M |u^\rho|_\epsilon \to \Delta_M |u|_\epsilon$ in the distribution sense over U (or, more precisely, over any relatively compact open subset of U). We now turn to the right-hand side of (5.15). Our object is to show that

$$\mathrm{Re}\left\langle \nabla^* \nabla u^\rho, \frac{u^\rho}{|u^\rho|_\epsilon} \right\rangle - \mathrm{Re}\left\langle \nabla^* \nabla u, \frac{u}{|u|_\epsilon} \right\rangle \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(U) \tag{5.17}$$

as $\rho \to 0+$. By adding and subtracting the term $\mathrm{Re}\langle \nabla^* \nabla u, |u^\rho|_\epsilon^{-1} u^\rho \rangle$ on the left-hand side of (5.17), we obtain

$$\mathrm{Re}\left\langle \nabla^* \nabla u^\rho - \nabla^* \nabla u, \frac{u^\rho}{|u^\rho|_\epsilon} \right\rangle + \mathrm{Re}\left\langle \nabla^* \nabla u, \frac{u^\rho}{|u^\rho|_\epsilon} - \frac{u}{|u|_\epsilon} \right\rangle. \tag{5.18}$$

Let us pass to the limit as $\rho \to 0+$. By Proposition 5.14, $\nabla^* \nabla u^\rho \to \nabla^* \nabla u$ in $L^1_{\mathrm{loc}}(E|_U)$. As we know, $||u^\rho|_\epsilon^{-1} u^\rho| < 1$. Therefore, the first term of the sum in (5.18) tends to 0 in $L^1_{\mathrm{loc}}(U)$ (and hence in the distribution sense).

As is known, $\nabla^* \nabla u \in L^1_{\mathrm{loc}}(E)$. We have already shown that $|u^\rho|_\epsilon \to |u|_\epsilon$ in $L^1_{\mathrm{loc}}(U)$ (as $\rho \to 0+$) and hence, after passing to a subsequence, almost everywhere. Therefore, for the same subsequence we have $|u^\rho|_\epsilon^{-1} u^\rho \to |u|_\epsilon^{-1} u$ almost everywhere. Thus, it follows from the dominated convergence theorem that the second term of the sum in (5.18) tends to 0 in $L^1_{\mathrm{loc}}(U)$ (and hence in the distribution sense) along some sequence of positive numbers ρ.

Therefore, for any $u \in L^1_{\mathrm{loc}}(E)$ such that $\nabla^* \nabla u \in L^1_{\mathrm{loc}}(E)$ we get that

$$\Delta_M |u|_\epsilon \leqslant \mathrm{Re}\left\langle \nabla^* \nabla u, \frac{u}{|u|_\epsilon} \right\rangle. \tag{5.19}$$

Let us pass to the limit in this inequality in the distribution sense as $\epsilon \to 0+$. We take a real-valued function $\phi \in C_c^\infty(M)$ and consider the relation

$$\int \phi \Delta_M |u|_\epsilon \, d\mu = \int |u|_\epsilon \Delta_M \phi \, d\mu \to \int |u| \Delta_M \phi \, d\mu. \tag{5.20}$$

The limit in (5.20) can be regarded as the value of the distribution $\Delta_M |u|$ on the test function ϕ. Therefore, $\Delta_M |u|_\epsilon \to \Delta_M |u|$ in the distribution sense as $\epsilon \to 0+$.

On the right-hand side of (5.19) we have $|u|_\epsilon^{-1} u \to \operatorname{sign} u$ almost everywhere and with the uniform bound $\big| |u|_\epsilon^{-1} u \big| \leqslant 1$. Since $\nabla^* \nabla u \in L^1_{\mathrm{loc}}(E)$, the dominated convergence theorem shows that the right-hand side tends to $\mathrm{Re}\,\big\langle \nabla^* \nabla u, \operatorname{sign} u \big\rangle$ in $L^1_{\mathrm{loc}}(M)$ (and hence in the distribution sense). This proves the theorem.

§ 6. The case in which the support of V is small

In this section we prove the special case of Theorem 2.7 in which $D = \nabla$ and the potential V is supported in a coordinate neighbourhood $W \subset M$.

Proposition 6.1. *Suppose that M is a manifold endowed with a Riemannian metric g^{TM} and a smooth measure $d\mu$. Let $W \subset M$ be a relatively compact coordinate neighbourhood in M and let $V = V_+ + V_-$ be as in Assumption* A. *Suppose in addition that* $\operatorname{supp} V \subset W$. *Then the operator*

$$H_V = \nabla^* \nabla + V$$

is essentially self-adjoint on $C_c^\infty(E)$.

We precede the proof with several lemmas which we also use in subsequent sections.

Lemma 6.2. *If Assumption* A *is satisfied and* $\operatorname{supp} V_- \subset W$, *then there are positive constants $a < 1$ and C_a such that*

$$|(V_- u, u)| \leqslant a \, \|\nabla u\|^2 + C_a \|u\|^2, \qquad u \in C_c^\infty(E). \tag{6.1}$$

Proof. Let $\overline{W}$ be the closure of W in M. By (2.2), there are constants $a_{\overline{W}}$, $0 \leqslant a_{\overline{W}} < 1$, and $C_{\overline{W}} > 0$ such that

$$\| \, |V_-| v\| \leqslant a_{\overline{W}} \|\Delta_M v\| + C_{\overline{W}} \|v\|, \qquad v \in C_c^\infty(M).$$

In other words, the operator $|V_-|$ is dominated by Δ_M. Hence, by Theorem X.18 in [64], the quadratic form of $|V_-|$ is dominated by the quadratic form of Δ_M. More precisely, for any $a \in (a_{\overline{W}}, 1)$ there is a constant $C_a > 0$ such that

$$(|V_-| v, v) \leqslant a(\Delta_M v, v) + C_a \|v\|^2 = a\|dv\|^2 + C_a \|v\|^2, \qquad v \in C_c^\infty(M).$$

Let us apply this inequality to $v = |u|_\epsilon$, where $u \in C_c^\infty(E)$ and the function $|u|_\epsilon$ is defined as in (5.7). By the inequality (5.10) we get that

$$|(V_- u, u)| \leqslant (|V_-| \, |u|_\epsilon, |u|_\epsilon) \leqslant a\|d|u|_\epsilon\|^2 + C_a \| \, |u|_\epsilon\|^2 \leqslant a\|\nabla u\|^2 + C_a \| \, |u|_\epsilon\|^2.$$

Passing to the limit as $\epsilon \to 0+$, we obtain (6.1). This completes the proof of the lemma.

Corollary 6.3. *If Assumption* A *holds and if* $\operatorname{supp} V_- \subset W$, *then the operator H_V is semibounded below on $C_c^\infty(E)$, that is, there is a constant $C > 0$ such that*

$$(H_V u, u) \geqslant -C\|u\|^2 \quad \textit{for any} \quad u \in C_c^\infty(E).$$

The next lemma follows immediately from (2.3).

Lemma 6.4. *Suppose that $H_V = D^* D + V$ is as in* (2.1). *The distribution equality*

$$H_V(\psi u) = \psi H_V u + D^* \big(\widehat{D}(d\psi) u\big) - \big(\widehat{D}(d\psi)\big)^* D u$$

holds for any $\psi \in C^2(M)$ and $u \in L^2_{\mathrm{loc}}(E)$.

Corollary 6.5. *Suppose that* $u \in \mathrm{Dom}(H_{V,\max})$. *Let* $\psi \in C^2(M)$ *be a function for which the support* $\mathrm{supp}(d\psi)$ *is compact. If the restriction of the function* u *to an open neighbourhood of* $\mathrm{supp}(d\psi)$ *belongs to* $W^{1,2}_{\mathrm{loc}}$, *then* $\psi u \in \mathrm{Dom}(H_{V,\max})$.

6.6. Proof of Proposition 6.1. Let us fix an open neighbourhood $U \subset W$ of $\mathrm{supp}\, V$ with closure $\overline{U}$ contained in W. Suppose that $\phi, \widetilde{\phi}\colon M \to [0,1]$ are smooth functions such that $\phi^2 + \widetilde{\phi}^2 \equiv 1$, the restriction of ϕ to U is identically equal to 1, and $\mathrm{supp}\, \phi \subset W$. Therefore, $\mathrm{supp}\, \widetilde{\phi} \subset M \setminus U$.

By the IMS localization formula (see [23], §3.1 or [76], Lemma 3.1) we have

$$\nabla^*\nabla = \phi\nabla^*\nabla\phi + \widetilde{\phi}\nabla^*\nabla\widetilde{\phi} + \frac{1}{2}\big[[\nabla^*\nabla, \phi], \phi\big] + \frac{1}{2}\big[[\nabla^*\nabla, \widetilde{\phi}], \widetilde{\phi}\big], \tag{6.2}$$

where $[\,\cdot\,,\cdot\,]$ stands for the commutator bracket. We write

$$J = \frac{1}{2}\big[[\nabla^*\nabla, \phi], \phi\big] + \frac{1}{2}\big[[\nabla^*\nabla, \widetilde{\phi}], \widetilde{\phi}\big] = -\frac{1}{2}|d\phi|^2 - \frac{1}{2}|d\widetilde{\phi}|^2.$$

Then J is a smooth function on M whose support is contained in $W \setminus U$. In this notation, the formula (6.2) becomes

$$\nabla^*\nabla = \phi\nabla^*\nabla\phi + \widetilde{\phi}\nabla^*\nabla\widetilde{\phi} + J. \tag{6.3}$$

Let the constant $C > 0$ be as in Corollary 6.3. Let $b \gg C$ be a large number (which will be chosen below). Suppose that $u \in L^2(E)$ satisfies the equality

$$(H_V + b)u = 0. \tag{6.4}$$

Here the expression $H_V u$ is understood in the distribution sense. To prove the proposition, it suffices to show that $u = 0$ (see, for instance, [32] or Theorem X.26 in [64]). Let us argue by contradiction and assume that $u \neq 0$.

Since the restriction $V|_{M\setminus U}$ of V to $M \setminus U$ vanishes, it follows from the formula (6.4) that $\nabla^*\nabla u|_{M\setminus U} = -bu|_{M\setminus U} \in L^2(E)$. Hence, $u|_{M\setminus U} \in W^{2,2}_{\mathrm{loc}}$ by elliptic regularity. According to Corollary 6.5, the sections ϕu and $\widetilde{\phi}u$ belong to $\mathrm{Dom}(H_{V,\max})$. We also note that $H_V = \nabla^*\nabla$ on the support of $\widetilde{\phi}$. As is well known (see, for example, [35]; this fact also follows from Proposition 8.11 below), the operator $\nabla^*\nabla$ is essentially self-adjoint (we note that Assumption A of § 2.1 is true by assumption in this case, whereas Assumption B follows from elliptic regularity). Hence, by taking the closure (see also the arguments in the proof of Proposition 8.13), we get that

$$(\widetilde{\phi}\nabla^*\nabla(\widetilde{\phi}u), u) = \|\nabla(\widetilde{\phi}u)\|^2 \geqslant 0, \tag{6.5}$$

with equality only if $\widetilde{\phi}u = 0$.

Let $\widetilde{b} = b - \max_{x\in M} |J(x)|$, and take b large enough that $\widetilde{b} > 0$. It follows from (6.3) and (6.4) that

$$0 = \big((H_V + b)u, u\big) \geqslant \big(\phi H_V(\phi u), u\big) + \big(\widetilde{\phi}\nabla^*\nabla(\widetilde{\phi}u), u\big) + \widetilde{b}\|u\|^2. \tag{6.6}$$

Since $u \neq 0$ by assumption, we can see from (6.5) and (6.6) that

$$\big(\phi(H_V + \widetilde{b})(\phi u), u\big) \leqslant \big(\phi H_V(\phi u), u\big) + \widetilde{b}\|u\|^2 \leqslant 0, \qquad \phi u \neq 0. \tag{6.7}$$

We can and will regard the coordinate neighbourhood W as a subset of $\mathbb{R}^n$. Let us fix a Riemannian metric $\widetilde{g}$ on $\mathbb{R}^n$ such that the restriction of $\widetilde{g}$ to W coincides with the metric induced by g^{TM} and $\widetilde{g}$ is equal to the standard Euclidean metric on $\mathbb{R}^n$ outside some compact set $K \supset W$. We also endow $\mathbb{R}^n$ with a measure $\widetilde{\mu}$ such that the restriction of $\widetilde{\mu}$ to W coincides with the measure induced by the measure μ on M and the restriction of $\widetilde{\mu}$ to $\mathbb{R}^n \setminus K$ is Lebesgue measure on $\mathbb{R}^n$.

Let $\widetilde{E} := \mathbb{R}^n \times \mathbb{C}^m$ be the trivial vector bundle over $\mathbb{R}^n$, endowed with the standard Hermitian metric. Choosing an orthonormal frame in the restriction $E|_W$ of E to W, we can and will identify the restrictions $E|_W$ and $\widetilde{E}|_W$. Then the connection ∇ induces a Hermitian connection on $\widetilde{E}|_W$. Let $\widetilde{\nabla}$ be a Hermitian connection on $\widetilde{E}$ whose restriction to W coincides with the connection induced by ∇ and whose restriction to $M \setminus K$ is the trivial flat connection. Then by (6.7) we have

$$\big(\phi(\widetilde{\nabla}^*\widetilde{\nabla} + V + \widetilde{b})(\phi u), u\big) \leqslant 0, \qquad \phi u \neq 0.$$

Let us choose the number $\widetilde{b}$ large enough that $\widetilde{H}_V = \widetilde{\nabla}^*\widetilde{\nabla} + V + \widetilde{b} > 0$ on $C_c^\infty(\widetilde{E})$ (this is possible by Corollary 6.3). By Theorem X.26 of [64], it follows that the operator $\widetilde{H}_V$ *is not* essentially self-adjoint on $C_c^\infty(\widetilde{E})$. Thus, we have reduced the proof of the proposition to the similar problem for the operator $\widetilde{H}_V$ on $\mathbb{R}^n$.

Since the operator $\widetilde{H}_V$ is not essentially self-adjoint, we can see from Theorem X.26 of [64] that there is a non-zero section $v \in L^2(\widetilde{E})$ for which

$$(\widetilde{H}_V + \widetilde{b})v = 0. \tag{6.8}$$

Let $\widetilde{\Delta} := d^*d$ be the scalar Laplacian on $\mathbb{R}^n$ associated with the metric $\widetilde{g}$ and the measure $\widetilde{\mu}$; see Definition 5.4. It follows from (6.8) and from Kato's inequality (5.6) that

$$\widetilde{\Delta}|v| \leqslant \operatorname{Re}\langle\widetilde{\nabla}^*\widetilde{\nabla}v, \operatorname{sign} v\rangle = -\langle(V + \widetilde{b})v, \operatorname{sign} v\rangle \leqslant (|V_-| - \widetilde{b})|v|,$$

where $|V_-(x)|$ is the norm of the endomorphism $V_-(x)\colon \widetilde{E}_x \to \widetilde{E}_x$. Thus,

$$(\widetilde{\Delta} + \widetilde{b})|v| \leqslant |V_-|\,|v|. \tag{6.9}$$

As is well known (and also explained in Appendix B), the Schwartz kernel of the operator $(\widetilde{\Delta} + \widetilde{b})^{-1}$ is positive. A somewhat more subtle argument (see Proposition B.3 of Appendix B) shows that the distribution inequality (6.9) implies that

$$|v| \leqslant (\widetilde{\Delta} + \widetilde{b})^{-1}|V_-|\,|v|, \tag{6.10}$$

where the inverse operator $(\widetilde{\Delta} + \widetilde{b})^{-1}$ can be regarded, for instance, as an operator on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

Let $\overline{W}$ be the closure of W in M. Since $\operatorname{supp} V_- \subset \overline{W}$, it follows from (2.2) that

$$\|\,|V_-|s\| \leqslant a_{\overline{W}}\|\widetilde{\Delta}s\| + C_{\overline{W}}\|s\| \quad \text{for any} \quad s \in C_c^\infty(\mathbb{R}^n).$$

Hence,

$$\| \, |V_-|(\widetilde{\Delta}+\widetilde{b})^{-1}s\| \leqslant a_{\overline{W}}\|\widetilde{\Delta}(\widetilde{\Delta}+\widetilde{b}\,)^{-1}s\| + C_{\overline{W}}\|(\widetilde{\Delta}+\widetilde{b}\,)^{-1}s\| \leqslant (a_{\overline{W}}+C_{\overline{W}}/\widetilde{b}\,)\|s\|.$$

Thus, we can choose $\widetilde{b}$ large enough that

$$\| \, |V_-|(\widetilde{\Delta}+\widetilde{b})^{-1}\| < 1.$$

Since

$$\big((\widetilde{\Delta}+\widetilde{b}\,)^{-1}|V_-|\big)^* = |V_-|(\widetilde{\Delta}+\widetilde{b})^{-1},$$

we can conclude that $\|(\widetilde{\Delta}+\widetilde{b})^{-1}|V_-|\,\| < 1$. Then it follows from the inequality (6.10) together with our assumption $v \neq 0$ that $\|v\| < \|v\|$. The contradiction thus obtained proves the proposition.

Remark 6.7. One of the main steps in the above proof is the reduction to the case $M = \mathbb{R}^n$ in which we used Proposition B.3 to conclude that the inequality (6.9) implies (6.10). The proof would be much simpler if this fact were valid on an arbitrary complete Riemannian manifold, that is, if Conjecture P in Appendix B were true. Unfortunately, the validity of the conjecture is an open problem (see Appendix B, where we explain the difficulties arising when trying to prove the conjecture).

Proposition 6.1 and Corollary 6.3 imply (see Theorem X.26 in [64]) the following assertion.

Corollary 6.8. *Let V and C be as in Corollary* 6.3. *Then*

$$(H_V u, u) \geqslant -C\|u\|^2 \quad \textit{for any} \quad u \in \operatorname{Dom}(H_{V,\max}).$$

Let $W^{1,2}(E)$ be the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_{W^{1,2}}$ determined by the inner product

$$(u,v)_{W^{1,2}} := (u,v) + (\nabla u, \nabla v), \qquad u, v \in C_c^\infty(E). \tag{6.11}$$

(It can readily be seen that $W^{1,2}(E)$ coincides with the set of all elements $u \in L^2(E)$ such that $\nabla u \in L^2(T^*M \otimes E)$, but we do not need this fact.)

Since the operator H_V is essentially self-adjoint, it follows that the domain $Q(H_V)$ of the quadratic form of H_V coincides with the closure of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the formula

$$\|u\|_1^2 := (H_V u, u) + (C+1)\|u\|^2.$$

By Theorem X.23 in [64], we have $\operatorname{Dom}(H_{V,\max}) \subset Q(H_V)$.

Corollary 6.9. *Under the assumptions of Proposition* 6.1, $Q(H_V) \subset W^{1,2}(E)$. *In particular*, $\operatorname{Dom}(H_{V,\max}) \subset W^{1,2}(E)$.

Proof. By Lemma 6.2, there are constants $a < 1$ and $C_a > 0$ such that

$$(H_V u, u) = \|\nabla u\|^2 + (V_+ u, u) + (V_- u, u) \geqslant (1-a)\|\nabla u\|^2 - C_a\|u\|^2, \qquad u \in C_c^\infty(E).$$

Thus, the domain $Q(H_V)$ of the quadratic form of the operator H_V coincides with the closure of the space $C_c^\infty(E)$ with respect to the norm

$$\|u\|_1^2 := (H_V u, u) + (C_a+1)\|u\|^2 \geqslant (1-a)\|\nabla u\|^2 + \|u\|^2.$$

Hence, $Q(H_V)$ is contained in the closure of $C_c^\infty(E)$ with respect to the norm $(1-a)\|\nabla u\|^2 + \|u\|^2$, and this closure coincides with the space $W^{1,2}(E)$. This proves the corollary.

§ 7. Proof of the second part of Theorem 2.3

In this section we assume that there is a function $s\colon T^*M \to \mathbb{R}$ of class C^∞ such that

$$\sigma(D^*D)(\xi) = s(\xi)\,\mathrm{Id}$$

for every $\xi \in T^*M$. Then

$$\sigma(D^*D) = |\xi|^2\,\mathrm{Id},$$

where $|\xi|$ stands for the norm on T^*M determined by the inner product (2.4). In this situation we say that the operator D^*D *has scalar principal symbol.* It follows from Proposition 2.5 in [6] that there exist a Hermitian connection ∇ on E and a linear self-adjoint bundle map $F \in C^\infty(\mathrm{End}\,E)$ such that

$$D^*D = \nabla^*\nabla + F.$$

Proposition 7.1. *Let D be as above. If H_V is the operator*

$$H_V = D^*D + V = \nabla^*\nabla + F + V,$$

where $V = V^+ + V^-$ is as in Assumption A, *then* $\mathrm{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$.

Proof. Let $u \in \mathrm{Dom}(H_{V,\max})$. We must prove that $u \in W^{1,2}_{\mathrm{loc}}(E)$. Clearly, it suffices to show that for every $x \in M$ there exist an open coordinate neighbourhood $U \ni x$ and a constant C_U such that

$$\big|\big(\nabla_{\frac{\partial}{\partial x_j}} u, z\big)\big| \leqslant C_U \|z\|$$

for any $j = 1, \dots, n$ and $z \in C_c^\infty(E|_U)$. Let us fix an $x \in M$ and choose a coordinate neighbourhood W_1 of x. Let $\phi\colon M \to [0,1]$ be a smooth function whose support is contained in W_1 and let ϕ be identically equal to 1 near x. In other words, we assume that there is an open neighbourhood W_2 of x such that $\phi|_{W_2} \equiv 1$. We write

$$H_\phi := \nabla^*\nabla + \phi(F+V)\phi.$$

Since F is smooth, it follows that the potential $\phi(F+V)\phi$ satisfies Assumption A. Hence, it follows from Proposition 6.1 and Corollaries 6.8 and 6.9 that the operator H_ϕ is essentially self-adjoint and bounded below and that $\mathrm{Dom}(H_{\phi,\max}) \subset W^{1,2}(E)$.

Let us choose an open neighbourhood $U \subset W_2$ of x and a smooth function $\psi\colon M \to [0,1]$ such that $\psi|_U \equiv 1$ and the support of ψ is contained in W_2.

Lemma 7.2. *There is a constant $C_1 = C_1(u, \phi, \psi)$ such that*

$$|(\psi u, H_\phi s)| \leqslant C_1(\|s\| + \|\nabla s\|) \quad \textit{for any} \quad s \in \mathrm{Dom}(H_{\phi,\max}).$$

Proof. Let us fix some $s \in \mathrm{Dom}(H_{\phi,\max})$. Since the operator H_ϕ is essentially self-adjoint on $C_c^\infty(E)$, there is a sequence $s_k \in C_c^\infty(E)$ that converges to s in the graph norm of H_ϕ. Since $\mathrm{Dom}(H_{\phi,\max}) \subset W^{1,2}(E)$, we have $s_k \to s$ in the topology of $W^{1,2}(E)$ as well. Hence,

$$\lim_{k\to\infty} |(\psi u, H_\phi s_k)| = |(\psi u, H_\phi s)|, \qquad \lim_{k\to\infty} (\|s_k\| + \|\nabla s_k\|) = \|s\| + \|\nabla s\|.$$

Therefore, it suffices to prove the lemma for $s \in C_c^\infty(E)$, and in what follows we assume this.

Since the support of ψ is compact, we see from Lemma 6.4 that the operator $\psi H_V - H_V\psi$ defines a continuous map $W^{1,2}(E) \to L^2(E)$. Hence, there is a constant $C > 0$ such that

$$\|\psi H_V s - H_V(\psi s)\| \leqslant C(\|s\| + \|\nabla s\|). \tag{7.1}$$

Since $\phi|_{\operatorname{supp}\psi} \equiv 1$, it follows that $\psi H_\phi = \psi H_V$. Thus, by (7.1) we get that

$$\begin{aligned}|(\psi u, H_\phi s)| = |(u, \psi H_V s)| &\leqslant |(u, H_V(\psi s))| + C(\|s\| + \|\nabla s\|)\|u\| \\ &\leqslant |(H_V u, \psi s)| + C(\|s\| + \|\nabla s\|)\|u\| \\ &\leqslant \|H_V u\|\,\|\psi s\| + C(\|s\| + \|\nabla s\|)\|u\| \leqslant C_1(\|s\| + \|\nabla s\|),\end{aligned}$$

where $C_1 = C\|u\| + \|H_V u\|$. This proves the lemma.

Lemma 7.3. *There are positive constants $a < 1$ and C_2 such that*

$$(H_\phi s, s) \geqslant (1-a)\|\nabla s\|^2 - C_2\|s\|^2, \qquad s \in \operatorname{Dom}(H_{\phi,\max}). \tag{7.2}$$

Proof. As in the proof of Lemma 7.2, it suffices to prove the inequality (7.2) for $s \in C_c^\infty(E)$, and in what follows we assume this. By Lemma 6.2, there are positive constants $a' < 1$ and C' such that

$$|(\phi V_-(\phi s), s)| \leqslant a'\|\nabla(\phi s)\|^2 + C'\|s\|^2, \qquad s \in C_c^\infty(E). \tag{7.3}$$

Let us fix an $\epsilon > 0$ such that $a := a'(1+\epsilon) < 1$. We write $m_1 := \max_{x\in M}|d\phi(x)|$. Then

$$\|\nabla(\phi s)\|^2 \leqslant (\|\nabla s\| + m_1\|s\|)^2 \leqslant (1+\epsilon)\|\nabla s\|^2 + (1+1/\epsilon)m_1^2\|s\|^2. \tag{7.4}$$

Combining the formulae (7.3) and (7.4), we have

$$|(\phi V_-(\phi s), s)| \leqslant a\|\nabla s\|^2 + C''\|s\|^2, \tag{7.5}$$

where $C'' = C' + a'(1+1/\epsilon)m_1^2$. Let $m_2 = \max_{x\in M}|\phi F\phi|$. It follows from (7.5) and the inequality $V_+ \geqslant 0$ that

$$(H_\phi s, s) \geqslant (\nabla^*\nabla s, s) - |(\phi V_-\phi s, s)| - |(\phi F\phi s, s)| \geqslant (1-a)\|\nabla s\|^2 - (C'' + m_2)\|s\|^2.$$

This proves the lemma.

Let us return to the proof of Proposition 7.1. We fix a function $z \in C_c^\infty(E|_U)$ and an index $j \in \{1, \dots, n\}$. Since the operator H_ϕ is essentially self-adjoint and bounded below, it follows that the map $H_\phi + \lambda\colon \operatorname{Dom}(H_{\phi,\max}) \to L^2(E)$ is bijective for sufficiently large $\lambda \gg 0$. Let $s = (H_\phi+\lambda)^{-1}\nabla^*_{\frac{\partial}{\partial x_j}} z$, where $\nabla^*_{\frac{\partial}{\partial x_j}}$ stands for the formal adjoint of $\nabla_{\frac{\partial}{\partial x_j}}$. Using Lemma 7.3, we get that

$$((H_\phi + \lambda)s, s) \geqslant (1-a)\|\nabla s\|^2 + (\lambda - C_2)\|s\|^2. \tag{7.6}$$

Thus, if $\lambda > C_2$, then $((H_\phi + \lambda)s, s) > 0$. Since $s \in \mathrm{Dom}(H_{\phi,\max}) \subset W^{1,2}(E)$, it follows that

$$((H_\phi + \lambda)s, s) = (\nabla^*_{\frac{\partial}{\partial x_j}} z, s) = (z, \nabla_{\frac{\partial}{\partial x_j}} s).$$

For any $\epsilon > 0$

$$\begin{aligned} ((H_\phi + \lambda)s, s) = (z, \nabla_{\frac{\partial}{\partial x_j}} s) &\leqslant \|z\| \left(\int_U |\nabla_{\frac{\partial}{\partial x_j}} s|^2 \, d\mu \right)^{1/2} \\ &\leqslant \frac{\epsilon}{2} \int_U |\nabla_{\frac{\partial}{\partial x_j}} s|^2 \, d\mu + \frac{1}{2\epsilon} \|z\|^2 \leqslant \frac{\epsilon C_3}{2} \|\nabla s\|^2 + \frac{1}{2\epsilon} \|z\|^2, \end{aligned} \tag{7.7}$$

where C_3 is the least upper bound of the length of the vector $\dfrac{\partial}{\partial x_j}$ on U. Combining (7.6) and (7.7), we see that

$$\|\nabla s\| + \|s\| \leqslant C_4 \|z\| \tag{7.8}$$

for some constant C_4. Using Lemma 7.2 and (7.8), we obtain

$$\left|\left(u, \nabla^*_{\frac{\partial}{\partial x_j}} z\right)\right| = \left|\left(\psi u, \nabla^*_{\frac{\partial}{\partial x_j}} z\right)\right| = |(\psi u, (H_\phi + \lambda)s)| \leqslant C_1(\|\nabla s\| + \|s\|) + C_5 \|s\| \leqslant C_6 \|z\|,$$

where $C_5 = \|\psi u\|$ and $C_6 = (C_1 + C_5)C_4$. This completes the proof.

§ 8. Quadratic forms and the essential self-adjointness of H_{V_+}

Suppose that the operator $H_V = D^*D + V$ satisfies Assumptions A and B. Throughout the section we assume that

$$(H_V u, u) \geqslant \|u\|^2 \quad \text{for any} \quad u \in C_c^\infty(E). \tag{8.1}$$

8.1. We recall that $W^{-1,2}_{\mathrm{loc}}(E)$ is the dual space of $W^{1,2}_{\mathrm{comp}}(E)$. Hence, the intersection $W^{-1,2}_{\mathrm{loc}}(E) \cap L^1_{\mathrm{loc}}(E)$ consists of the sections $v \in L^1_{\mathrm{loc}}(E)$ having the following property: for any compact set $K \subset M$ there is a constant $C_{K,v} > 0$ such that

$$\int \langle v, s \rangle \, d\mu \leqslant C_{K,v} \|s\|_{W^{1,2}} \quad \text{for any} \quad s \in C_c^\infty(E) \quad \text{with} \quad \operatorname{supp} s \subset K,$$

where $\|\cdot\|_{W^{1,2}}$ stands for the norm in the space $W^{1,2}(E)$; see (6.11).

Lemma 8.2. *$V_- u \in W^{-1,2}_{\mathrm{loc}}(E) \cap L^1_{\mathrm{loc}}(E)$ for any $u \in W^{1,2}_{\mathrm{loc}}(E)$.*

Proof. It follows from Lemma 6.2 by polarization that for any compact set $K \subset M$ there is a constant $C_K > 0$ such that

$$\left| \int \langle V_- s_1, s_2 \rangle \, d\mu \right| \leqslant C_K \|s_1\|_{W^{1,2}} \|s_2\|_{W^{1,2}}, \qquad s_1, s_2 \in C_c^\infty(E), \quad \operatorname{supp} s_2 \subset K. \tag{8.2}$$

Since the statement of the lemma is local, we can assume that u is supported in a coordinate neighbourhood. Let us choose such an element $u \in W^{1,2}_{\mathrm{comp}}(E)$. Let $u^\rho = \mathcal{J}^\rho u$ be as in § 5.12. It follows from Lemma 5.13 that $u^\rho \to u$ as $\rho \to 0$ both in

$W^{1,2}(E)$ and in $L^2(E)$. In particular, $\|u^\rho\|_{W^{1,2}}$ is bounded for $0 < \rho < 1$. Hence, by (8.2), for every compact $K \subset M$ there is a constant $C_{K,u} > 0$ such that

$$\left|\int \langle V_- u^\rho, s\rangle \, d\mu\right| \leqslant C_{K,u}\|s\|_{W^{1,2}}, \qquad s \in C_c^\infty(E), \quad \operatorname{supp} s \subset K, \quad 0 < \rho < 1. \tag{8.3}$$

Since $u^\rho \to u$ in $L^2_{\mathrm{loc}}(E)$, we have $V_- u^\rho \to V_- u$ in $L^1_{\mathrm{loc}}(E)$. Hence,

$$\lim_{\rho\to 0+} \int \langle V_- u^\rho, s\rangle \, d\mu = \int \langle V_- u, s\rangle \, d\mu. \tag{8.4}$$

Combining the formulae (8.3) and (8.4), we get that

$$\left|\int \langle V_- u, s\rangle \, d\mu\right| \leqslant C_{K,u}\|s\|_{W^{1,2}}, \qquad s \in C_c^\infty(E), \quad \operatorname{supp} s \subset K,$$

which completes the proof.

Below we often denote by $(\,\cdot\,,\cdot\,)$ the duality between $W^{-1,2}_{\mathrm{comp}}(E)$ and $W^{1,2}_{\mathrm{loc}}(E)$. The next lemma shows that this notation leads to no confusion.

Lemma 8.3. *Let $u \in L^2_{\mathrm{comp}}(E) \subset W^{-1,2}_{\mathrm{comp}}(E)$ and $v \in W^{1,2}_{\mathrm{loc}}(E)$. Then*

$$(u, v) := \int \langle u, v\rangle \, d\mu = (u, v)',$$

where $(\,\cdot\,,\cdot\,)'$ stands for the duality between $W^{-1,2}_{\mathrm{comp}}(E)$ and $W^{1,2}_{\mathrm{loc}}(E)$ continuously extending the L^2-inner product on $C_c^\infty(E)$.

Proof. Using a partition of unity, we can assume again that u and v are supported in a coordinate neighbourhood, and hence we can use Friedrichs mollifiers. Let $u^\rho = \mathcal{J}^\rho u$ and $v^\rho = \mathcal{J}^\rho v$ be as in § 5.12. It follows from Lemma 5.13 that $u^\rho \to u$ as $\rho \to 0$ both in $W^{-1,2}(E)$ and in $L^2(E)$. Similarly, $v^\rho \to v$ in $W^{1,2}(E)$ and in $L^2(E)$. Therefore, $(u, v) = \lim_{\rho\to 0+}(u^\rho, v^\rho) = \lim_{\rho\to 0+}(u^\rho, v^\rho)' = (u, v)'$. This completes the proof of the lemma.

The following lemma is due to Brézis and Browder [9].

Lemma 8.4. *Let $A \in L^2_{\mathrm{loc}}(\operatorname{End} E)$ be a non-negative bundle map (that is, for every $x \in M$ the endomorphism $A_x\colon E_x \to E_x$ has non-negative quadratic form). Let $u \in W^{1,2}_{\mathrm{comp}}(E)$. Suppose that $Au \in W^{-1,2}_{\mathrm{comp}}(E)$, and hence in fact $Au \in W^{-1,2}_{\mathrm{comp}}(E) \cap L^1_{\mathrm{comp}}(E)$. Then $\langle Au, u\rangle \in L^1_{\mathrm{comp}}(M)$ and*

$$\int \langle Au, u\rangle \, d\mu = (Au, u), \tag{8.5}$$

where $(\,\cdot\,,\cdot\,)$ stands for the duality between $W^{-1,2}_{\mathrm{comp}}(E)$ and $W^{1,2}_{\mathrm{loc}}(E)$ continuously extending the L^2-inner product on $C_c^\infty(E)$.

Proof. For the convenience of the reader we present a proof which differs slightly from that given in [9]. Using a partition of unity, we can assume that A is supported by a coordinate neighbourhood. Since the values of the section u are relevant only in some neighbourhood of supp A, we can assume that u is supported in the same coordinate neighbourhood. Let $v \in W^{1,2}(E) \cap L^\infty(E)$. We claim that

$$\int \langle Au, v\rangle\, d\mu = (Au, v). \tag{8.6}$$

Indeed, let $(Au)^\rho = \mathcal{J}^\rho(Au)$. It follows from Lemma 5.2 that $(Au)^\rho \to Au$ as $\rho \to 0$ both in $W^{-1,2}_{\mathrm{comp}}(E)$ and in $L^1_{\mathrm{comp}}(E)$. Hence,

$$(Au, v) = \lim_{\rho\to 0+}((Au)^\rho, v) = \lim_{\rho\to 0+}\int \langle (Au)^\rho, v\rangle\, d\mu = \int \langle Au, v\rangle\, d\mu.$$

For every $R > 0$ we define the truncation u_R of u by the formula

$$u_R(x) = \begin{cases} u(x) & \text{for } |u(x)| \leqslant R,\\ R\dfrac{u(x)}{|u(x)|} & \text{for } |u(x)| > R.\end{cases}$$

It follows from Theorem A of the appendix in [52] that $u_R \in W^{1,2}_{\mathrm{comp}}(E)$ for all $R > 0$ and that $u_R \to u$ in $W^{1,2}_{\mathrm{comp}}(E)$ as $R \to \infty$. Hence,

$$(Au, u) = \lim_{R\to\infty}(Au, u_R) = \lim_{R\to\infty}\int \langle Au, u_R\rangle\, d\mu, \tag{8.7}$$

where the last equality follows from (8.6). By our assumption, $\langle A(x)u(x), u(x)\rangle \geqslant 0$ for almost all $x \in M$. Hence, $\langle A(x)u(x), u_R(x)\rangle$ increases as R increases for almost all $x \in M$. The lemma now follows from the monotone convergence theorem.

Remark 8.5. More general results can be found, for instance, in [10], [11], [15]. It should be noted that the assertion of the lemma is not completely trivial. For example, if w and v are scalar functions, $w \in W^{-1,2}_{\mathrm{loc}} \cap L^1_{\mathrm{loc}}$ and $v \in W^{1,2}_{\mathrm{comp}}$, then it can happen that wv is not in L^1_{comp}, and thus the integral $\int wv\, d\mu$ can fail to be well defined (even though (w, v) is perfectly well defined by the duality between $W^{-1,2}$ and $W^{1,2}$); see, for instance, the example in [10]. In fact, if $w \in L^1_{\mathrm{loc}}$ is fixed, then the condition that wv be in L^1_{loc} for every $v \in W^{1,2}_{\mathrm{comp}}$ is equivalent to the condition $|w| \in W^{-1,2}_{\mathrm{loc}}$ (this follows, for instance, from [57], § 8.4.4, Theorem 2).

8.6. Applying Lemma 8.4 to the bundle map $A = -V_-$ and using Lemma 8.2, we see that the integral

$$\int \langle V_- u, u\rangle\, d\mu = (V_- u, u) \tag{8.8}$$

is finite for all $u \in W^{1,2}_{\mathrm{comp}}(E)$. Let us consider the expression $h_V(u)$ defined by the formula

$$h_V(u) = \|Du\|^2 + \int \langle V_- u, u\rangle\, d\mu + \int \langle V_+ u, u\rangle\, d\mu \leqslant +\infty, \qquad u \in W^{1,2}_{\mathrm{comp}}(E).$$

The main result of the section is the following assertion.

Proposition 8.7. *Suppose that $h_V(u) \geqslant \|u\|^2$ for any $u \in C_c^\infty(E)$. Then*

$$h_V(u) \geqslant \|u\|^2 \quad \textit{for any} \quad u \in W^{1,2}_{\mathrm{comp}}(E). \tag{8.9}$$

The proof of this proposition occupies the rest of the section.

The following simple lemma on 'integration by parts' follows, for instance, from Theorem 7.7 in [74].

Lemma 8.8. *The equality $(Du, v) = (u, D^*v)$ holds if one of the sections u and v is compactly supported and either $u \in L^2_{\mathrm{loc}}(E)$ and $v \in W^{1,2}_{\mathrm{loc}}(F)$ or $u \in W^{1,2}_{\mathrm{loc}}(E)$ and $v \in L^2_{\mathrm{loc}}(F)$. Here $(\,\cdot\,,\cdot\,)$ is understood as either the inner product in L^2 or as the duality between $W^{-1,2}_{\mathrm{comp}}$ and $W^{1,2}_{\mathrm{loc}}$ continuously extending the L^2-inner product on $C_c^\infty(E)$.*

The following well-known lemma [30] (whose proof we reproduce for completeness) gives sufficiently many 'cut-off' functions to be used below.

Lemma 8.9. *Suppose that g is a complete Riemannian metric on a manifold M. Then there is a sequence $\{\phi_k\}$ of Lipschitz functions with compact support on M such that*:

(i) $0 \leqslant \phi_k \leqslant 1$ *and $|d\phi_k| \leqslant 1/k$, where $|d\phi_k|$ stands for the length* (*induced by the metric g*) *of the cotangent vector $d\phi_k$*;

(ii) $\lim_{k\to\infty} \phi_k(x) = 1$ *for any $x \in M$.*

Proof. Let d be the distance function with respect to the metric g. We fix $x_0 \in M$ and set $P(x) = d(x, x_0)$. Then $P(x)$ is Lipschitz, and hence it is differentiable almost everywhere. Moreover, $|dP| \leqslant 1$. The completeness condition $\int^\infty ds = \infty$, where ds is the arc-length element associated with g, means that $P(x) \to \infty$ as $x \to \infty$.

Let us consider a function $\chi \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ near 0, and $|\chi'| \leqslant 1$. We set $\phi_k(x) = \chi(P(x)/k)$. Clearly, ϕ_k has the desired properties, and the lemma is proved.

The following lemma is an analogue of (5.3) in [79].

Lemma 8.10. *For any $u \in \mathrm{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}(E)$ and any compactly supported Lipschitz function $\psi\colon M \to \mathbb{R}$,*

$$h_V(\psi u) = \mathrm{Re}(\psi H_V u, \psi u) + \|\widehat{D}(d\psi)u\|^2. \tag{8.10}$$

Proof. Since $u \in W^{1,2}_{\mathrm{loc}}(E)$, it follows that $D^*Du \in W^{-1,2}_{\mathrm{loc}}(E)$. By Lemma 8.2, $V_-u \in W^{-1,2}_{\mathrm{loc}}$ as well. Hence, $V_+u = H_Vu - D^*Du - V_-u \in W^{-1,2}_{\mathrm{loc}}$. Therefore, by Lemma 8.4,

$$\int \langle V_+(\psi u), \psi u\rangle\, d\mu = (V_+(\psi u), \psi u) < \infty.$$

In particular, $h_V(\psi u) < \infty$. Using Lemma 8.3, we see that

$$\begin{aligned}(\psi H_V u, \psi u) &= (\psi D^* Du, \psi u) + (V_+(\psi u), \psi u) + (V_-(\psi u), \psi u)\\ &= (\psi D^* Du, \psi u) + \int \langle V_+(\psi u), \psi u\rangle\, d\mu + \int \langle V_-(\psi u), \psi u\rangle\, d\mu. \end{aligned}\tag{8.11}$$

Here the symbol $(\,\cdot\,,\cdot\,)$ on the left-hand side stands for the L^2-inner product, whereas the same symbol stands for the duality between $W^{-1,2}_{\mathrm{comp}}(E)$ and $W^{1,2}_{\mathrm{loc}}(E)$ in other terms of the equality.

Using Lemma 8.8 on integration by parts, we get that

$$\begin{aligned}
&(D(\psi u), D(\psi u))\\
&\quad= (\widehat{D}(d\psi)u, D(\psi u)) + (\psi Du, D(\psi u)) = (\widehat{D}(d\psi)u, D(\psi u)) + (Du, \psi D(\psi u))\\
&\quad= (\widehat{D}(d\psi)u, \widehat{D}(d\psi)u) + (\widehat{D}(d\psi)u, \psi Du) - (Du, \widehat{D}(d\psi)\psi u) + (Du, D(\psi^2 u))\\
&\quad= \|\widehat{D}(d\psi)u\|^2 + 2i\,\mathrm{Im}(\widehat{D}(d\psi)u, \psi Du) + (\psi D^*Du, \psi u).
\end{aligned}$$

Adding this formula to its complex conjugate and dividing by 2, we have

$$(D(\psi u), D(\psi u)) = \|\widehat{D}(d\psi)u\|^2 + \mathrm{Re}(\psi D^*Du, \psi u).$$

The equality (8.10) now follows from (8.11). This proves the lemma.

Proposition 8.11. *Let H_{V_+} (where $V_+ \in L^2_{\mathrm{loc}}(\mathrm{End}\,E)$ and $V_+ \geqslant 0$) satisfy Assumption* B. (*In particular, this holds if $n \leqslant 3$ or if D^*D has scalar principal symbol, and also if $V_+ \in L^p_{\mathrm{loc}}(\mathrm{End}\,E)$ with $p > n/2$ for $n \geqslant 4$.*) *In this case, the operator $H_{V_+} = D^*D + V_+$ is essentially self-adjoint on $C_c^\infty(E)$.*

Proof. By Theorem X.26 in [64], it suffices to prove that if $(H_{V_+}+1)u = 0$ for some $u \in L^2(E)$, then $u = 0$.

Indeed, for any such element u we have $u \in \mathrm{Dom}(H_{V_+,\max})$. Let ϕ_k be as in Lemma 8.9. By Lemma 8.10 (for $V = V_+$),

$$h_{V_+}(\phi_k u) = \mathrm{Re}(\phi_k H_{V_+}u, \phi_k u) + \|\widehat{D}(d\phi_k)u\|^2 = -\|\phi_k u\|^2 + \|\widehat{D}(d\phi_k)u\|^2. \quad (8.12)$$

Since $h_{V_+}(\phi_k u) \geqslant 0$, it follows from the equation (8.12) that

$$\|\phi_k u\|^2 \leqslant h_{V_+}(\phi_k u) + \|\phi_k u\|^2 = \|\widehat{D}(d\phi_k)u\|^2 \leqslant \frac{m}{k^2}\|u\|^2, \quad (8.13)$$

where $m = \dim(E_x)$. The last inequality in (8.13) holds by Lemma 8.9 and (2.5). Passing to the limit as $k \to \infty$, we see that $\|u\| = 0$.

We need the following well-known abstract assertion.

Lemma 8.12. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{D} \subset \mathcal{H}$ be a linear subspace. Let $A\colon \mathcal{D} \to \mathcal{H}$ be an essentially self-adjoint operator acting in $\mathcal{H}$. Suppose that h is a closed positive quadratic form on $\mathcal{H}$ with domain $Q(h) \supset \mathcal{D}$ for which*

$$h(u,v) = (Au, v) \quad \text{for any} \quad u \in \mathcal{D}, \quad v \in Q(h), \quad (8.14)$$

where $h(u,v)$ is the sesquilinear form determined by h via polarization. Then the closure $\overline{A}$ of A is the operator associated with h by the Friedrichs construction (*see* [64], § *X.23*). *In particular, $\mathcal{D}$ is a core of h.*

Proof. Let B be the self-adjoint operator associated with h by the Friedrichs construction. We recall that the domain $\mathrm{Dom}(B)$ consists of the vectors $u \in Q(h)$ such that the map

$$l_u\colon v \mapsto h(u,v), \qquad v \in Q(h),$$

is continuous if the space $Q(h)$ is endowed with the norm inherited from $\mathcal{H}$. In this case, Bu is determined by the formula

$$h(u,v) = (Bu, v), \qquad v \in Q(h). \quad (8.15)$$

We are now ready to prove the lemma. It follows from (8.14) that the map l_u is continuous for all $u \in \mathcal{D}$. In other words, $\mathcal{D} \subset \mathrm{Dom}(B)$. Comparing (8.14) and (8.15), we conclude that $Bu = Au$ for any $u \in \mathcal{D}$. Thus, B is a self-adjoint extension of A. Since A is essentially self-adjoint, it has a unique self-adjoint extension, and $\overline{A} = B$. This proves the lemma.

Let us consider the expression

$$h_{V^+}(u) = \|Du\|^2 + \int \langle V_+ u, u\rangle \, d\mu,$$

where h_{V^+} is regarded as a quadratic form with the domain

$$Q(h_{V^+}) = \bigl\{ u \in L^2(E) \cap W^{1,2}_{\mathrm{loc}}(E) : h_{V^+}(u) < +\infty \bigr\}.$$

Clearly, h_{V^+} is a closed positive form. We recall (see Proposition 8.11) that the operator H_{V^+} is essentially self-adjoint on $C_c^\infty(E)$.

Proposition 8.13. *The closure of H_{V^+} is the operator associated with the form h_{V^+} by the Friedrichs construction. In particular, the space $C_c^\infty(E)$ is a core of the form h_{V^+}, that is, $Q(h_{V^+})$ is the closure of $C_c^\infty(E)$ with respect to the norm*

$$\|u\|_1^2 := \|Du\|^2 + \int \langle V_+ u, u\rangle \, d\mu + \|u\|^2.$$

Proof. Since $Q(h_{V^+}) \subset W^{1,2}_{\mathrm{loc}}(E)$, it follows from Lemma 8.8 on integration by parts that

$$(H_{V^+}u, v) = (D^*Du, v) + \int \langle V_+ u, v\rangle \, d\mu = (Du, Dv) + \int \langle V_+ u, v\rangle \, d\mu = h_{V^+}(u, v)$$

for any $u \in C_c^\infty(E)$ and $v \in Q(h_{V^+})$. Hence, applying Lemma 8.12, we complete the proof.

For any compact set $K \subset M$ we define

$$V_-^{(K)}(x) = \left\{ \begin{matrix} V_-(x) & \text{for } x \in K \\ 0 & \text{otherwise} \end{matrix} \right\}, \qquad V^{(K)} = V_-^{(K)} + V_+.$$

We note that $\left| \int \langle V_-^{(K)} u, u\rangle \, d\mu \right| = |(V_-^{(K)}u, u)| < \infty$ for any $u \in W^{1,2}_{\mathrm{loc}}(E)$ by (8.8).

Lemma 8.14. *Suppose that $v_k \to v$ in $\|\cdot\|_1$. Then*

$$\lim_{k\to\infty} (V_-^{(K)} v_k, v_k) = (V_-^{(K)} v, v).$$

Proof. By the definition of the norm $\|\cdot\|_1$ we have

$$\lim_{k\to\infty} \|D(v - v_k)\| = 0, \qquad \lim_{k\to\infty} \|v - v_k\| = 0, \tag{8.16}$$

where $\|\cdot\|$ stands for the L^2 norm. Let us fix a function $\phi \in C_c^\infty(M)$ such that $\phi = 1$ on a neighbourhood of K. Then by (8.16) we obtain

$$\lim_{k\to\infty} \|D(\phi v - \phi v_k)\| \leqslant \lim_{k\to\infty} \|\widehat{D}(d\phi)(v - v_k)\| + \lim_{k\to\infty} \|\phi D(v - v_k)\| = 0.$$

In other words, ϕv_k converges to ϕv in $W^{1,2}_{\mathrm{comp}}(E)$. It follows now from Lemma 6.2 (or, more precisely, from the inequality (8.2)) that

$$\lim_{k\to\infty}(V_-^{(K)}v_k, v_k) = \lim_{k\to\infty}(V_-^{(K)}\phi v_k, \phi v_k) = (V_-^{(K)}\phi v, \phi v) = (V_-^{(K)}v, v).$$

8.15. Proof of Proposition 8.7. Let us fix some $u \in W^{1,2}_{\mathrm{comp}}(E)$. Let $K \subset M$ be a compact set which contains a neighbourhood of $\operatorname{supp} u$. We write

$$h_{V^{(K)}}(u) = \|Du\|^2 + \int \langle V_+u, u\rangle\, d\mu + \int \langle V_-^{(K)}u, u\rangle\, d\mu = h_{V_+}(u) + (V_-^{(K)}u, u).$$

If $h_V(u) = +\infty$, then the inequality (8.9) is trivially true. Hence, we can assume that $h_V(u) < +\infty$. In particular, $u \in Q(h_{V_+})$. By Proposition 8.13, there is a sequence $u_k \in C_c^\infty(E)$ such that $\|u_k - u\|_1 \to 0$ as $k \to \infty$. Using (8.1) and Lemma 8.14, we get that

$$\begin{aligned} h_{V_+}(u) + (V_-^{(K)}u, u) &= \lim_{k\to\infty} h_{V_+}(u_k) + \lim_{k\to\infty}(V_-^{(K)}u_k, u_k) = \lim_{k\to\infty} h_{V^{(K)}}(u_k) \\ &\geqslant \lim_{k\to\infty} h_V(u_k) \geqslant \lim_{k\to\infty}\|u_k\|^2 = \|u\|^2. \end{aligned}$$

Corollary 8.16. *Let δ, q, and V be as in Theorem* 2.7. *Then*

$$\delta(Du, Du) + (Vu, u) \geqslant -(qu, u) \quad \textit{for any} \quad u \in W^{1,2}_{\mathrm{comp}}(E).$$

Proof. It suffices to prove that

$$(Du, Du) + \delta^{-1}((V+q)u, u) \geqslant 0 \quad \text{for any} \quad u \in W^{1,2}_{\mathrm{comp}}(E).$$

This follows immediately from Proposition 8.7.

§ 9. Proof of Theorem 2.7

Throughout the section we assume that the potential V satisfies the conditions of Theorem 2.7.

Lemma 9.1. *Let ψ, $0 \leqslant \psi \leqslant q^{-1/2} \leqslant 1$, be a compactly supported Lipschitz function and let*

$$C = \sqrt{m}\ \operatorname*{ess\,sup}_{x\in M}|d\psi(x)| := \operatorname*{ess\,sup}_{x\in M}\big(\operatorname{Re}\operatorname{Tr}((\widehat{D}(d\psi))^*\widehat{D}(d\psi))\big)^{1/2}.$$

Then

$$\|\psi Du\|^2 \leqslant \frac{2}{1-\delta}\left(\left(1 + \frac{2C^2(1+\delta)^2}{1-\delta}\right)\|u\|^2 + \|u\|\|H_V u\|\right) \tag{9.1}$$

for any $u \in \operatorname{Dom}(H_{V,\max})$, where $\|\cdot\|$ stands for the L^2-norm.

Proof. We first note that

$$\operatorname*{ess\,sup}_{x\in M}|\widehat{D}(d\psi)| \leqslant C$$

by Remark 2.6. Let $u \in \operatorname{Dom}(H_{V,\max})$. Then $u \in W^{1,2}_{\mathrm{loc}}(E)$ by Assumption B. Hence, $\psi^2 Du \in L^2_{\mathrm{comp}}(F)$. We can apply Lemma 8.8 on integration by parts, which implies that

$$\begin{aligned}\|\psi Du\|^2 &= (D^*(\psi^2 Du), u) = (\psi^2 D^* Du, u) + 2(\psi \widehat{D^*}(d\psi) Du, u)\\ &= \operatorname{Re}(\psi^2 D^* Du, u) + 2\operatorname{Re}(\psi \widehat{D^*}(d\psi) Du, u)\\ &\leqslant \operatorname{Re}(\psi^2 D^* Du, u) + 2C\|\psi Du\|\,\|u\|.\end{aligned}$$

Moreover,

$$\begin{aligned}\operatorname{Re}(\psi^2 D^* Du, u) &\geqslant (\psi Du, \psi Du) - 2C\|\psi Du\|\|u\|\\ &= (D(\psi u), D(\psi u)) - (\widehat{D}(d\psi)u, \psi Du) - (\psi Du, \widehat{D}(d\psi)u) - 2C\|\psi Du\|\|u\|\\ &\geqslant (D(\psi u), D(\psi u)) - 4C\|\psi Du\|\|u\|.\end{aligned}$$

Hence, by Corollary 8.16 we have

$$\begin{aligned}(1-\delta)\|\psi Du\|^2 &\leqslant (1-\delta)\operatorname{Re}(\psi^2 D^* Du, u) + 2C(1-\delta)\|\psi Du\|\|u\|\\ &= \operatorname{Re}(\psi^2 H_V u, u) - \delta \operatorname{Re}(\psi^2 D^* Du, u) - (\psi^2 V u, u) + 2C(1-\delta)\|\psi Du\|\|u\|\\ &\leqslant \|H_V u\|\|u\| - \delta(D(\psi u), D(\psi u)) - (V(\psi u), \psi u) + 2C(1+\delta)\|\psi Du\|\|u\|\\ &\leqslant \|H_V u\|\|u\| + (q\psi u, \psi u) + 2C(1+\delta)\|\psi Du\|\|u\|\\ &\leqslant \|H_V u\|\|u\| + \|u\|^2 + 2C(1+\delta)\|\psi Du\|\|u\|.\end{aligned}$$

Using the inequality $2ab \leqslant \varepsilon a^2 + \dfrac{b^2}{\varepsilon}$ $(a, b \in \mathbb{R})$ with $\varepsilon = \dfrac{1-\delta}{2(1+\delta)^2}$, we get that

$$2(1+\delta)C\|\psi Du\|\,\|u\| \leqslant \frac{1-\delta}{2}\|\psi Du\|^2 + 2C^2\frac{(1+\delta)^2}{1-\delta}\|u\|^2.$$

Therefore,

$$(1-\delta)\|\psi Du\|^2 \leqslant \frac{1-\delta}{2}\|\psi Du\|^2 + \left(1 + \frac{2C^2(1+\delta)^2}{1-\delta}\right)\|u\|^2 + \|u\|\|H_V u\|,$$

and the inequality (9.1) follows immediately. This proves the lemma.

Lemma 9.2. *Suppose that the metric g^{TM} is complete (as described in § 2.5). Let q be as in Theorem 2.7, and let $u \in \operatorname{Dom}(H_{V,\max})$. Then $q^{-1/2}Du \in L^2(F)$ and*

$$\|q^{-1/2}Du\| \leqslant \frac{2}{1-\delta}\left(\left(1 + \frac{2L^2(1+\delta)^2}{1-\delta}\right)\|u\|^2 + \|u\|\|H_V u\|\right). \tag{9.2}$$

Proof. Using a sequence of Lipschitz functions ϕ_k given by Lemma 8.9, we set $\psi_k = \phi_k \cdot q^{-1/2}$. Then $0 \leqslant \psi_k \leqslant q^{-1/2}$, and $|d\psi_k| \leqslant |d\phi_k| \cdot q^{-1/2} + \phi_k |dq^{-1/2}|$. Therefore, $|d\psi_k| \leqslant 1/k + L$, where L is the Lipschitz constant for $q^{-1/2}$. Since $\psi_k(x) \to q^{-1/2}(x)$ as $k \to \infty$, the dominated convergence theorem applied to (9.1) with $\psi = \psi_k$ immediately yields (9.2). This proves the lemma.

9.3. Proof of Theorem 2.7. Let $u, v \in \operatorname{Dom}(H_{V,\max})$ and let $\phi \geqslant 0$ be a compactly supported Lipschitz function. By Remark 2.8, the metric g^{TM} is complete. It follows from Lemma 9.2 that $q^{-1/2}Du$ and $q^{-1/2}Dv$ are in $L^2(F)$.

By Assumption B, the sections u and v belong to $W^{1,2}_{\mathrm{loc}}(E)$. Hence, $Du, Dv \in L^2_{\mathrm{loc}}(F)$. Using Lemma 8.8, we get that

$$(\phi u, D^*Dv) = (D(\phi u), Dv) = (\widehat{D}(d\phi)u, Dv) + (\phi Du, Dv),$$
$$(D^*Du, \phi v) = (Du, D(\phi v)) = (Du, \widehat{D}(d\phi)v) + (\phi Du, Dv).$$

By (2.5),

$$\operatorname*{ess\,sup}_{x\in M} |\widehat{D}(d\phi)| \leqslant \sqrt{m}\, \operatorname*{ess\,sup}_{x\in M} |d\phi(x)|,$$

where $|d\phi|$ stands for the length of the covector $d\phi$ in g^{TM}. Therefore,

$$\begin{aligned} &|(\phi u, H_V v) - (H_V u, \phi v)| \\ &\qquad \leqslant |(\widehat{D}(d\phi)u, Dv)| + |(Du, \widehat{D}(d\phi)v)| \\ &\qquad \leqslant \sqrt{m}\, \operatorname*{ess\,sup}_{x\in M}(|d\phi| q^{1/2}) \cdot \bigl(\|u\|\, \|q^{-1/2}Dv\| + \|v\|\, \|q^{-1/2}Du\|\bigr). \end{aligned} \tag{9.3}$$

Let us consider the metric $g := q^{-1}g^{TM}$; denote the length of a covector in this metric by $|\cdot|_g$. By the condition (iii) of the theorem, this metric is complete. Using g, we take a sequence $\{\phi_k\}$ of Lipschitz functions as in Lemma 8.9. Since $|d\phi_k|_g = q^{1/2}|d\phi_k|$, it follows that $q^{1/2}|d\phi_k| \leqslant 1/k$. Therefore,

$$\operatorname*{ess\,sup}_{x\in M}(|d\phi_k| q^{1/2}(x)) \leqslant \frac{1}{k}\,.$$

Using (9.3), we get that

$$|(\phi_k u, H_V v) - (H_V u, \phi_k v)| \leqslant \frac{\sqrt{m}}{k}\bigl(\|u\|\, \|q^{-1/2}Dv\| + \|q^{-1/2}Du\|\, \|v\|\bigr) \to 0 \ \text{ as } \ k \to \infty.$$

On the other hand, by the dominated convergence theorem we have

$$(\phi_k u, H_V v) - (H_V u, \phi_k v) \to (u, H_V v) - (H_V u, v) \quad \text{as} \quad k \to \infty.$$

Thus, $(H_V u, v) = (u, H_V v)$ for any $u, v \in \mathrm{Dom}(H_{V,\max})$. Therefore, H_V is essentially self-adjoint. This completes the proof of the theorem.

§ 10. Proof of Theorem 2.13

We proceed as in § 5.1 of [79]. Since the operator H_V is semibounded below, there is a constant $C > 0$ such that $H_V \geqslant -C$ on $C^\infty_c(E)$.

Adding $(C+1)I$ to H_V, we can assume that $H_V \geqslant I$ on $C^\infty_c(E)$, that is,

$$(H_V u, u) \geqslant \|u\|^2, \qquad u \in C^\infty_c(E).$$

As is well known (see, for instance, Theorem X.26 in [64]), the operator H_V is essentially self-adjoint if and only if the equation $H_V u = 0$ has no non-trivial solutions in $L^2(E)$ (in the sense of distributions). We assume that $H_V u = 0$ for some $u \in L^2(E)$. In particular, $u \in \mathrm{Dom}(H_{V,\max})$. By assumption, $u \in W^{1,2}_{\mathrm{loc}}(E)$.

Let ϕ_k be a sequence of compactly supported Lipschitz functions given by Lemma 8.9. Then $\phi_k u$ belongs to $W^{1,2}_{\mathrm{comp}}(E)$ for any $k = 1, 2, \dots$.

Using Proposition 8.7, Lemma 8.10, and the equality $H_V u = 0$, we get that

$$\|\phi_k u\|^2 \leqslant h_V(\phi_k u) = \|\widehat{D}(d\phi_k)u\|^2 \leqslant \frac{m}{k^2}\|u\|^2.$$

Here the inequality follows from Lemma 8.9 and (2.5). Since

$$\|u\| = \lim_{k\to\infty} \|\phi_k u\|,$$

we conclude that $u = 0$. This completes the proof of the theorem.

Appendix A. Friedrichs mollifiers

Let $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ be a vector function on $\mathbb{R}^n$. Let $\mathcal{J}$ be the integral operator whose integral kernel is $j_\rho(x-y)\,\mathrm{Id}$, where Id is the $m \times m$ identity matrix and $j_\rho(x-y)$ is as in § 5.12. We set $u^\rho = \mathcal{J}^\rho u$.

The main result of this appendix is the following proposition, which generalizes results of Friedrichs [29] and Kato [47] (§ 5, Lemma 2) to our vector setting.

Proposition A.1. *Let us consider a second-order differential operator*

$$L = \sum_{i,k} \partial_i a_{ik}(x)\partial_k + \sum_i b_i(x)\partial_i + c(x), \tag{A.1}$$

where $a_{ik}(x)$, $b_i(x)$, and c are $m \times m$ matrices such that the entries of $a_{ik}(x)$ and $b_i(x)$ are locally Lipschitz functions on $\mathbb{R}^n$ and the entries of $c(x)$ belong to $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose that $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ and $Lu \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$, and assume in addition that

$$\partial_k u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m \text{ if } a_{ik}(x) \neq 0 \text{ for some } x \in \mathbb{R}^n \text{ and } i \in \{1,\dots,n\}. \tag{A.2}$$

Then $Lu^\rho \to Lu$ in $(L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$.

Remark A.2. We note that the assumption (A.2) is trivially true if L is a first-order operator, that is, if $a_{ik} \equiv 0$ for all i and k. On the other hand, if L is an elliptic second-order differential operator with smooth coefficients, then (A.2) is a consequence of the other assumptions (that $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ and $Lu \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$). This follows from standard elliptic regularity results; see, for instance, the arguments at the end of the proof of Lemma 4.1. In fact, in this case it suffices to assume that $a_{ik}(x)$ and $b_i(x)$ are locally Lipschitz and $c(x)$ is in L^∞_{loc} (see the sketch of the proof in [47], § 5, Lemma 2).[3]

In the proof we use the following version of Friedrichs' result (see the equation (3.8) in the proof of the main theorem in [29]).

Proposition A.3. *Let J^ρ be as in (5.13). Fix an index $i \in \{1,\dots,n\}$ and let $T = b(x)\partial_i$, where $b(x)$ is a locally Lipschitz function on $\mathbb{R}^n$. Suppose that $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$(J^\rho T - TJ^\rho)v \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^n) \tag{A.3}$$

as $\rho \to 0+$. The same holds if we replace $T = b(x)\partial_i$ by $\tilde{T} = \partial_i \cdot b$, where b is regarded as the operator of multiplication by $b(x)$.

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[3] In [47] it is assumed that a_{ik} and b_i belong to C^1, but the same argument can be carried out assuming only that these functions are Lipschitz.

A.4. Proof of Proposition A.3. Since the statement of Proposition A.3 is local, it suffices to prove that it holds if the support of b is compact and $v \in L^1(\mathbb{R}^n)$, and we assume from now on that these conditions are satisfied. In this case, $K^\rho := J^\rho T - TJ^\rho$ is a continuous operator $L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. Let $\|K^\rho\|$ be the norm of this operator.

We recall that any locally Lipschitz function is differentiable almost everywhere and its pointwise derivative coincides with the distribution derivative (in particular, the derivative belongs to L^∞_{loc}). This enables us to integrate by parts if the functions are Lipschitz just as if they were in C^1 (see, for instance, [57], §§ 1.1 and 6.2).

We begin by establishing an analogue of Proposition A.3 for zero-order operators.

Lemma A.5. *If $c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $cu^\rho - (cu)^\rho \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $\rho \to 0+$. The same holds if u is a vector function (with values in $\mathbb{C}^m$) and c takes values in the space of $m \times m$ matrices.*

Proof. By Lemma 5.13 (ii), both terms on the right-hand side of the equation

$$cu^\rho - (cu)^\rho = c(\mathcal{J}^\rho u - u) + (cu - \mathcal{J}^\rho(cu))$$

tend to 0 in $(L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ as $\rho \to 0+$.

Lemma A.6. *The operator $K^\rho \colon L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is an integral operator, and its Schwartz kernel is*

$$k_\rho(x,y) = \frac{\partial}{\partial y^i}\bigl((b(x) - b(y))j_\rho(x-y)\bigr). \tag{A.4}$$

Proof. For any $v \in C_c^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
(TJ^\rho v)(x) &= b(x)\frac{\partial}{\partial x^i}\int j_\rho(x-y)v(y)\,dy = \int b(x)\frac{\partial}{\partial x^i}\bigl(j_\rho(x-y)\bigr)v(y)\,dy \\
&= \int b(x)\biggl(-\frac{\partial}{\partial y^i}j_\rho(x-y)\biggr)v(y)\,dy = \int\biggl(-\frac{\partial}{\partial y^i}\bigl(b(x)j_\rho(x-y)\bigr)\biggr)v(y)\,dy, \\
(J^\rho Tv)(x) &= \int j_\rho(x-y)b(y)\frac{\partial}{\partial y^i}v(y)\,dy = \int\biggl(-\frac{\partial}{\partial y^i}\bigl(b(y)j_\rho(x-y)\bigr)\biggr)v(y)\,dy.
\end{aligned}$$

Hence,

$$K^\rho v(x) = \int k_\rho(x,y)v(y)\,dy,$$

where k_ρ is given by (A.4). Since K^ρ is a continuous operator $L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, the lemma is proved.

Lemma A.7. *There is a constant $C > 0$ such that*

$$\int |k_\rho(x,y)|\,dx \leqslant C \quad \text{for any} \quad y \in \mathbb{R}^n, \quad \rho > 0. \tag{A.5}$$

In particular, $\|K^\rho\| \leqslant C$ for any $\rho > 0$.

Proof. It follows from (A.4) that

$$k_\rho(x,y) = -\frac{\partial b(y)}{\partial y^i}\,j_\rho(x-y) + \bigl(b(x) - b(y)\bigr)\frac{\partial}{\partial y^i}j_\rho(x-y). \tag{A.6}$$

Let us now estimate the integral of the absolute values of the terms on the right-hand side of (A.6). Since $\int j_\rho(x-y)\,dx = 1$, we get that

$$\int \left| \frac{\partial b(y)}{\partial y^i}\, j_\rho(x-y) \right| dx \leqslant \operatorname*{ess\,sup}_{y\in \operatorname{supp} b} \left| \frac{\partial b(y)}{\partial y^i} \right| \leqslant \operatorname*{ess\,sup}_{y\in \operatorname{supp} b} |\nabla b(y)|. \tag{A.7}$$

We recall that $j_\rho(x) = \rho^{-n} j(\rho^{-1}x)$; see § 5.12. Hence,

$$\int \left| \big(b(x)-b(y)\big) \frac{\partial}{\partial y^i} j_\rho(x-y) \right| dx \leqslant \Big(\rho \operatorname*{ess\,sup}_{\xi\in \operatorname{supp} b} |\nabla b(\xi)| \Big) \cdot \rho^{-1} \int \left| \frac{\partial j(x)}{\partial x^i} \right| dx. \tag{A.8}$$

It follows from (A.7) and (A.8) that the estimate (A.5) holds with the constant

$$C = \left(1 + \int |\nabla j(x)|\, dx \right) \operatorname*{ess\,sup}_{y\in \operatorname{supp} b} |\nabla b(y)|.$$

This completes the proof of the lemma.

Let us recall that the function b is compactly supported by assumption. The next lemma summarizes some additional properties of $k_\rho(x,y)$.

Lemma A.8.

(i) *The support of* k_ρ *is contained in the* ρ*-neighbourhood of* $\operatorname{supp} b \times \operatorname{supp} b \subset \mathbb{R}^n \times \mathbb{R}^n$;

(ii) $k_\rho(x,y) = 0$ *for* $|x-y| > \rho$;

(iii) *there is a constant* $C_1 > 0$ *such that* $\int |k_\rho(x,y)| dx\, dy \leqslant C_1$ *for any* $\rho > 0$;

(iv) $\int k_\rho(x,y)\, dy = 0$ *for any* $x \in \mathbb{R}^n$.

Proof. The assertions (i) and (ii) follow immediately from (A.4). The assertion (iii) follows from (i) and (A.5), and the assertion (iv) holds because k_ρ is the derivative of the compactly supported function $\big(b(x)-b(y)\big) j_\rho(x-y)$.

We now return to the proof of Proposition A.3. By virtue of Lemma A.7, it suffices to prove the relation (A.3) for $v \in C_c^\infty(\mathbb{R}^n)$. Using Lemmas A.7 and A.8, we get that

$$\begin{aligned} \int |(K^\rho v)(x)|\, dx &= \int \left| \int k_\rho(x,y) v(y)\, dy \right| dx = \int \left| \int k_\rho(x,y) \big(v(y)-v(x)\big)\, dy \right| dx \\ &\leqslant C_1 \max_{|x-y|\leqslant\rho} |v(y)-v(x)| \leqslant C_1 \rho \max_{\xi\in \operatorname{supp} v} |\nabla v(\xi)| \to 0 \text{ as } \rho \to 0+\,. \end{aligned}$$

This proves the first part of Proposition A.3. To prove the second part, we note that

$$\widetilde{T} u = \partial_i(bu) = Tu + (\partial_i b) u,$$

and thus the desired result follows from the first part and from Lemma A.5.

A.9. Proof of Proposition A.1. It follows from the assertion (ii) in Lemma 5.13 that $(Lu)^\rho - Lu \to 0$ in $(L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ as $\rho \to 0+$. Thus, it suffices to show that

$$Lu^\rho - (Lu)^\rho = (Lu^\rho - Lu) - ((Lu)^\rho - Lu) \to 0 \tag{A.9}$$

in $(L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ as $\rho \to 0+$. Let us estimate the left-hand side of (A.9) separately for the zero-, first-, and second-order terms of the operator L; see (A.1). We can use Lemma A.5 for the zero-order terms. Proposition A.3 gives the desired result for the first-order terms (for vector functions we must apply the proposition separately to each component of the vector). The following lemma treats the second-order terms.

Lemma A.10. *Fix indices $i, k \in \{1, \dots, n\}$ and a function $u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ such that $\partial_k u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$. Let $a(x)$ be an $m \times m$ matrix with locally Lipschitz entries. Then $\partial_i(a\partial_k u^\rho) - (\partial_i(a\partial_k u))^\rho \to 0$ in $(L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$ as $\rho \to 0+$.*

Proof. Since $\mathcal{J}^\rho$ commutes with $\partial_k u$, we have

$$\begin{aligned}\partial_i(a\partial_k\mathcal{J}^\rho u) - \mathcal{J}^\rho(\partial_i(a\partial_k u)) &= \partial_i(a\mathcal{J}^\rho\partial_k u) - \mathcal{J}^\rho(\partial_i(a\partial_k u)) \\ &= \partial_i(a\mathcal{J}^\rho v) - \mathcal{J}^\rho(\partial_i(av)),\end{aligned} \tag{A.10}$$

where $v = \partial_k u \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^m$. The lemma now follows from (A.10) and Proposition A.3.

Combining Lemma A.5, Proposition A.3, and Lemma A.10, we obtain Proposition A.1.

A.11. Proof of Proposition 5.14. Since the statement is local, we can work in a coordinate neighbourhood U in which $\nabla^*\nabla$ is given by (5.14). Using the standard elliptic regularity argument (see, for instance, the arguments at the end of the proof of Lemma 4.1), we see that $u \in W^{1,1}_{\mathrm{loc}}(E)$, that is, $\partial_i u \in L^1_{\mathrm{loc}}$ for any $i = 1, \dots, n$ in any local coordinates. Hence, it is immediate from Proposition A.1 that $\nabla^*\nabla u^\rho - (\nabla^*\nabla u)^\rho \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)(E|_U)$ as $\rho \to 0+$.

Appendix B. Positivity

The content of this appendix is mostly well known, but we could not find an adequate reference in the literature.

Let us briefly describe the classical positivity results for the Green's function of the operator $b + \Delta_M$ on a complete Riemannian manifold (M, g) with a smooth positive measure $d\mu$. Here $b > 0$, and $\Delta_M = d^*d \geqslant 0$ is the scalar Laplacian (see Definition 5.4). We note that, if $d\mu$ is the Riemannian volume form on M, then $\Delta_M = -\Delta_g$, where Δ_g is the *metric* Laplacian, $\Delta_g u = \operatorname{div}(\operatorname{grad} u)$.

The classical result of Gaffney [30] claims that the operator Δ_g is essentially self-adjoint. Gaffney's argument works for Δ_M without any modifications (for any measure $d\mu$). We reproduced this argument in the proof of Lemma 8.9. A more general statement about the essential self-adjointness of $\nabla^*\nabla$ follows from Proposition 8.11. (We note that Assumption A is automatically true in this case, whereas Assumption B follows from elliptic regularity.)

Hence, the operator $b + \Delta_M$ is also essentially self-adjoint and strictly positive. More precisely,

$$((b + \Delta_M)u, u) \geqslant b(u, u), \qquad u \in C_c^\infty(M). \tag{B.1}$$

Therefore, this inequality holds for any $u \in \mathrm{Dom}(\Delta_{M,\max})$. In particular, the spectrum of $b + \Delta_M$ is a subset of the semi-axis $[b, +\infty)$, and thus the operator $(b + \Delta_M)^{-1}$ is everywhere defined and bounded. We denote by $G_b = G_b(\cdot, \cdot)$ the Schwartz kernel of $(b + \Delta_M)^{-1}$. The kernel G_b is locally integrable on $M \times M$, C^∞ smooth outside the diagonal in $M \times M$, and has singularities on the diagonal that can readily be described by a variety of methods (for instance, by using the technique of pseudodifferential operators).

Theorem B.1. *In the above notation,*

$$G_b(x,y) \geqslant 0 \quad \textit{for} \quad \textit{any} \quad x,y \in M, \quad x \neq y. \tag{B.2}$$

Proof. This proof was communicated to us by A. Grigor'yan. It suffices to establish that

$$u(x) := \int_M G_b(x,y)\phi(y)\,d\mu(y) \geqslant 0, \qquad x \in M, \tag{B.3}$$

for any $\phi \in C_c^\infty(M)$ such that $\phi \geqslant 0$. Clearly, $u = (b+\Delta_M)^{-1}\phi \in L^2(M)$, and u satisfies the equation

$$(b+\Delta_M)u = \phi. \tag{B.4}$$

In particular, $u \in C^\infty(M)$.

The equation (B.4) has a unique solution $u \in L^2(M)$ (in the sense of distributions). We now construct a positive solution $v \in L^2(M)$ of this equation, which will complete the proof because then we will have $v = u$ by uniqueness.

Let us take a sequence of relatively compact open subsets with smooth boundary in M,

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega_k \Subset \Omega_{k+1} \Subset \cdots$$

(that is, Ω_k is relatively compact in Ω_{k+1}), such that they exhaust M, which means that their union is M. For any k we denote by v_k the solution of the following Dirichlet problem in Ω_k:

$$(b+\Delta_M)v_k = \phi, \qquad v_k\big|_{\partial\Omega_k} = 0. \tag{B.5}$$

Let us consider only indices k large enough that $\operatorname{supp}\phi \subset \Omega_k$. It follows from the maximum principle that $v_k \geqslant 0$ and

$$0 \leqslant v_k \leqslant v_{k+1} \quad \text{for any} \quad k.$$

Thus, the sequence of functions $v_1, v_2, \dots$ is increasing, and hence there is a pointwise limit

$$v(x) = \lim_{k\to\infty} v_k(x), \qquad x \in M.$$

We prove that this limit is in fact everywhere finite and locally bounded.

To this end, we multiply the equation in (B.5) by $v_k\,d\mu$ and integrate over Ω_k. Using the Stokes formula, we get that

$$\int_{\Omega_k} (bv_k^2 + |dv_k|^2)\,d\mu = \int_{\Omega_k} \phi v_k\,d\mu \leqslant \frac{b}{2}\int_{\Omega_k} v_k^2\,d\mu + \frac{1}{2b}\int_{\Omega_k} \phi^2\,d\mu,$$

and hence

$$\int_{\Omega_k} \left(\frac{b}{2}v_k^2 + |dv_k|^2\right) d\mu \leqslant \frac{1}{2b}\int_{\Omega_k} \phi^2\,d\mu,$$

which gives, in particular, a uniform estimate for the L^2 norm of v_k on any compact set in M. By standard interior elliptic estimates (see, for instance, [87], § 5.3), this implies that every derivative of v_k is bounded uniformly with respect to k on any compact set $L \subset M$, and hence the sequence v_k converges in the topology of $C^\infty(M)$. Therefore, v is everywhere finite and positive, and it satisfies the condition (B.4). Applying Fatou's lemma, we also see that $v \in L^2(M)$. Therefore, $v = u$, and hence $u \geqslant 0$. This proves the theorem.

In a more general context, it can be a problem to prove that a solution u of the equation (B.4) is positive. Namely, we assume that

$$(b + \Delta_M)u = \nu \geqslant 0, \qquad u \in L^2(M),$$

where the inequality $\nu \geqslant 0$ means that ν is a positive distribution, that is, $(\nu, \phi) \geqslant 0$ for any $\phi \in C_c^\infty(M)$. It follows that ν is in fact a positive Radon measure (see, for instance, [31], Theorem 1 in § 2 of Chap. II).

Conjecture P. In this case, $u \geqslant 0$ (almost everywhere or, equivalently, as a distribution).

At first glance it seems that the above proof (with $\phi \in C_c^\infty(M)$ instead of ν) could work in this case as well. However, it does not work without additional conditions on u or M (see Proposition B.2 below). Let us see what the difficulty is.

Taking a test function $\phi \in C_c^\infty(M)$, $\phi \geqslant 0$, we must prove that $(u, \phi) \geqslant 0$. Let us solve the equation $(b+\Delta_M)\psi = \phi$, where $\psi \in L^2(M)$. We can see that $\psi \in C^\infty(M)$ and $\psi \geqslant 0$, according to Theorem B.1. Thus, we can write

$$(u, \phi) = \bigl(u, (b + \Delta_M)\psi\bigr).$$

The right-hand side can now be represented as

$$\bigl(u, (b + \Delta_M)\psi\bigr) = \bigl((b + \Delta_M)u, \psi\bigr)_S = (\nu, \psi)_S,$$

where the Hermitian form $(\,\cdot\,,\cdot\,)_S$ on the right-hand side is obtained by continuously extending the inner product on $L^2(M)$ from $C_c^\infty(M) \times C_c^\infty(M)$ to the (non-degenerate) Hermitian duality

$$\widetilde{H}^{-2}(M) \times \widetilde{H}^2(M) \to \mathbb{C} \tag{B.6}$$

of the Sobolev spaces $\widetilde{H}^{-2}(M) = (b + \Delta_M)L^2(M)$ and $\widetilde{H}^2(M) = \mathrm{Dom}(\Delta_{M,\max}) = \mathrm{Dom}(\Delta_{M,\min})$ with the Hilbert structures transferred from $L^2(M)$ by means of the operator $b + \Delta_M$ acting from $L^2(M)$ to $\widetilde{H}^{-2}(M)$ and from $\widetilde{H}^2(M)$ to $L^2(M)$, respectively. The norms in $\widetilde{H}^2(M)$ and $\widetilde{H}^{-2}(M)$ are given by the formulae

$$\|v\|_2 = \|(b + \Delta_M)v\|, \qquad \|(b + \Delta_M)f\|_{-2} = \|f\|, \tag{B.7}$$

where $\|\cdot\|$ is the norm in $L^2(M)$.

The continuous extension of the duality (B.6) from $C_c^\infty(M) \times C_c^\infty(M)$ is well defined because $C_c^\infty(M)$ is dense in both spaces $\widetilde{H}^{-2}(M)$ and $\widetilde{H}^2(M)$ with respect to the corresponding norms (B.7). Indeed, the denseness of $C_c^\infty(M)$ in $\widetilde{H}^2(M)$ simply means that $(b + \Delta_M)C_c^\infty(M)$ is dense in $L^2(M)$. To establish this fact, let us take $f \in L^2(M)$ orthogonal to $(b + \Delta_M)C_c^\infty(M)$ in $L^2(M)$, which means that

$(b+\Delta_M)f = 0$ in the sense of distributions, that is, f is in the null-space of the maximal operator $(b+\Delta_M)_{\max}$. This implies that $f=0$ due to the above essential self-adjointness and the strict positivity of $b+\Delta_M$.

Similarly, the denseness of $C_c^\infty(M)$ in $\widetilde{H}^{-2}(M)$ means that $(b+\Delta_M)^{-1}C_c^\infty(M)$ is dense in $L^2(M)$. To prove this fact, we take $h \in L^2(M)$ orthogonal to $(b+\Delta_M)^{-1}C_c^\infty(M)$. Since $(b+\Delta_M)^{-1}$ is a bounded self-adjoint operator, this would imply that $(b+\Delta_M)^{-1}h = 0$, and hence $h=0$.

We note that the space $\widetilde{H}^2(M)$ differs (at least formally) from the space $H_2^2(M)$, whose norm includes arbitrary second-order covariant derivatives (see [33], § 2.2). It is apparently still unknown whether $C_c^\infty(M)$ is dense in $H_2^2(M)$ (see § 3.1 in [33]).

We must now find out whether or not the formula

$$(\nu,\psi)_S = \int_M \psi\nu \tag{B.8}$$

holds (the integral on the right-hand side makes sense as the integral of a positive measure, though this integral can be infinite). If this formula holds, then we are done because the integral is obviously non-negative.

A possible way to establish (B.8) is to present the function ψ as a limit

$$\psi = \lim_{k\to\infty} \psi_k,$$

where $\psi_k \in C_c^\infty(M)$, $\psi_k \geqslant 0$, $\psi_k \leqslant \psi_{k+1}$, and the limit is taken with respect to the norm $\|\cdot\|_2$. The equality (B.8) obviously holds if we replace ψ by ψ_k, and hence we would obtain the equality for ψ by passing to the limit.

We can try to take $\psi_k = \chi_k\psi$, where $\chi_k \in C_c^\infty(M)$, $0 \leqslant \chi_k \leqslant 1$, $\chi_k \leqslant \chi_{k+1}$, and for every compact set $L \subset M$ there is a k such that $\chi_k|_L = 1$. Then we obviously have $\psi_k \to \psi$ in $L^2(M)$ as $k \to +\infty$. It is also desirable to have the limit relation $\Delta_M\psi_k \to \Delta_M\psi$ in $L^2(M)$. Clearly,

$$\Delta_M\psi_k = \chi_k\Delta_M\psi - 2\langle d\chi_k, d\psi\rangle + (\Delta_M\chi_k)\psi.$$

Obviously, $\chi_k\Delta_M\psi \to \Delta_M\psi$ in $L^2(M)$. We can now note that

$$\|d\psi\|^2 = (\Delta_M\psi,\psi) \leqslant \frac{1}{2}\|\Delta_M\psi\|^2 + \frac{1}{2}\|\psi\|^2$$

and, on the other hand, $d\chi_k \to 0$ and $\Delta_M\chi_k \to 0$ in $C^\infty(M)$. We have achieved our goal if we can construct χ_k such that

$$\sup_{x\in M} |d\chi_k(x)| \leqslant C, \qquad \sup_{x\in M} |\Delta_M\chi_k(x)| \leqslant C, \tag{B.9}$$

where $C > 0$ does not depend on k. This leads to the following statement communicated to us by E. B. Davies.

Proposition B.2. *Let us assume that (M,g) is a complete Riemannian manifold with a positive smooth measure such that there are cut-off functions χ_k, $k = 1,2,\ldots$, satisfying the above assumptions. Then Conjecture* P *holds.*

It is always possible (on any complete Riemannian manifold) to construct cut-off functions χ_k satisfying the above conditions except for the uniform estimate for $\Delta_M\chi_k$ (that is, except for the second estimate in (B.9)). This was proved by Karcher [45] (see also [79]). However, the estimate for $\Delta_M\chi_k$ presents a difficulty. The following sufficient condition provides an important class of examples.

Proposition B.3. *Let (M, g) be a manifold of bounded geometry with arbitrary measure $d\mu$ having positive smooth density. Then Conjecture* P *holds.*

Proof. For the definition and properties of manifolds of bounded geometry, see, for instance, [67] and [75]. In particular, a construction of cut-off functions χ_k satisfying the necessary properties on any manifold of bounded geometry can be found in [75], p. 61.

*Remark B.*4. We note that the bounded geometry conditions are not imposed on the measure $d\mu$, which can be an arbitrary measure with positive smooth density.

B.5. Example. Let us consider the manifold $M = \mathbb{R}^n$ with a metric g coinciding with the standard flat metric g_0 outside a compact set. Then Conjecture P holds. This can also be proved by using the invertibility of $b + \Delta_M$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, which, in turn, implies the invertibility of $b + \Delta_M$ in the dual space $\mathcal{S}'(\mathbb{R}^n)$ formed by the tempered distributions.

Appendix C. Stummel and Kato classes

In this appendix we briefly survey the definitions and most important properties of the Stummel and Kato classes of functions on $\mathbb{R}^n$ and on manifolds.

C.1. Stummel classes. The uniform Stummel classes on $\mathbb{R}^n$ were introduced in [84]. For more details about the Stummel classes and proofs, see § 1.2 in [23], Chaps. 5 and 9 in [72], and also [2], [83], [84].

The (uniform) Stummel class S_n consists of the measurable real-valued functions V on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\lim_{r\downarrow 0}\left[\sup_x \int_{|x-y|\leqslant r} |x-y|^{4-n}|V(y)|^2\,dy\right] = 0 \qquad &&\text{for} \quad n \geqslant 5,\\
&\lim_{r\downarrow 0}\left[\sup_x \int_{|x-y|\leqslant r} \log(|x-y|^{-1})|V(y)|^2\,dy\right] = 0 \qquad &&\text{for} \quad n = 4,\\
&\sup_x \int_{|x-y|\leqslant r_0} |V(y)|^2\,dy < \infty &&\text{for} \quad n \leqslant 3.
\end{aligned}$$

Here $r_0 > 0$ is an arbitrarily fixed constant. Clearly, this class S_n is invariant under multiplication by any real-valued bounded measurable function. We can also readily see that if $f \in S_n$ and a diffeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear near infinity, then $\phi^* f = f \circ \phi \in S_n$. Therefore, the corresponding local version $S_{n,\mathrm{loc}}(M)$ is well defined for any manifold with $\dim M = n$. We denote by $S_{n,\mathrm{comp}}(M)$ the class of compactly supported functions $f \in S_{n,\mathrm{loc}}(M)$. Both the classes $S_{n,\mathrm{loc}}(M)$ and $S_{n,\mathrm{comp}}(M)$ are invariant under diffeomorphisms of M and also under multiplication by any real-valued locally bounded measurable function.

There is a relationship between the local Stummel classes and the spaces L^p:

$$L^p_{\mathrm{loc}}(M) \subset S_{n,\mathrm{loc}}(M) \quad \text{if} \quad p > n/2 \ \text{for} \ n \geqslant 4 \quad \text{and if} \quad p = 2 \ \text{for} \ n \leqslant 3. \tag{C.1}$$

The same relationship holds for the classes of compactly supported functions and for the uniform classes S_n on $\mathbb{R}^n$ (if $L^p(\mathbb{R}^n)$ is replaced by the class $L^p_{\mathrm{unif}}(\mathbb{R}^n)$ consisting of the functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ whose L^p norms are bounded on any unit ball).

An important property of the uniform Stummel classes is that, if $u \in S_k$ on $\mathbb{R}^k$ and $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is a surjective linear projection, then $\pi^* u \in S_n$. This obviously implies the corresponding property for the local Stummel classes with respect to submersions of manifolds.

The following majorization property makes the Stummel classes important in the theory of Schrödinger operators: if $V \in S_n$ on $\mathbb{R}^n$, then for any $a > 0$ there is a constant $C > 0$ such that

$$\|Vu\| \leqslant a\|\Delta u\| + C\|u\|, \qquad u \in C_c^\infty(\mathbb{R}^n), \tag{C.2}$$

where Δ is the flat Laplacian. Hence, if $V \in S_{n,\mathrm{loc}}(M)$, where $\dim M = n$, then the inequality (2.2) holds for the function V (instead of $|V_-|$) because, locally, all elliptic operators of the same order have equal strength.

C.2. Example. We have $1/|x| \in L^2_{\mathrm{unif}}(\mathbb{R}^3)$, and hence $1/|x| \in S_3$. It follows that $1/|x_k - x_l| \in S_{3N}$ in $\mathbb{R}^{3N}$ if $1 \leqslant k < l \leqslant N$, $x_k, x_l \in \mathbb{R}^3$, and the points in $\mathbb{R}^{3N}$ are represented in the form $x = (x_1, \dots, x_N)$ with $x_j \in \mathbb{R}^3$. Therefore, the inequality (C.2) implies the majorization relation

$$\|\,|x_k - x_l|^{-1}u\| \leqslant a\|\Delta u\| + C\|u\|, \qquad u \in C_c^\infty(\mathbb{R}^{3N}), \tag{C.3}$$

with an arbitrary $a > 0$ and with $C = C(a)$. However, it should be noted that $|x_k - x_l|^{-1} \notin L^{3N/2}(\mathbb{R}^{3N})$ already for $N = 2$.

C.3. Kato classes. The definitions of the uniform Kato classes K_n are similar to those of the Stummel classes S_n and play the same role for majorization of quadratic forms as the Stummel classes for majorization of operators. They were introduced by Beals [3] as a special case of more general classes in Schechter's paper [71] (see also Schechter's book [72] or its first edition published in 1971). A year later, this class was reinvented by Kato in his famous paper [47], where Kato's inequality appeared for the first time.

A good introduction to the Kato classes K_n can be found in § 1.2 of [23]. These classes were extensively studied by Aizenman and Simon ([2], [82]).

The (uniform) Kato class K_n consists of the measurable real-valued functions V on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\lim_{r\downarrow 0}\left[\sup_x \int_{|x-y|\leqslant r} |x-y|^{2-n}|V(y)|\,dy\right] = 0 && \text{for} \quad n \geqslant 3,\\
&\lim_{r\downarrow 0}\left[\sup_x \int_{|x-y|\leqslant r} \log(|x-y|^{-1})|V(y)|\,dy\right] = 0 && \text{for} \quad n = 2,\\
&\sup_x \int_{|x-y|\leqslant r_0} |V(y)|\,dy < \infty && \text{for} \quad n = 1.
\end{aligned}$$

Here $r_0 > 0$ is an arbitrarily fixed number. The class K_n is again invariant under multiplication by any real-valued bounded measurable function, as well as under any diffeomorphism of $\mathbb{R}^n$ that is linear near infinity. Thus, the corresponding local class $K_{n,\mathrm{loc}}(M)$ is well defined for any manifold of dimension $\dim M = n$, and we denote by $K_{n,\mathrm{comp}}(M)$ the class of compactly supported functions $f \in K_{n,\mathrm{loc}}(M)$. Both the classes $K_{n,\mathrm{loc}}(M)$ and $K_{n,\mathrm{comp}}(M)$ are invariant under diffeomorphisms of M and also under multiplication by any real-valued locally bounded measurable function.

The relation similar to (C.1) also holds for the Kato classes:

$$L^p_{\mathrm{loc}}(M) \subset K_{n,\mathrm{loc}}(M) \quad \text{if} \quad p > n/2 \ \text{ for } \ n \geqslant 2 \quad \text{and if} \quad p = 2 \ \text{ for } \ n = 1.$$

The same holds for the classes of compactly supported functions and for the uniform classes K_n on $\mathbb{R}^n$ (if $L^p(\mathbb{R}^n)$ is replaced by the class $L^p_{\mathrm{unif}}(\mathbb{R}^n)$).

As far as the uniform Stummel classes are concerned, it is also true that if $u \in K_s$ on $\mathbb{R}^s$ and $\pi\colon \mathbb{R}^n \to \mathbb{R}^s$ is a surjective linear projection, then $\pi^* u \in K_n$. This obviously implies the corresponding property for the local Kato classes with respect to submersions of manifolds.

The following majorization property for quadratic forms makes the Kato classes important: if $V \in K_n$ on $\mathbb{R}^n$, then for any $a > 0$ there is a $C > 0$ such that

$$|(Vu, u)| \leqslant a(\Delta u, u) + C\|u\|^2, \qquad u \in C_c^\infty(\mathbb{R}^n).$$

The arguments given in Example 8.2 can be repeated literally with the Stummel classes replaced by the Kato classes.

Appendix D. Some more history

In this appendix we describe some related results and provide bibliographical comments. This section complements § 2.16, where the most recent references were provided.

This survey is by no means complete. It would be next to impossible to make it complete. (For example, MathSciNet, the database of the American Mathematical Society based on Mathematical Reviews, lists more than 1000 related papers.) Thus, except for several landmark papers and papers which were explicitly important for our work or closely related to it, we concentrated on results concerning operators on manifolds. A comprehensive survey of self-adjointness results for one-dimensional operators can be found in [25]. Concerning the multidimensional case (for operators on $\mathbb{R}^n$), the reader can consult the books [64], Chap. X, and [26] and the survey papers [43], [41], and [83]. We tried not to repeat these sources, unless it was directly relevant to the main text of our paper.

This story was begun by Weyl (as many good stories in mathematics) in 1909–1910 in his pioneering papers on the spectral theory of one-dimensional symmetric singular differential operators (see [90]–[92] and also [20], Chap. IX, [34], Chap. 13, and [55], Chap. II, § 2). The term singular means here that one (or two) endpoints of an interval on which the operator is considered is either a point at infinity or a singular point of coefficients of the operator. Depending on the asymptotic behaviour of solutions of the corresponding ordinary differential equation at a singular endpoint of the interval, Weyl classifies the situation at this endpoint as either the *limit point case* or the *limit circle case.* In modern terminology (which emerged decades later, after the invention of quantum mechanics and von Neumann's mathematical formulation of it), the case of limit points at both endpoints corresponds to essential self-adjointness.

Among the first authors who wrote about multidimensional Schrödinger operators of the form $H_V = -\Delta + V$ in $\mathbb{R}^n$, we find Carleman [14] and Friedrichs [28], who proved independently the essential self-adjointness when the potential V is locally bounded and semibounded below. (Carleman's proof was reproduced in the book [32] of Glazman, Chap. 1, Theorem 34.)

Moving closer to this day, let us consider a magnetic Schrödinger operator (on $M = \mathbb{R}^n$) of the form

$$H_{g,b,V} = -\sum_{j,k=1}^{n} (\partial_j - ib_j(x))g^{jk}(x)(\partial_k - ib_k(x)) + V(x), \tag{D.13}$$

where the matrix $g^{jk}(x)$ (the metric) is assumed to be real, symmetric, and positive definite, and b_j and V are some real-valued functions. Additional regularity conditions can be imposed on g^{jk}, b_j, and V. Let us assume that the coefficients g^{jk} are sufficiently smooth (at least locally Lipschitz).

We denote by $H_{b,V}$ the operator $H_{g,b,V}$ for the flat metric g^{jk}, that is, $g^{jk} = \delta_{jk}$.

Assume first for simplicity that $b_j \in C^1(\mathbb{R}^n)$. It is desirable that the operator $H_{b,V}$ be naturally defined on $C_c^\infty(\mathbb{R}^n)$. Then we must assume that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

The most dramatic improvement of the Carleman–Friedrichs result in the case of potentials V semibounded below is due to Kato [47]. Kato established that the condition of local boundedness imposed on V can be completely removed, and hence it suffices to assume that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and V is semibounded below (even for the magnetic Schrödinger operator $H_{b,V}$ with $b_j \in C^1(\mathbb{R}^n)$). What is now called Kato's inequality appeared for the first time in this context in [47]. Namely, let L be the operator in (D.1) with $V = 0$ and let L_0 be the operator with $b_j = 0$ and $V = 0$ (thus, $L_0 = -\Delta$). We also assume that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $Lu \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the following inequality holds:

$$L_0|u| \leqslant \mathrm{Re}(Lu \cdot \mathrm{sign}\,\overline{u}\,),$$

where $(\mathrm{sign}\,u)(x)$ is equal to $u(x)/|u(x)|$ if $u(x) \neq 0$ and to 0 otherwise. The main result of Kato's paper [47] was the essential self-adjointness of the operator $H_{b,V}$ with $V = V_1 + V_2$, where $V_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $V_1 \geqslant -f(|x|)$, $f(r)$ is a monotone nondecreasing function of $r = |x|$ of order $o(r^2)$ as $r \to \infty$, and V_2 belongs to what is now called the Kato class (see Appendix C). In particular, Kato was the first who established the essential self-adjointness of the operator H_V with $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $V \geqslant 0$, without any global conditions on V.

Returning to a more general case of the operator $H_{b,V}$, we now omit the condition $b_j \in C^1(\mathbb{R}^n)$. Then the natural conditions for the minimal operator to be well defined are $b \in (L^4_{\mathrm{loc}}(\mathbb{R}^n))^n$, $\mathrm{div}\, b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Working under these minimal regularity conditions only, Leinfelder and Simader [52] improved Kato's result by admitting potentials of the form $V = V_1 + V_2$, where $V_1 \geqslant -c|x|^2$ and V_2 is Δ-bounded with a relative bound $a < 1$. In particular, if V is semibounded below, then the above minimal regularity conditions on b and V are sufficient for essential self-adjointness (see also [23], Chap. 1). A non-trivial technique of non-linear truncations is used in [52] to approximate functions in the maximal domain of the operator $H_{b,V}$ and of its quadratic form by bounded functions.

A natural question (probably first formulated by Glazman) is as follows: can we replace the semiboundedness below of the potential (in the above Carleman–Friedrichs result on essential self-adjointness) by the semiboundedness of the operator itself (that is, the semiboundedness of the corresponding quadratic form) on $C_c^\infty(\mathbb{R}^n)$? (In this case, the potential can fail to be semibounded below on some relatively 'small' sets.)

It turns out that if we assume only that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, then the semiboundedness below of H_V does not imply its essential self-adjointness in general, as we can see from the following example (see [43], Example 1a, and [81], Appendix 2, and also [64], Example 4 in § X.2, for $n = 5$).

D.1. Example. Let $H_V = -\Delta + V$, where $V = \frac{\beta}{|x|^2}$ and $n \geqslant 5$. Then H_V is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$ if and only if $\beta \geqslant \beta_0 := 1 - \bigl(\frac{n-2}{2}\bigr)^2$. The operator H_V is semibounded below if and only if $\beta \geqslant -\bigl(\frac{n-2}{2}\bigr)^2$. The first assertion follows by separation of variables, and the second from Hardy's inequality (see [44]).

The first result about essential self-adjointness of semibounded differential operators is probably due to Rellich [65], who considered the one-dimensional case and a general Sturm–Liouville operator on an interval with one or two singular endpoints. In the case when $M = \mathbb{R}^n$, V is continuous, and $H_V = -\Delta + V$ is semibounded below, the essential self-adjointness of H_V was proved by Povzner [63] (Chap. I, Theorem 6). (Povzner indicates that the result was conjectured by Glazman in a conversation between them. This happened not later than 1952, the year in which Povzner's paper was submitted for publication. It is unclear whether Glazman knew of Rellich's paper.) About a year and a half after the publication of Povzner's paper, the essential self-adjointness problem for a semibounded Schrödinger operator H_V was posed by Rellich [66] in his talk at the Amsterdam International Congress of Mathematicians. Answering Rellich's question, Wienholtz [93] proved the following result.

Theorem D.2. *Let $H_{g,b,V}$ be elliptic, let $g^{jk} = g^{kj}$ be bounded functions of class $C^3(\mathbb{R}^n)$, let $b_j \in C^3(\mathbb{R}^n)$, and let V be a real continuous function. Suppose that the operator $H_{g,b,V}$ is semibounded below on $C_c^\infty(\mathbb{R}^n)$. Then $H_{g,b,V}$ is essentially self-adjoint.*

(Clearly, neither Rellich nor Wienholtz were aware of Povzner's paper. Wienholtz used a simpler method than Povzner. A simplified version of the Wienholtz result is explained in [32].) Wienholtz [93] also proved the same statement for any potential V belonging to a global Stummel-type class (V need not be continuous).

The case in which the operator H_V is semibounded differs substantially from the case in which the potential V is semibounded. This is true even for $n = 1$, as becomes clear from an example by Moser, which is described by Rellich [65] and also cited by Kalf [40]. This is an example of a semibounded Schrödinger operator H_V in $L^2(\mathbb{R})$ with smooth potential V such that $\mathrm{Dom}(H_{V,\max})$ is not contained in $W^{1,2}(\mathbb{R})$, which means that there is an element $u \in \mathrm{Dom}(H_{V,\max})$ for which

$u' \notin L^2(\mathbb{R})$. This cannot happen if V is semibounded below (for instance, see Proposition 8.13).

There are many papers in which the smoothness assumptions on the coefficients in Theorem D.2 were relaxed in diverse directions (see, for instance, the results cited in [43]). In particular, an important step was made by Simader [80] who considered an operator $H_V = -\Delta + V$ that is semibounded below with $V = V_1 + V_2$, where $0 \leqslant V_1 \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and V_2 satisfies a local Stummel-type condition or a local majorization condition. The proof is based on the observation that $\mathrm{Dom}(H_{V,\max}) \subset W^{1,2}_{\mathrm{loc}}$ in this case (this is the most difficult part of the proof), and this condition is sufficient for essential self-adjointness. Following [79], we have used Simader's ideas in the geometric context of the present paper.

Brézis [8] introduced yet another majorization-type local condition on V ('localized' self-adjointness), which also implies the essential self-adjointness of a semibounded operator H_V in $\mathbb{R}^n$. We also note that Rofe-Beketov and Kalf [70] unified the results of Simader [80] and Brézis [8] by a refined use of localized self-adjointness.

The smoothness assumptions for the metric coefficients g^{jk} can also be relaxed. For example, Perel'muter and Semenov [62] proved the essential self-adjointness of the operator $H_{g,0,V}$ for $V \geqslant 0$ and $g^{jk} - \delta_{jk} \in W^{1,4}(\mathbb{R}^n)$.

A Povzner–Wienholtz-type result for matrix-valued Sturm–Liouville operators on intervals in $\mathbb{R}$ and with coefficients in L^1_{loc} was established by Clark and Gesztesy in their recent preprint [19], which appeared after our paper was submitted for publication.

We comment on the completeness condition in Theorem 2.13. This condition cannot be omitted. In fact, Ural'tseva [88] and Laptev [51] showed that there are elliptic operators $H_{g,0,0}$ acting in $L^2(\mathbb{R}^n)$, $n \geqslant 3$, that is, operators of the form

$$\sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(g^{jk}(x)\frac{\partial}{\partial x^k}\right)$$

with smooth positive-definite matrix g^{jk}, such that the space $C_c^\infty(\mathbb{R}^n)$ is not dense in the maximal domain $Q_{\max}$ of the corresponding quadratic form. If this non-completeness holds, then the operator $H_{g,0,0}$ is not essentially self-adjoint. This is caused by the 'rapid growth' of the functions g^{jk}, which implies that the elements of the inverse matrix g_{jk} are 'rapidly decreasing', which means that $\mathbb{R}^n$ is not complete with respect to the metric g_{jk}. (Laptev [51] also gives some sufficient conditions ensuring that $C_c^\infty(\mathbb{R}^n)$ is dense in $Q_{\max}$.)

Maz'ya [56] (see also [57], § 2.7) established the amazing fact that the cases $n = 1, 2$ are special here. Using the notion of capacity, he proved that $C_c^\infty(\mathbb{R}^n)$ is always dense in $Q_{\max}$ for $n = 1$ or $n = 2$.

The non-completeness of the metric can sometimes be compensated by a specific behaviour of V, in which case the operator $H_{g,b,V}$ can be self-adjoint even though the metric g is not complete (see, for instance, [12], [13], [22], Chap. 4, and the references therein).

We make some comments on Schrödinger-type operators which can fail to be semibounded below. The first result similar to Theorem 2.7 (however, on $M = \mathbb{R}^n$

with the standard metric and measure and with $q = q(|x|))$ for an operator $H_V = -\Delta + V$ with $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ is due to Sears [73] (see also [86], §§ 22.15 and 22.16, and [5], Chap. 3, Theorem 1.1), who followed an idea of an earlier paper by Titchmarsh [85]. Rofe-Beketov [68] was apparently the first who admitted a minorant $q(x)$ that is not radially symmetric, though he did not formulate the conditions geometrically. He also proved that the local inequality $V \geqslant -q$ can be replaced by the operator inequality $H_V \geqslant -\delta\Delta - q(x)$ with a constant $\delta > 0$. This fact enables us to consider potentials that are not bounded below even locally. Oleinik [58]–[60] showed that this phenomenon extends to the case of manifolds. He also provided a geometric self-adjointness condition relating this property to the classical completeness in the situation that is not radially symmetric. Oleinik's proof was simplified by Shubin [77], and then the result was extended to the case of magnetic Schrödinger operators in [78].

A first Sears-type result for an operator $H_{b,V}$ with a locally singular potential V was obtained by Ikebe and Kato [37] (still with a radially symmetric minorant for V). The paper [78] extends this result to complete Riemannian manifolds and admits non-radially symmetric minorants for the potential V with Oleinik-type completeness condition, though the potential V is assumed to be locally bounded. By now we can also admit appropriate singularities of the potential V for the magnetic Schrödinger operators considered in [78], since these operators are special cases of those in Theorem 2.7.

A remarkable Sears-type result of Iwatsuka [38] is seemingly the only result directly involving the magnetic field in the essential self-adjointness. In this result, the magnetic field grows at infinity in such a way that the scalar potential V can fall off to $-\infty$ more rapidly. (This fall can in fact be arbitrarily rapid, depending on the growth of the magnetic field.)

Levitan [54] suggested a new proof of the Sears theorem. This proof is based on the consideration of the hyperbolic Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} + Hu = 0, \qquad u\Big|_{t=0} = f, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = g, \tag{D.2}$$

where H is the symmetric differential operator in question (for instance, $H = H_{g,b,V}$ as in (D.1)). Let us assume for a moment that H is non-negative (or semibounded below) on $C_c^\infty(\mathbb{R}^n)$. Then the essential self-adjointness of H would follow if the problem (D.2) were known to have a unique solution belonging to $L^2(\mathbb{R}^n_x)$ for every $t \in \mathbb{R}$ (or even $t \in [0, t_0]$ with some $t_0 > 0$) for any $f, g \in C_c^\infty(\mathbb{R}^n)$, because different self-adjoint extensions would generally produce different solutions by the spectral theorem. In turn, the uniqueness follows (by the Holmgren principle) from the existence if a globally finite propagation speed is ensured, that is, if for every $t_0 > 0$, every bounded open set $G \subset \mathbb{R}^n$, and every $f, g \in C_c^\infty(G)$ there is a bounded open set $\Omega \supset G$ such that the problem (D.2) has a solution $u(t,x) \in C^\infty([0,t_0]\times\mathbb{R}^n)$ for which $\operatorname{supp} u(t,x) \subset \Omega$ for all $t \in [0, t_0]$. A modification of this argument enables one to consider non-semibounded operators as well. Establishing the globally finite propagation speed for a problem of the form (D.2) is crucial in proving the essential self-adjointness.

A good explanation of the approach based on the abstract hyperbolic equation can be found in Berezanskii's book [4] (Chap. VI, § 1.7). For details concerning the hyperbolic equation method see the paper [61] by Orochko, which also contains a good survey with many relevant references.

Berezanskii's book [4] also contains an extensive discussion of self-adjointness for operators generated by boundary-value problems for elliptic and more general operators.

Up to now we assumed that the potential V belongs (at least) to $L^2_{\rm loc}(\mathbb{R}^n)$, in which case the minimal operator is defined on $C_c^\infty(\mathbb{R}^n)$. However, Kato [49] pointed out that if $V \in L^1_{\rm loc}(\mathbb{R}^n)$, then one can still consider the maximal operator $H_{V,\max}$ associated with $H_V = -\Delta + V$ as an operator with the domain

$$\mathrm{Dom}(H_{V,\max}) = \big\{u \in L^2(\mathbb{R}^n) : Vu \in L^1_{\rm loc}(\mathbb{R}^n) \;\text{ and }\; H_V u \in L^2(\mathbb{R}^n)\big\},$$

where $H_V u$ is *a priori* defined as a distribution. The question which can then be asked is whether the operator $H_{V,\max}$ is self-adjoint. Moreover, the minimal operator $H_{V,\min}$ can now be defined as the restriction of $H_{V,\max}$ to the compactly supported elements of $\mathrm{Dom}(H_{V,\max})$. In the latter case one can ask whether the operator $H_{V,\min}$ is densely defined and whether $H^*_{V,\min} = H_{V,\max}$. More general operators of the form (D.1) can also be considered. We refer the reader to [50], [27], [42], and [61] for results and references in this direction. In particular, locally integrable potentials were treated in [61] by the hyperbolic equation method.

Let us discuss some results on essential self-adjointness for operators on manifolds. The essential self-adjointness of the Laplace–Beltrami operator on a complete Riemannian manifold was established by Gaffney [30] (not only in $L^2(M)$ but also in the standard L^2-spaces of differential forms). Cordes [21] (see also [22], Chap. 4) established the essential self-adjointness of powers of Schrödinger-type operators with positive potentials (as in the present paper, these operators act in L^2 spaces defined by measures not related to the metric). Cordes studied not only the case of complete Riemannian manifolds but also cases in which the non-completeness is compensated by an appropriate behaviour of the potential, for instance, in domains $G \subset M$ of a complete Riemannian manifold M if the potential $V(x)$ is bounded below by the quantity $\gamma(\mathrm{dist}(x, \partial G))^{-2}$ with appropriate constant $\gamma > 0$ (his approach is 'stationary'). Chernoff [16] used the hyperbolic equation approach to establish similar results. (He later extended his results to the case of singular potentials V [17].) We refer the reader to Cordes' book [22], Chap. 4, for other results concerning these topics. Kato [48] extended Chernoff's arguments used in [16] to prove the essential self-adjointness of powers of an operator H_V, where $H_V \geqslant -a - b|x|^2$ and V is smooth (for $H_V = -\Delta + V$ on $M = \mathbb{R}^n$ with the standard metric and measure). Chumak [18] used the hyperbolic equation method to prove the essential self-adjointness of operators $H_{g,0,V}$ semibounded below on complete Riemannian manifolds.

We also mention a subtle result of Donnelly and Garofalo [24], who proved the essential self-adjointness of the operator $H_V = -\Delta + V$ on a complete Riemannian manifold under the assumption that V is locally in a Stummel-type class except at a point x_0, and has a negative minorant $Q(x)$ such that $Q(x) = \beta_0 \,\mathrm{dist}(x, x_0)^{-2}$ near x_0 (which exactly agrees with the borderline case in Example 9.1, and thus the condition is practically optimal) and $Q(x) = -c\,\mathrm{dist}(x, x_0)^2$ at infinity (with a constant $c > 0$).

For other related results on Schrödinger-type operators on manifolds, see, for instance, the references in [22] and [77]–[79]. For higher-order operators in $\mathbb{R}^n$, see [72] and [69]. For higher-order operators on manifolds of bounded geometry, see [75].

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