Theorem 3. On a nonsingular algebraic variety, $c_k^d$ is the dimension of the $\mathbb{Q}$-submodule of $H^d(M, \mathbb{Q})$ generated by cohomology classes dual to $(2n - d)$-dimensional rational cycles formed by intersecting $(2n + 2k - d)$-dimensional rational cycles with $(n - k)$-dimensional algebraic subvarieties of $M_n$.

Thus the index $c_{n-k}^{2n-d}$ in a sense gives the dimension of the set of $d$-dimensional cycles lying on $k$-dimensional algebraic subvarieties of $M_n$. The proof of this result follows from a theorem of Lefschetz, which asserts that a $(2n - 2)$-dimensional cycle is represented by a divisor (effective or not) if and only if its dual cohomology class is represented by a differential form of type $(1, 1)$, and from a theorem of Severi, which asserts that an irreducible algebraic subvariety $V_k \subset M_n$ is the complete intersection of $r - k$ divisors (effective or not) on $M_n$.

3. This corresponds to the Riemann conditions on $\Omega$.
4. F. Severi, Serie, sistemi d'equivalenza e corrispondenza algebriche sulla varietà algebriche (Rome, 1942).

ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS

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Everything in this note will be considered from the semilinear point of view; i.e., any 3-manifold will be considered with a fixed triangulation (this is permissible according to Moise's work), any curve or line will be considered as polygonal, any surface as polyhedral, and so on.

The following theorem was first considered by Dehn, but it was pointed out by Kneser that Dehn's proof contains a gap.

Dehn's Lemma. Let $M$ be a 3-manifold, compact or not, with boundary which may be empty, and in $M$ let $D$ be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve $C$ and such that there exists a closed neighborhood of $C$ in $D$ which is an annulus (i.e., no point of $C$ is singular). Then there exists a 2-cell $D_0$ with boundary $C$, semilinearly imbedded in $M$.

Johansson proved that, if Dehn's lemma holds for all orientable 3-manifolds, it also holds for all nonorientable ones. We prove that Dehn's lemma holds for all orientable 3-manifolds, and by a modification of our method we also prove the following theorem.

Sphere Theorem. Let $M$ be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$, and which can be topologically imbedded in a 3-manifold $N$, having the following property: The first homology group of any nontrivial (but not necessarily proper) subgroup of $\pi_1(N)$, has an element of infinite order (note in particular that this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere $S$ semilinearly imbedded in $M$, such that $S$ is not homotopic to zero in $M$. 
From these two theorems follow others, which may be stated easily if we introduce
the following notions: Let $F$ be a nonempty proper closed subset in $S^3$. We say
that $F$ is geometrically splittable if there is a 2-sphere $S^2 \subset S^3 - F$ such that both
components of $S^3 - S^2$ contain points of $F$. We say that $F$ is algebraically splitt-
able if $\pi_1(S^3 - F)$ is the free product of two groups, each of which is nontrivial.

**Theorem 1.** Let $U$ be a nonempty proper open connected subset of the 3-sphere $S^3$.
Then $U$ is aspherical if and only if $S^3 - U$ is not geometrically splittable.

The above theorem provides us with a solution of a problem of Whitehead. From Theorem 1 follows easily the following:

**Corollary 1.** If $F$ is a nonempty proper closed connected subset of $S^3$, then each
component of $S^3 - F$ is aspherical.

The above corollary provides us with a solution of a problem of Eilenberg. An
immediate consequence of Corollary 1 is the following:

**Corollary 2.** If $F$ is a connected graph or knot, then $S^3 - F$ is aspherical.

The following theorem solves completely a problem initiated by Higman.

**Theorem 2.** Let $K$ be a link in $S^3$. The following three statements are equiva-

(i) $S^3 - K$ is not aspherical.

(ii) $K$ is geometrically splittable.

(iii) $K$ is algebraically splittable.

Let $K$ be a knot in $S^3$. According to Dehn (op. cit., Satz 2, p. 158), Specker, Papakyriakopoulos, and the above Corollary 2, the following theorem holds:

**Theorem 3.** (i) $K$ is unknotted if and only if $\pi_1(S^3 - K)$ is free cyclic. (ii) The number of ends of $\pi_1(S^3 - K)$ is either 1 or 2. (iii) $\pi_1(S^3 - K)$ has 1 end if $K$ is

The following statement is known as Hopf’s conjecture.

**Theorem 4.** If $U$ is an open connected subset of the 3-sphere, then $\pi_1(U)$ has no
element of finite order.

The proof is based on the sphere theorem and the following simple consequence of
a theorem due to P. A. Smith: The fundamental group of an aspherical poly-

A Sketch of the Proof of Dehn’s Lemma.—By a Dehn disk we mean a 2-cell, which
may have singularities, but not on its boundary. We suppose that $M$ is orientable
and that $D$ has no branch points (see Whitehead). Let $G$ be the inverse image of
$D$ under $f$, where $G$ is a 2-cell. There is a finite set $J$ of triples $(J', J'', \psi)$, where
$J'$, $J''$ are closed curves on $G$, called the $J$-curves, and $\psi: J' \to J''$ is a “nice map”
such that $f(r) = f\psi(r)$, for any point $r \in J'$, i.e., $f(J') = f(J'')$ is a double line of $D$.

Emphasize that, according to Johansson, $J' \neq J''$ because $M$ is orientable.

We have a map $f: (G, J) \to D$ called a realized diagram. We denote by $l(D)$ and
d($D$) the number of triple points and double lines of $D$, respectively. The couple
($l(D)$, $d(D)$) is called the complexity of $D$.

Let $V \subset M$ be a 3-manifold with boundary such that int $V$ is a neighborhood of
$D - C$, $C \subset$ bdry $V$, and $V$ is very small and “very nice,” so that each face of
bdry $V$ is “parallel” to a face of $D$. Such a $V$ is called a prismatic neighborhood
of $D$ in $M$. We observe that $D$ is a deformation retract of $V$. Let $p_\#: M_\# \to V$
be the universal covering of $V$, let $D_\#$ be a Dehn disk covering $D$ just once, and let
Finally, let $V_*$ be a prismatic neighborhood of $D_*$ in $M_*$. So we have the following diagram:

$$
\begin{align*}
M_* & \supset V_* \supset D_* \leftarrow (G, J_*) \\
p_* & \downarrow q_* \\
M & \supset V & D & \leftarrow (G, J)
\end{align*}
$$
called an elementary tower over $D \subset V \subset M$, where $f_*$ is the lifted map, and $f_* : (G, J_*) \rightarrow D_*$ is a realized diagram.

If $V$ is not simply connected, then there is a covering translation $\tau$ of $p_* : M_*) \rightarrow V$, such that $D_*) \cap \tau^{-1}(D_*)$ is not empty, and so it consists of a finite number of closed curves $T_i, i = s(\tau, 1), \ldots, s(\tau, d(\tau))$. These curves for all possible $\tau$'s are called the $T$-curves. Then

$$d(D) = d(D*) + \frac{1}{2} \sum d(\tau),$$

where the sum runs over all covering translations $\tau \neq 1$ (note that if $D_* \cap \tau^{-1}(D_*) = \emptyset$, then naturally $d(\tau) = 0$). Hence $d(D_*) < d(D)$. We emphasize that, in the special case where $D_*$ is a 2-cell, $q_* : (D_*, T_*) \rightarrow D$ is a realized diagram, where $T_*$ is now the set of all triples $(T', T'', \tau)$ such that $T' \in D_* \cap \tau^{-1}(D_*)$ and $T'' = \tau(T') \in \tau(D_*) \cap D_*$. If $V$ is simply connected, then each of the components of bdry $V$ is a 2-sphere, according to a result due to Seifert.\(^{31}\) In this case $d(D_*) = d(D)$.

The diagram

\[
\begin{align*}
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
M_m & \supset V_m \supset D_m \leftarrow f_m (G, J_m) \\
p_m & \downarrow q_m \\
M_{m-1} & \supset V_{m-1} \supset D_{m-1} \leftarrow f_{m-1} (G, J_{m-1}) \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
M_1 & \supset V_1 \supset D_1 \leftarrow f_1 (G, J_1) \\
p_1 & \downarrow q_1 \\
M & \supset V & D & \leftarrow (G, J)
\end{align*}
\]

is called a tower over $D \subset M$ and is defined as follows: The diagram $(m)$ is an elementary tower over $D_{m-1} \subset V_{m-1} \subset M_{m-1}$, for $m = 1, 2, \ldots$, where $M_0 = M$, $V_0 = V$, $D_0 = D$, $f_1 = f$, $J_0 = J$. Then

$$d(D_0) \geq d(D_1) \geq d(D_2) \geq \ldots \geq 0,$$

and there exists a number $n \geq 0$, such that $d(D_i) > d(D_{i+1})$, for $i < n$, and $d(D_j) = d(D_{j+1})$ for $j \geq n$, i.e., $V_n$ is simply connected but $V_1$ is not. This number $n$ is called
the height of the tower. The following two cases are possible: (1) \( d(D_n) > 0 \), i.e., \( D_n \) is not a 2-cell; (2) \( d(D_n) = 0 \), i.e., \( D_n \) is a 2-cell.

In case (1) \( V_n \) is simply connected, and so \( \text{bdry} \ V_n \) is composed of 2-spheres, according to the result of Seifert mentioned above. Let \( D_n' \) be one of the 2-cells bounded by \( C_n \), on the component of \( \text{bdry} \ V_n \) containing the boundary \( C_n \) of \( D_n \). Then \( D' = p_1 \ldots p_n(D_n') \subset M \) is a Dehn disk with boundary \( C \) and "roughly speaking" having the following property: Either \( t(D') < t(D) \), or \( t(D') = t(D) \) and \( d(D') < d(D) \), i.e., we may say that complexity of \( D' < \) complexity of \( D \).

In case (2) we prove that there exists a triple \((J', J'', \psi)\) of \( f: (G, J) \to D \) such that \( J' \) and \( J'' \) are disjoint simple closed curves. The proof of this is rather algebraic and makes use of the fact that \( q_n: (D_n, T_n) \to D_{n-1} \) is a realized diagram, as we have observed above, and that each element of \( T_n \) is a triple \((T', T'', \tau)\), where \( \tau \) is a covering translation of \( p_n: M_n \to V_{n-1} \). Then by a cut ("Umschaltung";\(^{14}\)) note that this is the only case in which we can apply Dehn's process without any danger [see Dehn, op. cit. p. 150, B, and Kneser, op. cit., p. 260] of \( D \) along \( J = f(J') = f(J'') \) we obtain a new Dehn disk \( D' \subset M \), with boundary \( C \) and such that either \( t(D') < t(D) \) or \( t(D') = t(D) \) and \( d(D') < d(D) \), i.e., we may say that complexity of \( D' < \) complexity of \( D \).

In the same way we obtain from \( D' \) a new Dehn disk \( D'' \subset M \) with boundary \( C \) such that complexity of \( D'' < \) complexity of \( D' \), and so on. Thus, after a finite number of repetitions of the above process, we finally obtain a Dehn disk in \( M \) with boundary \( C \) and complexity \((0, 0)\), i.e., we obtain a 2-cell in \( M \) with boundary \( C \).

As far as the proof of the sphere theorem is concerned, we restrict ourself to the remark that the method of proof makes use of the above process, standard Hurewicz theorems, and the Poincaré duality theorem.

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ON DEHN’S LEMMA AND THE ASPHERICITY OF KNOTS

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Dedicated to Professor N. Kritikos

§1. Introduction

The present paper contains a proof of Dehn’s lemma and an analogous result that we call the sphere theorem, from which other theorems follow.¹

Dehn’s Lemma. Let $M$ be a 3-manifold, compact or not, with boundary which may be empty, and in $M$ let $D$ be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve $C$ and such that there exists a closed neighborhood of $C$ in $D$ which is an annulus (i.e. no point of $C$ is singular). Then there exists a 2-cell $D_0$ with boundary $C$, semi-linearly imbedded in $M$.

Sphere Theorem. Let $M$ be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_1(M) \neq 0$, and which can be semi-linearly² imbedded in a 3-manifold $N$, having the following property: the commutator quotient group of any non-trivial (but not necessarily proper) finitely generated subgroup of $\pi_1(N)$ has an element of infinite order (n.b. in particular this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere $S$ semi-linearly imbedded in $M$, such that³ $S \not\cong 0$ in $M$.

Dehn’s lemma was included in a 1910 paper of M. Dehn [4] p. 147, but in 1928 H. Kneser [13] p. 260, observed that Dehn’s proof contained a serious gap. In 1935 and 1938 appeared two papers by I. Johansson [11], [12], on Dehn’s lemma. In the second one, p. 659, he proves that, if Dehn’s lemma holds for all orientable 3-manifolds, it then holds for all non-orientable ones. We now prove in this paper that Dehn’s lemma holds for all orientable 3-manifolds. Our proof makes use also of I. Johansson’s first paper.

As far as the sphere theorem is concerned we have to remark that, to the best knowledge of this author, the first one to attempt a theorem of this kind was H. Kneser in 1928, [13] p. 257; however his proof does not seem to be conclusive. In 1937 S. Eilenberg [5] p. 242, Remark 1, observed a relation between the non-vanishing of the second homotopy group and the existence of a non-contractible 2-sphere. Finally in 1939 J. H. C. Whitehead [25] p. 161, posed a problem which stimulated the author to prove the sphere theorem, stated above. We emphasize that, if $\pi_1(N)$ is a free group⁴ then the hypotheses of the sphere theorem are fulfilled, according to the following

Nielsen-Schreier Theorem. Every subgroup of a free group is itself a free group.⁵

¹ Numbers in brackets refer to the bibliography at the end of the paper.
³ $\cong$ means homotopic to.
⁴ On a number $(\geq 0, \leq \infty)$ of free generators.
⁵ See [14] p. 28.
Even more, the hypotheses of the sphere theorem are fulfilled, if \( \pi_1(N) \) is locally free.\(^6\)

Applications of Dehn’s lemma and especially of the sphere theorem are explained in §§6–8. These §§ can be read without the knowledge of §§2–5, where the proofs of those two theorems are explained. §6 contains, in No. 26, the solution of the problem of J. H. C. Whitehead mentioned above, from which follows the solution of a problem of S. Eilenberg [5] p. 241, and the asphericity of knots. §6 contains in No. 27 the solution of a problem proposed and partly solved by G. Higman [8], and No. 28 contains the relation between knots and ends, first discovered by E. Specker [23] p. 329. §7 contains a general theorem from which follows immediately a conjecture due to H. Hopf. In §8 we prove that, if Poincaré’s conjecture is true then, the orientable closed\(^3\) 3-manifolds whose fundamental groups are free groups on \( h \geq 0 \) free generators, are completely characterized by the number \( h \), i.e. any two such 3-manifolds are homeomorphic if and only if they have the same \( h \). We emphasize that, throughout §§6–8 the key theorem is the sphere theorem, except in No. 28, where we need also Dehn’s lemma.

The §§3–5 form the main part of this paper. In those §§ the proofs of Dehn’s lemma and the sphere theorem are explained. We emphasize that the sphere theorem goes parallel to Dehn’s lemma, both in content and in proof. The condition \( \pi_2(M) \neq 0 \), in the sphere theorem, means the existence in \( M \) of a 2-sphere with self-intersections (singularities), which is not contractible in \( M \). From this, and the imbeddability of \( M \) in \( N \), we conclude the existence of a 2-sphere semi-linearly imbedded in \( M \), which is not contractible in \( M \). Thus obviously the sphere theorem is parallel to Dehn’s lemma in content. But also the proofs of those two theorems are parallel to each other. In fact the proof of the sphere theorem is a modification of that of Dehn’s lemma.

In §3 is explained the construction of a certain diagram called the tower. The tower forms actually the back-bone of the proofs of Dehn’s lemma and the sphere theorem. The whole §3 serves for both, for the proof of Dehn’s lemma and that of the sphere theorem. In §4 we prove Dehn’s lemma, in the case where \( M \) is orientable, and in §5 we prove the sphere theorem. §5 is a modification of §4. In §2 we explained the preliminaries, which are used in §§3–5.

The method of proving Dehn’s lemma and the sphere theorem is actually constructive, i.e. a method is given to construct the 2-cell and the 2-sphere, though in this paper we present the method in a different way. The method will be better understood if the sketch of the proof given in [20] is kept in mind. In that sketch the method is presented as constructive. We also emphasize that this paper contains two more results not contained in [20]: First, the general theorem (31.2), from which Hopf’s conjecture (31.8) follows as immediate corollary, and second, the theorem (32.1) mentioned above, concerning the orientable closed 3-manifolds, whose fundamental groups are free groups.

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\(^6\) Locally free means any finitely generated subgroup is free, [14] p. 166, and actually has a finite number \( \geq 0 \) of free generators (n.b. the trivial group is supposed to be locally free).

\(^7\) Closed means compact without boundary.
In §9 some problems arising from the previous §§ are explained.

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§2. Preliminaries

1. Normal curves. Any 3-manifold to be considered in this paper is supposed to have a fixed triangulation. This is permissible according to E. E. Moise's work, [15] and [16]. Moreover everything will be considered from the semi-linear point of view, i.e. any curve will be considered as polygonal, any surface as polyhedral and so on. By a curve we mean a polygonal curve which may or may not be simple, but by a surface we always mean a simple polyhedral surface (i.e. one that has no self-intersections).

A closed curve on a surface will be called normal, if a point describing it passes at most twice through any of its points, and at each double point the two branches of the curve actually cross one another. A finite system of closed curves on a surface is called normal, if each curve is normal, no point lies on more than two of the curves of the system, and each multiple point of the system is a double point at which the two branches involved actually cross one another. The double points and the intersection points of those curves, are called the crossing-points of the system.

2. Singular maps and surfaces. Let \( f: S \to M \) be a map such that \( S \) is a triangulated orientable surface of type \((p, r)\), i.e. an orientable compact surface of genus \( p \) and with \( r \) boundary curves, and \( f(\sigma) \) is a rectilinear 2-simplex in a 3-simplex of \( M \), for any 2-simplex \( \sigma \) of \( S \). Then \( f \) is called singular locally-linear map, and \( f(S) \) is called singular orientable surface of type \((p, r)\), since it may have self-intersections, cf. [4] p. 147, [11] p. 312, [13] p. 248, [17] p. 656. In the special case where \( S \) is of type \((0, 1)\) whence it is a disc (i.e. 2-cell), or of type \((0, 0)\) whence it is a 2-sphere, we call \( f(S) \) a singular disc or singular 2-sphere respectively.

A singular orientable surface \( f(S) \) is called normal, if its singularities (i.e. self-intersections) consist of double curves (fig. 1a) along which two sheets cross, triple points (fig. 3) at which three sheets cut, and branch points (Verzweigungspunkte, Windungspunkte) (fig. 2a). A point \( b \) is called a branch point if \( f(S) \) cuts a small sphere with \( b \) as its center in a single non-simple curve, cf. [4] p. 148, [11] p. 312, [17] pp. 656–657, and especially [24] first footnote on p. 66.

In some places in this paper we shall use cuts (Umschaltungen). Let \( ab \) be an arc on a double curve of a normal singular surface \( f(S) \) (fig. 1a). Let us consider an orientation of \( S \) and the induced orientation of \( f(S) \). We observe that \( ab \) belongs to the boundary of four small rectangles \( x, y, z, w \) each one of which has an orientation induced by that of \( f(S) \). Let us suppose that the orientation of \( ab \)
induced by that of $x$ (or $y$) is opposite to the orientation of $ab$ induced by that of $z$ (or $w$). We cut $x \cup y$ and $z \cup w$ along $ab$, and we match $x$ with $z$ along $ab$, and we also match $y$ with $w$ along $ab$ (fig. 1b). So we obtain a new singular orientable surface $g(S')$. The operation described is called an orientation preserving cut along $ab$. In the case that we do not take care about orientations we have a cut along $ab$. The points $a, b$ may be branch points of $f(S)$ (fig. 2a). We emphasize that $g(S')$ need not be normal, this happens for instance in (fig. 2b) where the normality is destroyed at $b$, but by a small modification of $g(S')$ we always can obtain a new normal singular orientable surface. We also emphasize that if $g(S')$ is normal, then the points $a$ and $b$ are branch points of it (fig. 1b). Regarding the cuts see [4] p. 150, [7] pp. 785–786, [11] p. 314, [17] pp. 658–659, [24] second footnote on p. 66.

3. Canonical Dehn surfaces. By a Dehn surface of type $(p, r)$ we mean a singular orientable surface of type $(p, r)$ such that no point of the boundary of it is singular, i.e. if $C$ is a component of its boundary, then there is a small tube $T$ around $C$ such that $T$ intersects the surface in an annulus. Obviously, any
singular orientable surface of type \((p, 0)\), is a Dehn surface. Therefore the above definition has real value only if \(r > 0\). In this paper we shall confine ourselves to Dehn surfaces of type \((0, 1)\) and \((0, 0)\), i.e. to Dehn discs and singular 2-spheres. However Nos. 2 and 3 contain more general things, which we have in mind to investigate in later papers.

A Dehn surface of type \((p, r)\) is called canonical if it is normal and has no branch points, i.e. its only singularities are double curves and triple points.

Let \(R = f(S)\) be a normal Dehn surface of type \((p, r)\), and let \(b\) be a branch point or a regular point of \(R\). Without loss of generality we may suppose that \(b\) is an interior point of a 3-simplex \(\sigma\) of \(M\). Let \(B\) be a sufficiently small convex 2-sphere in \(\sigma\) containing \(b\) in its interior, and let \(K = R \cap B\). This is a normal closed curve on \(B\), the number of whose double points is called the multiplicity of \(b\), and is denoted by \(m(b)\) (\(\geq 0\)). Obviously, \(m(b)\) does not depend on \(B\). If \(m(b) > 0\) then \(b\) is a branch point of \(R\), and if \(m(b) = 0\) then \(b\) is a regular point of \(R\).

Let \(b\) be a branch point of \(R\), such that \(m(b) > 1\). By making a small cut of \(R\) starting at \(b\) and ending at a point \(b' \in B\), we obtain from \(R\) a new normal Dehn surface \(R'\) of type \((p, r)\) such that the part of \(R\) outside \(B\) is not affected by the cut. By a small modification of \(R'\), which leaves untouched everything outside \(B\), we obtain from \(R'\) a new normal Dehn surface \(R''\) of type \((p, r)\) such that: (i) To the point \(b\) of \(R\) correspond two points \(b''\) and \(b'''\) of \(R''\), (ii) \(m(b') = 1, m(b'')\) and \(m(b''') \geq 0\), and \(m(b'') + m(b''') < m(b)\), (iii) \(R''\) is obtained from \(R\) by a deformation.

Properties (i)–(ii) are obvious, to prove (iii) we only need to prove that \(R\) is obtained from \(R'\) by a deformation. Let \(A\) be a convex solid cone with vertex \(b \in R'\), whose base lies on \(B\), contains \(b'\) in its interior, and is very small. Let now \(A'\) be another convex solid cone with vertex \(b\), containing \(A\) in its interior, very near and nicely placed in respect to \(A\). Keeping things fixed outside \(A'\), and shrinking \(A\) to the point \(b\), we obtain from \(R'\) precisely \(R\), i.e. \(R\) is obtained from \(R'\) by a deformation. This proves property (iii).
Applying if necessary the above process to the points $b''$ and $b'''$ of $R''$ etc., we finally obtain a new normal Dehn surface $R_\beta$ of type $(p, r)$ such that: $R_\beta = R$ outside $B$, on $B$ there are branch points of $R_\beta$ all having multiplicity 1, the part of $R_\beta$ inside $B$ is formed by a finite number of disjoint open discs, and $R_\beta$ is obtained from $R$ by a deformation.

Repeating this process for all branch points of multiplicity $> 1$, we obtain finally a normal Dehn surface $R_0$ of type $(p, r)$ such that: All branch points of $R_0$ have multiplicity 1, $R_0$ is obtained from $R$ by deformation, and $\text{bd } R_0 = \text{bd } R$ (note that if $\text{bd } R = \emptyset$, then $\text{bd } R_0 = \emptyset$). Hence the following lemma holds:

**Lemma (3.1).** If $R \subset M$ is a normal Dehn surface of type $(p, r)$, then there is a normal Dehn surface $R_0 \subset M$ of type $(p, r)$, obtained from $R$ by deformation and such that $\text{bd } R_0 = \text{bd } R$, and every branch point of $R_0$ has multiplicity 1.

Let us now suppose that $p \geq 0$ and $r = 1$, i.e. $\text{bd } R = \text{bd } R_0$ is a simple closed curve $C$. Let $\omega(R_0) > 0$ be the number of branch points of $R_0$, all of which are of multiplicity 1 by (3.1), and let $b$ be a branch point of $R_0$. Let us start a cut at $b$, and let us continue it until the end. Such a cut must end at a branch point. There results from this cut a normal Dehn surface $R'$ of type $(p', 1)$, where $\text{bd } R' = C$, and a normal Dehn surface $R''$ of type $(p'', 0)$ such that $p' + p'' = p$, and $w(R')$ and $w(R'') < w(R_0)$. Repeating a finite number of times the above process we finally obtain a normal Dehn surface $R^*$ of type $(p^*, 1)$ such that, $\text{bd } R^* = C$, $p^* \leq p$, and $w(R^*) = 0$, i.e. $R^*$ is canonical. Hence the following lemma holds:

**Lemma (3.2).** If $R \subset M$ is a normal Dehn surface of type $(p, 1)$ with boundary $C$, then there is a canonical Dehn surface in $M$ with the same boundary $C$ and of type $(p^*, 1)$, where $p^* \leq p$.

Let us now suppose that $p = 0$ and $r = 0$, i.e. $R_0$ is a normal singular 2-sphere, and that $R_0 \not\subset 0$ in $M$. Let $\omega(R_0) > 0$ be the number of branch points of $R_0$, which all are of multiplicity 1 by (3.1), and let $b$ be a branch point of $R_0$. Let us start a cut at $b$, and let us continue it till the end. Such a cut must end at a branch point. However we are careful not to disconnect the surface at $b$. So we obtain two normal singular 2-spheres $R'$, $R''$ in $M$ such that, $w(R')$ and $w(R'') < w(R_0)$, and $R_0 = R'R''$ (product of two 2-spheres in the sense of homotopy groups). Thus at least one of $R'$ and $R''$ is $\not\subset 0$ in $M$. Repeating a finite number of times the above process we finally obtain a normal singular 2-sphere $R^* \not\subset 0$ in $M$, such that $w(R^*) = 0$, i.e. $R^*$ is canonical. Hence the following lemma holds:

**Lemma (3.3).** If $R \subset M$ is a normal singular 2-sphere $\not\subset 0$ in $M$, then there exists in $M$ a canonical singular 2-sphere $\not\subset 0$ in $M$.

**4. Realized diagrams.** Let $G$ be an orientable surface of type $(p, r)$, and on $G$ let $(J'_i, J''_i, \psi_i), i = 1, \cdots, s$, be triples, where all $J$-curves are distinct and form a normal system of curves, and let $\psi_i: J'_i \to J''_i$ be a map such that:

(4.1) If a point $u$ "moves smoothly" on $J'_i$, then $\psi_i(u)$ moves smoothly on $J''_i$.

---

*bd = boundary, cl = closure, int = interior.*

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(4.2) If \( u \) is a crossing-point of the \( J \)-curves, i.e. \( u = r' \in J'_1 \cap J''_1 \), then \( \psi_i(u) \) is a crossing-point of the \( J \)-curves, i.e. \( r'' = \psi_i(r') \in J'_1 \cap J''_1 \) (fig. 4).

(4.3) \( r' = \psi_h(r'''') \), where \( r''' = \psi_j(r'') \in J'_1 \cap J''_1 \). Hence the following relation holds:

(4.4) 
\[
r' = \psi_h \psi_j (r').
\]

The pair \((G, J)\) consisting of \( G \) and the set \( J \) of all triples will be called a **diagram**. We also write \( s = N(J) \).

Let now \( D \) be a canonical Dehn surface of type \((p, r)\) in an orientable 3-manifold \( M \). Then there is a singular locally-linear map \( f : G \to D \), where \( G \) is an orientable surface of type \((p, r)\). The singularities of \( D \) consist of double lines and triple points, whose number is \( d(D) \) and \( t(D) \) respectively. If \( L_k \) is a double line of \( D \), then \( f^{-1}(L_k) \) consists of two distinct closed curves \( J'_k \) and \( J''_k \), according to [11] p. 319, 2 Satz.\footnote{Johansson studies only the Dehn discs, but the proof of 2 Satz holds also for our case, because \( G \) and \( M \) are orientable.} If \( r \) is a triple point of \( D \), then \( f^{-1}(r) \) consists of three points \( r', r'', r''' \) (fig. 4). We now observe that, if a point \( v \) moves smoothly on \( L_k \), then the two points \( f^{-1}(v) = u' \) and \( u'' \) move smoothly on \( J'_k \) and \( J''_k \) respectively. So we have a map \( \psi_k : J'_k \to J''_k \) such that \( f \psi_k = f \), and the set \( J \) of all triples \((J'_k, J''_k, \psi_k), k = 1, \cdots, d(D)\), give rise to a diagram \((G, J)\). Hence we actually have a map

(4.5) 
\[
f : (G, J) \to D
\]
called a **realized diagram** of \( D \). Obviously

(4.6) 
\[
N(J) = d(D).
\]

The pair \((t(D), d(D))\) will serve as a kind of “complexity” of \( D \). Obviously, \( D \) is a (non-singular) orientable surface of type \((p, r)\) if and only if \( d(D) = 0 \). It is easily proved that the realized diagram of \( D \) is essentially unique, i.e. if \( f' : (G', J') \to D \) is another realized diagram, then there exists an obvious homeo-
morphism \( h: G \to G' \) such that \( f = f'h. \) \( h \) is called a natural homeomorphism compatible with \( f, f' \).

Let us draw in the interior of \( G \) curves \( J'_{i1}, J'_{i2} \) or \( J''_{i1}, J''_{i2} \) parallel to and on either side of \( J'_i \) or \( J''_i \) respectively, so close to \( J'_i \) or \( J''_i \) that two of them do not intersect, except in the vicinity of a crossing-point, \( i = 1, \ldots, N(J), \) (fig. 5) and cf. [11] p. 316, fig. 7 and [12] p. 666, fig. 8. These curves divide \( G \) into (closed) regions of the following three types:

(4.7) **Triple regions** \( Z \), quadrilaterals in the vicinity of one of the points of intersection of the \( J \)-curves.

(4.8) **Double regions** \( Y \), long ribbon shaped quadrilaterals.

(4.9) **Single regions** \( X \), which may have many sides and need not be simply connected. If \( G \) is of type \((p, 1)\), then one of the single regions has \( \text{bd} \ G \) on its boundary, and we denote this region by \( X_0 \).

The closures of the components of \( G-(J\text{-curves}) \) are called the *regions* of \((G, J)\). If \( X_i \) is a single region, we then denote by \( W_i \) the region containing it.

5. **Prismatic neighbourhoods.** Let \( G \) be an orientable surface of type \((p, \tau)\), semi-linearly imbedded in the euclidean 3-space. Let \( IG \) be the topological product of \( G \) with the unit interval such that “base” and “top” of \( IG \) consists of two surfaces \( G' \) and \( G'' \) “parallel” to the “middle” surface \( G \), and that the “height” of \( IG \) is very small with respect to \( G \). We apply, in particular, this construction to the surface \( G \) of the previous No. 4.
Let us now extend the map \( f:G \to D \subset M \) to a map \( F:IG \to M \), by mapping each “vertical” line segment \( l \) of \( IG \) into the “adjustend” normal of \( f(l \cap G) \), or rather into a very small segment of it.

The (closed) regions \( IX, IY, IZ \) of \( IG \) that correspond to the (closed) regions \( X, Y, Z \) of \( G \) are bounded by the boundary of \( IG \) and the cylindrical walls \( IJ_i', IJ_i'' \) that correspond to the curves \( J_i', J_i'' \) of \( G \). Denote by \( IX' \) and \( IX'' \) the parts of \( IX \) above and below \( G \subset IG \), i.e. \( IX' \) or \( IX'' \) lies between \( G \) or \( G' \) or \( G'' \) respectively, and \( IX' \cap G' = X' \), \( IX'' \cap G'' = X'' \). The \( IY', IY'', Y', Y'' \) and \( IZ', IZ'', Z', Z'' \) are defined in a similar way. It is not difficult to see that the map \( f:G \to M \) can be extended to a map \( F:IG \to M \) that maps each region \( IX, IY, IZ \) semi-linearly into \( M \) in such a way that the following properties hold:

(5.1) The triple regions \( IZ \) are matched together in triples as in [11] p. 318, fig. 9a, [12] p. 662, fig. 4a.

(5.2) The double regions \( IY \) are matched together in pairs as in [11] p. 318, fig. 9b, [12] p. 662, fig. 5a.

(5.3) \( F \mid IX \) is semi-linear.\(^1\)

Thus \( IG, IJ \) consists of \( IG \) together with the cylindrical walls over the \( J \)-curves, cf. [11] p. 317, fig. 8.

We call \( V = F(IG) \) a prismatic neighbourhood of \( D \) in \( M \). The \( V \) is a compact 3-manifold, and since \( M \) is orientable so also is \( V \). Furthermore

\[
\begin{align*}
\text{bd } V &= F(\text{lateral sides of } IG) \cup F(X' \cup X'' \text{'s}), \quad \text{if } r > 0 \\
\text{bd } V &= F(X' \cup X'' \text{'s}), \quad \text{if } r = 0,
\end{align*}
\]

where \((p, r)\) is the type of \( G \). We emphasize that the single, double and triple regions \( X, Y, Z \) of \((G, J)\) are completely defined by \( V = F(IG) \). So they are corresponding to \( V \).

6. A lemma. The following lemma will be used in the following \( \S\S \).

Lemma (6.1). Let \( Q \) be a 3-manifold, and let \( P \subset Q \) be a 3-manifold whose boundary is composed of a finite number of 2-spheres. Then the injection \( \pi_1(P) \subset \pi_1(Q) \) is an isomorphism. In particular if \( Q \) is simply connected so is \( P \).

Proof. Let \( Q_i, i = 1, \cdots, h \), be the closures of the components of \( Q - P \). Then\(^1\)

\[
\pi_1(Q) = \pi_1(P) \ast \prod_{i=1}^h \pi_1(Q_i) \ast H_i
\]

where \( H_i \) is a free group on \( h_i - 1 \) generators, \( h_i \) being the number of components of \( P \cap Q_i \). From this formula it follows that the injection \( \pi_1(P) \subset \pi_1(Q) \) is an isomorphism. If \( \pi_1(Q) = 1 \), then \( \pi_1(P) = 1 \) and \( h_i = 1 \). This proves (6.1).

§3. The tower

7. Some remarks. Let \( D \) be a canonical Dehn disc with boundary \( C \) or a canonical singular 2-sphere in an orientable 3-manifold \( M \), and let \( d(D) \) and \( t(D) \) be the

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\(^{10}\) Semi-linear maps are supposed to be homeomorphisms.

\(^{11}\) \( \prod \) and \( \ast \) means free product.
number of double curves and triple points of $D$. Then $d(D) = 0$ if and only if $D$ is a disc or 2-sphere, whence $t(D) = 0$.

Let $V$ be a prismatic neighborhood of $D$ in $M$, cf. No. 5. $V$ is a compact 3-manifold with boundary. Hence, according to [22] p. 222, Satz I, $\text{bd} V$ consists of a finite number of orientable closed surfaces. The following two properties hold:

1. $D$ is a deformation retract of $V$, in a very nice and natural way.
2. If $V$ is simply connected, then each one of the components of $\text{bd} V$ is a 2-sphere.

Property (7.1) is fairly obvious, and (7.2) follows from [22] p. 223, Satz IV.

8. The elementary tower. Let $p_*: M_* \to V$ be the universal covering of $V$, and let $D_*$ be a singular disc or 2-sphere covering $D$ just once. Then $D_*$ is a canonical Dehn disc or canonical singular 2-sphere in $M_*$, because the projection of any branch point of $D_*$ will be a branch point of $D$. Let $q_* = p_*/D_*$. Let $V_*$ be the prismatic neighbourhood of $D_*$ in $M_*$ covering $V$, and let $v_* = p_*/V_*$. So we obtain the following diagrams

$$
M_* \supset V_* \supset D_* \xrightarrow{f_*} G \xrightarrow{V_*}
$$

$$
M \supset V \supset D \xleftarrow{f} (G, J) \xleftarrow{V} IG
$$

where $f: (G, J) \to D$ is a realized diagram of $D$, cf. No. 4, and $f_*$ is the lifted map. $V_*$ is $F_*(IG)$, where $V = F(IG)$, cf. No. 5, and $F_*$ is the map obtained by lifting $F: IG \to M$.

Let now $(J', J'', \psi)$ be a triple of $J$, then $J' \neq J''$, cf. No. 4. Either $f_*(J') = f_*(J'')$ is a double curve $L_*$ of $D_*$, or $f_*(J') = T'$ and $f_*(J'') = T''$ are neither of them double curves of $D_*$. In the latter case there is a unique covering translation $\tau$ such that $\tau(T') = T''$. Since $T'$ and $T'' \subset D_*$ so $T'' = \tau(T') \subset D_* \cap \tau(D_*)$. Conversely every double curve $L_*$ of $D_*$ is projected by $p_*$ into a double curve $L$ of $D$. From the above we conclude easily that, $f_*: (G, J_*) \to D_*$ is a realized diagram, where $J_*$ is the set of all triples $(J', J'', \psi)$ of $J$, such that $f_*(J') = f_*(J'')$. Thus we have the diagrams:

$$
M_* \supset V_* \supset D_* \xrightarrow{f_*} (G, J_*) \supset V_*
$$

$$
M \supset V \supset D \xleftarrow{f} (G, J) \xleftarrow{V} IG
$$

(8.1)

called an elementary tower over $D \subset V \subset M$, where $d(D_*) \leq d(D)$.

9. A remark. If $V$ is simply connected we may suppose that $M_* = V_* = V$, $D_* = D$, $J_* = J$, $p_*$ and $q_*$ and $v_* =$ identities, $f_* = f$, $F_* = F$. Thus $d(D_*) = d(D)$.
Let us now suppose that \( V \) is not simply connected. Then \( \pi_1(D) = p^{-1}(D) \) is connected, by (7.1). There is a covering translation \( \tau \), such that \( D \cap \tau^{-1}(D) \neq \emptyset \), because \( V \) is not simply connected and \( \pi_1(D) \) is connected. Then \( D \cap \tau^{-1}(D) \) consists of a finite number of closed curves such that, if \( T' \) is one of them then \( T' = f_*^n(J') \), where \( J' \) belongs to a triple \( (J', J'', \psi) \subset J - J_* \), because \( p_*(T') \) is a double curve of \( D \). Then, by (4.6),

\[
d(D) = N(J_*) < N(J) = d(D).
\]

Hence the following property holds:

(9.1) \( d(D) = d(D) \) if \( V \) is simply connected, and \( d(D) < d(D) \) if \( V \) is not simply connected.

10. The tower. The diagram

\[
\begin{align*}
M_m & \ni V_m \ni D_m \xleftarrow{f_m} (G, J_m) \quad (m) \\
M_{m-1} & \ni V_{m-1} \ni D_{m-1} \xleftarrow{f_{m-1}} (G, J_{m-1}) \\
M_1 & \ni V_1 \ni D_1 \xleftarrow{f_1} (G, J_1) \\
M_0 & \ni V_0 \ni D_0 \xleftarrow{f_0} (G, J_0)
\end{align*}
\]

is called a tower over \( D_0 \subseteq M_0 \) and is defined as follows: The diagram (\( m \)) is an elementary tower over \( D_{m-1} \subseteq V_{m-1} \subseteq M_{m-1} \), for \( m = 0, 1, 2, \cdots \), where \( M_0 = M, V_0 = V, D_0 = D, f_0 = f, F_0 = F, J_0 = J \). By (9.1),

\[
d(D_0) \geq d(D_1) \geq d(D_2) \geq \cdots \geq 0
\]

and there is a number \( n \geq 0 \), such that \( d(D_i) > d(D_{i+1}) \) for \( i < n \), and \( d(D_i) = d(D_{j+1}) \) for \( j \geq n \), i.e. \( V_n \) is simply connected but \( V_{n-1} \) is not. The number \( n \) is called the height of the tower. The following property holds:

(10.1) If \( n (\geq 0) \) is the height of a tower over \( D_0 \subseteq M_0 \), then \( V_n \) is simply connected but \( V_{n-1} \) is not.

11. Two lemmas. Let us now suppose that \( n \) is the height of a tower over \( D_0 \subseteq M_0 \). Let \( \chi: \pi_1(V_{n-1}) \to H_1(V_{n-1}) \) be the natural homomorphism, and let \( \Lambda_n \) be the torsion subgroup of \( H_1(V_{n-1}) \) (n.b. if there is no torsion then \( \Lambda_n = 0 \in H_1(V_{n-1}) \)). Let \( \Theta_n \) be the group of covering translations of \( p_n: M_n \to V_{n-1} \), let \( \varphi: \Theta_n \approx \pi_1(V_{n-1}) \) be the natural isomorphism, and let \( B_n = \varphi^{-1}(B_n) \) (n.b. \( \Gamma_n \approx \Theta_n \) because \( \Gamma_n \) contains always the 1 \( \in \Theta_n \)).
Lemma (11.1). If at least one of the components of bd $V_{n-1}$ is not a 2-sphere, then $\Gamma_n \neq \Theta_n$.

Proof. According to [22] p. 223, Satz IV, $\Lambda_n \neq H_1(V_{n-1})$ which implies $\Gamma_n \neq \Theta_n$. This proves (11.1).

Lemma (11.2). If $\Gamma_n \neq \Theta_n$ then there exists a $\tau \in \Theta_n$ such that, order $\tau = \infty$ and $\tau(D_n) \cap D_n \neq \emptyset$.

Proof. Since $p_n^{-1}(D_{n-1})$ is a connected subset and

$$p_n^{-1}(D_{n-1}) = F' \cup F'', \quad F' = \cup_{\tau \in \Gamma_n} \tau(D_n), \quad F'' = \cup_{\tau \in \Gamma_n} \tau(D_n)$$

where $\tau \in \Theta_n$, we must have $F' \cap F'' \neq \emptyset$. Hence there exists a $\delta \in \Gamma_n$, and a $\sigma \in \Theta_n$ but $\sigma \notin \Gamma_n$, such that $\delta(D_n) \cap \sigma(D_n) \neq \emptyset$. The covering translation we are asking for is $\tau = \delta^{-1}\sigma$.

Evidently $\tau(D_n) \cap D_n \neq \emptyset$. Let us suppose that $\tau^m = 1$. Then

$$-m\chi\varphi(\delta) + m\chi\varphi(\sigma) = 0.$$ 

From this equality and $\chi\varphi(\delta) \in \Lambda_n$ and $\chi\varphi(\sigma) \in \Lambda_n$ follows that $m = 0$. This proves (11.2).

12. The special case $d(D_n) = 0$. Let us now suppose that the height of a tower over $D_0 \subset M_0$ is $n$, and that the following property holds:

(12.1) $D_n$ is a disc or 2-sphere, whence $d(D_n) = 0$ and $d(D_{n-1}) > 0$.

Let $\Theta_n$ be the group of covering translations of $p_n : M_n \rightarrow V_{n-1}$. The set $\Phi_n$ of all $\tau \in \Theta_n$, such that $\tau \neq 1$ and $D_n \cap \tau^{-1}(D_n) \neq \emptyset$, is not empty. As we can easily see, because $D_{n-1}$ is a canonical Dehn disc or a canonical singular 2-sphere and $D_n$ covers $D_{n-1}$, the following property holds:

(12.2) $D_n \cap \tau^{-1}(D_n)$ consists of a positive finite number of disjoint simple closed curves lying in the interior of $D_n$, if $\tau \in \Phi_n$.

Let now $T'$ be one of the simple closed curves of $D_n \cap \tau^{-1}(D_n)$. Then $\tau(T') = T''$ is a simple closed curve of $D_n \cap \tau(D_n)$, and we have the map $\tau : T' \rightarrow T''$.

From these, we can easily see that, $q_n : (D_n, T_n) \rightarrow D_{n-1}$ is a realized diagram of $D_{n-1}$, where $T_n$ is the set of all triples $(T', T'', \tau)$ for $\tau \in \Phi_n$. So we have the following diagram

$$\begin{array}{cc}
\left(D_n, T_n\right) \\
\downarrow q_n \\
D_{n-1} \leftarrow f_{n-1}^{-1} (G, J_{n-1})
\end{array}$$

where $g_n$ is a natural homeomorphism compatible with $q_n$ and $f_{n-1}$.

Let $r$ be a triple point of $D_{n-1}$, and let $r', r'', r'''$ be the inverse (i.e. under $q_n^{-1}$) images of $r$. Let us suppose that $r' \in S'' \cap T'$, $r'' \in T'' \cap R'$, $r''' \in R'' \cap S'$, where $(T', T'', \tau), (R', R'', \rho), (S', S'', \sigma)$ are triples of $T_n$. Then $r'' = \tau(r'), r''' = \rho(r''), r' = \sigma(r''') = \sigma\tau(r')$, therefore
Let \( T' \), \( T'' \), \( \tau \) has the property \( T' \cap T'' \neq \emptyset \), and let \( r' \in T' \cap T'' \), i.e. we suppose that the triples \( (T', T'', \tau) \) and \( (S', S'', \sigma) \) are the same. Then the triple \( (R', R'', \rho) \) is called adjacent to the triple \( (T', T'', \tau) \), in the same way as \( \rho = \tau^{-2} \).

It may also be possible that, the triples \( (T', T'', \tau), (S', S'', \sigma), (R', R'', \rho) \) are the same, whence we call \( (T', T'', \tau) \) a self-adjacent triple, in respect to the point \( r' \). Then, by (12.4),

\[
\rho = \tau^{-2}.
\]

LEMMA (12.5). If \( (T', T'', \tau) \in T_n \) such that \( T' \cap T'' \neq \emptyset \), then there exists a triple \( (R', R'', \rho) \) adjacent to \( (T', T'', \tau) \), and \( \rho = \tau^{-2} \).

LEMMA (12.6). If (12.1) holds, and if \( (T', T'', \tau) \in T_n \) where order \( \tau = \infty \), then there is a triple of \( T_n \), whose curves are simple and disjoint.

Proof. We observe that the curves of any triple of \( T_n \) are simple and closed, by (12.2). So the only thing to be proved is that there is a triple of \( T_n \) whose curves are disjoint. If \( T' \cap T'' = \emptyset \) we are through. If \( T' \cap T'' \neq \emptyset \), there is a triple \( (T'_1, T''_1, \tau_1 = \tau^{-2}) \) adjacent to \( (T', T'', \tau) \). If \( T'_1 \cap T''_1 = \emptyset \) we are through. If \( T'_1 \cap T''_1 \neq \emptyset \) there is a triple \( (T'_2, T''_2, \tau_2 = \tau^{-1} = \tau^{-2}) \) adjacent to \( (T'_1, T''_1, \tau_1) \). Let us suppose that we have already obtained \( (T'_m, T''_m, \tau_m = \tau^{-1}^{m-1}) \). If \( T'_m \cap T''_m = \emptyset \) we are through. If \( T'_m \cap T''_m \neq \emptyset \) there is a triple

\[
(T'_{m+1}, T''_{m+1}, \tau_{m+1} = \tau_{m}^{-2} = \tau^{-1}^{m+1})
\]

adjacent to \( (T'_m, T''_m, \tau_m) \) and so on.

If we suppose that in this way we never obtain the desired triple, then there are triples \( (T'_{r}, T''_{r}, \tau_{r}) \) and \( (T'_s, T''_s, \tau_s) \) which are the same and \( r < s \), because \( T_n \) is finite. Thus

\[
\tau^{-1}^{r} = \tau_r = \tau_s = \tau^{-1}^{s} = 1,
\]

which contradicts order \( \tau = \infty \). This proves (12.6).

§4. Dehn's lemma

13. Let \( M_0 \) be an orientable 3-manifold, and let \( C_0 \) be the boundary of a canonical Dehn disc in \( M_0 \). Let \( \Delta \) be the set of all canonical Dehn discs in \( M_0 \) with boundary \( C_0 \). If \( D' \) and \( D'' \in \Delta \), we say that \( D' \) is simpler than \( D'' \) if either \( t(D') < t(D'') \), or \( t(D') = t(D'') \) and \( d(D') < d(D'') \). Let \( D_0 \in \Delta \) be such that:

(13.1) There is no element of \( \Delta \) simpler than \( D_0 \).

It is to be proved that \( d(D_0) = 0 \), i.e. \( D_0 \) is a disc.

14. Let \( f : (G, J_0) \to D_0 \) be a realized diagram of \( D_0 \), and let us suppose that there is a triple \( (J', J'', \psi) \) such that \( J' \) and \( J'' \) are disjoint simple closed curves. Then \( f(J') = f(J'') = L \) is a simple double curve of \( D_0 \). Let \( E' \) and \( E'' \) be the discs on \( G \) bounded by \( J' \) and \( J'' \). The following two cases are possible.

(14.1) \( E' \cap E'' = \emptyset \).
(14.2) $E'' \subset \text{int } E'$.

Let us now start a cut on $L$, such that $\text{bd}(G - E') - \text{bd} G$ is matched with $\text{bd} E''$. Thus in case (14.1), $\text{bd}(G - E'') - \text{bd} G$ is matched with $\text{bd} E'$, and in case (14.2), the two components of $\text{bd}(E' - \text{int } E'')$ are matched together. The cut can be continued throughout $L$ smoothly, because $L$ is simple. Thus in case (14.1) we obtain a new canonical Dehn disc $D^*$ with boundary $C_0$, and in case (14.2) we obtain a new canonical Dehn disc $D^*$ with boundary $C_0$ and a singular closed surface which has no special interest for us. In both cases, if there are three points of $D_0$ on $L$ then $t(D^*) < t(D_0)$, and if there are no such points then $t(D^*) \leq t(D_0)$ and $d(D^*) < d(D_0)$. This means that $D^*$ is simpler than $D_0$, which contradicts (13.1). Thus the following lemma holds:

**Lemma (14.3).** There are no double curves on $D_0$ which are simple, i.e. there is no triple of the realized diagram of $D_0$, whose curves are simple closed and disjoint.

15. Let us now suppose that $d(D_0) > 0$. Let us consider a tower over $D_0 \subset M_0$, and let $n$ be the height of it. The following two cases are possible:

(15.1) $d(D_n) > 0$, i.e. $D_n$ is not a disc, whence $n \geq 0$.

(15.2) $d(D_n) = 0$, i.e. $D_n$ is a disc, whence $n > 0$.

16. **Case (15.1).** $V_n$ is simply connected but $D_n$ is not a disc $n \geq 0$. By (7.2), $\text{bd} V_n$ is composed of 2-spheres. Then the component of $\text{bd} V_n$ containing $C_n$ the boundary of $D_n$ is a 2-sphere. Let $D'_n$ be one of the two discs, on that component, bounded by $C_n$. So $D'_n$ is formed by a small strip $I'_n$, by $F_n(X'_n)$ say, and by some of the $F_n(X'_j)$ and $F_n(X''_j)$, $j = 1, \ldots, a$, where $X_i$, $i = 0, 1, \ldots, a$, are the single regions of $(G, J_n)$ corresponding to $V_n$, i.e.

$$D'_n = I'_n \cup F_n(X'_0) \cup \bigcup_{k \neq A} F_n(X'_k) \cup \bigcup_{k \in B} F_n(X''_k)$$

where $A$ and $B$ are subsets of the set $(1, \ldots, a)$, and they are disjoint, according to the following lemma.

**Lemma (16.2).** There is no $i$ such that both $F_n(X'_i)$ and $F_n(X''_i)$ belong to $D'_n$.

**Proof.** Let us suppose that $F_n(X'_\alpha)$ and $F_n(X''_\alpha)$ belong to $D'_n$ $(1 \leq \alpha \leq a)$. Let $P$ be the closed 3-manifold obtained attaching 3-cells to $\text{bd} V_n$. Then $P$ is simply connected, because so is $V_n$. There exists a loop $K$ in $P$ such that

$$\text{sc}(K, D'_n) = 0, \quad \text{sc}(K, D_n) \equiv 1 \mod 2.$$  

Then

$$0 \equiv \text{sc}(K, D'_n) = \text{lk}(K, C_n) = \text{sc}(K, D_n) \equiv 1 \mod 2$$

which is impossible. This proves (16.2).

Let $D^*_n = \text{cl}(\text{bd} V'_n - D'_n)$, where

$$V'_n = F_n(IX'_0) \cup \bigcup_{k \neq A} F_n(IX'_k) \cup \bigcup_{k \in B} F_n(IX''_k)$$

the $A$ and $B$ are defined by (16.1). $D^*_n$ is a disc by (16.2). Thus$^{13}$ $D^* = p_1 \cdots$.

---

$^{12}$ sc and lk means intersection and linking numbers in $P$.

$^{13}$ If $n = 0$, then $p_1 \cdots p_n = \text{identity}$. 

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proves joint. Thus the existence of \( (15.2) \), \( (17.2) \), \( (G, J, n) \), cf. No. 4.

**LEMMA (16.3).** If \( D_n' \) contains a triple point of \( D_n \), then \( D* \) is simpler than \( D_0 \).

**Proof.** Let \( r_n \) be such a triple point. Then \( \tau = p_1 \cdots p_n(r_n) \) is a triple point of \( D_0 \), it lies on \( D" \), but it is not a triple point of \( D* \). We now observe that any triple point of \( D* \) is a triple point of \( D_0 \), because of the very constructions involved. From the above it follows that \( t(D*) < t(D_0) \). This proves (16.3).

**LEMMA (16.4).** On \( f_n(W_0) \subset D_n' \) there is a triple point of \( D_n \).

**Proof.** Let us suppose that this does not hold. Then on bd \( W_0 \) there is no crossing-point of the \( J_n \)-curves. Thus each component of bd \( W_0 \) = bd \( G \) is one of the curves of a triple \( (J', J'', \psi) \in J_n \subset J_0 \) such that, \( J' \) and \( J'' \) are simple closed and disjoint. This contradicts (14.3). Hence (16.4) is proved.

According to (16.3) and (16.4), \( D* \) is simpler than \( D_0 \), which contradicts (13.1). Hence case (15.1) leads to a contradiction.

**17. Case (15.2).** \( V_n \) is simply connected and \( D_n \) is a disc, \( n > 0 \). We now shall make use of the Nos. 11–12.

**LEMMA (17.1).** There is a triple of \( T_n \), whose curves are simple closed and disjoint.

**Proof.** Let us first suppose that \( n = 1 \). We observe that \( d(D_0) > 0 \), and the component of bd \( V_0 \) containing \( C_0 \) is not a 2-sphere; because otherwise there would exist a disc \( D' \subset M_0 \) with boundary \( C_0 \), where \( D' \) is simpler than \( D_0 \), which contradicts (13.1). By (11.1), \( \Gamma_1 \neq \Theta_1 \). Thus by (11.2), there exists a \( \tau \in \Theta_1 \) such that order \( \tau = \infty \) and \( \tau(D_1) \cap D_1 \neq \emptyset \). The last relations imply the existence of a triple \( (T', T'', \tau) \in T_1 \). By (12.6), (17.2) is proved if \( n = 1 \).

Let us now suppose that \( n > 1 \). Then \( d(D_{n-1}) > 0 \) by No. 10, and at least one of the components of bd \( V_{n-1} \) is not a 2-sphere; because otherwise it would follow from (6.1), that \( \pi_1(V_{n-1}) = 1 \), which contradicts (10.1). By (11.1), \( \Gamma_n \neq \Theta_n \). Thus by (11.2) there exists a \( \tau \in \Theta_n \) such that order \( \tau = \infty \) and \( \tau(D_n) \cap D_n \neq \emptyset \). The last relations imply the existence of a triple \( (T', T'', \tau) \in T_n \). By (12.6), (17.2) is proved if \( n > 1 \).

The triple provided us by (17.2) is carried by \( g_n \) of (12.3) into a triple \( (J', J'', \psi) \in J_{n-1} \subset J_0 \) such that \( J' \) and \( J'' \) are disjoint simple closed curves. This contradicts (14.3). Hence case (15.2) leads to a contradiction.

**18.** The assumption \( d(D_0) > 0 \) stated in No. 15 leads to the cases (15.1) and (15.2), each one of which leads to a contradiction. Therefore \( d(D_n) = 0 \), which proves Dehn’s lemma for all orientable 3-manifolds.

**§5. The sphere theorem**

**19.** Let \( M_0 \) be an orientable 3-manifold such that \( \pi_2(M_0) \neq 0 \), and which can be imbedded semi-linearly in a 3-manifold \( N_0 \), having one of the following properties:
(19.1) $N_0$ is simply connected.

(19.2) The commutator quotient group of any non-trivial (but not necessarily proper) finitely generated subgroup of $\pi_1(N_0)$ has an element of infinite order.

By (3.3), the set $\Delta$ of all canonical singular 2-spheres in $M_0$, which are $\not\approx 0$ in $M_0$, is not empty because $\pi_1(M_0) \neq 0$. If $D'$ and $D''$ $\epsilon \Delta$, we say that $D'$ is simpler than $D''$ if either $t(D') < t(D'')$, or $t(D') = t(D'')$ and $d(D') < d(D'')$. Let $D_0 \epsilon \Delta$ be such that:

(19.3) There is no element of $\Delta$ simpler than $D_0$.

It is to be proved that $d(D_0) = 0$, i.e. $D_0$ is a 2-sphere.

20. Let $f: (G, J_0) \to D_0$ be a realized diagram of $D_0$, and let us suppose that there is a triple $(J', J'', \psi)$ such that $J'$ and $J''$ are disjoint simple closed curves. Then $f(J') = f(J'') = L$ is a simple double curve of $D_0$. Let $E'$ and $E''$ be the two disjoint discs on $G$ bounded by $J'$ and $J''$ respectively. Thus $D' = f(E') \cup f(E'')$ is a canonical singular 2-sphere simpler than $D_0$, because if there are triple points of $D_0$ on $L$ then $t(D') < t(D_0)$, and if there are no such points then $t(D') = t(D_0)$ and $d(D') < d(D_0)$. By (19.3), $D' \approx 0$ in $M_0$. Thus there exists a very nice and obvious homotopy $h_t (0 \leq t \leq 1)$ of $D_0$ in $M_0$ such that

$$h_t(f(E'))) = f(E''), \quad h_t(f(E'')) = f(E')$$

and all other points of $D_0$ stay pointwise fixed during the homotopy. By a small modification of $h_t(D_0)$ along the simple curve $L$, we obtain a canonical singular 2-sphere $D'' \approx D_0$ in $M_0$, and simpler than $D_0$, because if there are triple points of $D_0$ on $L$ then $t(D'') < t(D_0)$, and if there are no such points then $t(D'') = t(D_0)$ and $d(D'') < d(D_0)$. Thus $D'' \epsilon \Delta$ and is simpler than $D_0$. Thus the following lemma holds.

Lemma (20.1). There are no double curves on $D_0$ which are simple, i.e. there is no triple of the realized diagram of $D_0$, whose curves are simple closed and disjoint.

21. Let us now suppose that $d(D_0) > 0$. Let us consider a tower over $D_0 \subset M_0$, and let $n$ be the height of it. The following two cases are possible:

(21.1) $d(D_n) > 0$, i.e. $D_n$ is not a 2-sphere, whence $n \geq 0$.

(21.2) $d(D_n) = 0$, i.e. $D_n$ is a 2-sphere, whence $n > 0$.

22. Case (21.1). $V_n$ is simply connected but $D_n$ is not a 2-sphere, $n \geq 0$. By (7.2), bd $V_n$ is composed of 2-spheres $D_{nj}$, $j = 1, \ldots, e$. So $D_{nj}$ is formed by some of $F_n(X_i')$ and $F_n(X_i'')$, $i = 1, \ldots, a$, where $X_i$ are the single regions of $(G, J_n)$ corresponding to $V_n$, i.e.

$$D_{nj} = U_{kA(i)} F_n(X_i') \cup U_{kB(j)} F_n(X_i'')$$

where $A(j)$ and $B(j)$ are subsets of the set $(1, \ldots, a)$, and they are disjoint, according to the following lemma.

Lemma (22.2). There is no $i$ such that both $F_n(X_i')$ and $F_n(X_i'')$ belong to $D_{nj}$.

Proof. Let us suppose that $F_n(X_i')$ and $F_n(X_i'')$ belong to $D_{nj}$ ($1 \leq \alpha \leq a$).

Let $P$ be the closed 3-manifold we obtain attaching 3-cells to bd $V_n$. Then $P$ is simply connected, because so is $V_n$. There exists a loop $K$ in $P$ such that

$$\text{sc}(K, D_n) \equiv 1 \text{ mod } 2.$$
But\textsuperscript{14} $K \sim 0$ in $P$. Therefore
\[ sc(K, D_n) \equiv 0 \mod 2. \]

We arrived at a contradiction. This proves (22.2).

Let $D_{n,j}^* = \text{cl}(\text{bd } V_{n,j} - D_{n,j})$, where
\[ V_{n,j} = \bigcup_{k \in A(j)} F_n(IX^k_h) \cup \bigcup_{k \in B(j)} F_n(IX^\nu_h) \]
the $A(j)$ and $B(j)$ are defined by (22.1). $D_{n,j}^*$ is a 2-sphere by (22.2). Thus\textsuperscript{13} $D_{n,j}^* = p_1 \cdots p_n(D_{n,j}^*)$ is a canonical singular 2-sphere in $M_0$. Let us write $D_{n}^\nu = p_1 \cdots p_n(D_{n,j}^*)$, where
\[ D_{n,j}^\nu = \bigcup_{k \in A(j)} f_n(W_h) \cup \bigcup_{k \in B(j)} f_n(W_k) \]
the $A(j)$ and $B(j)$ are defined by (22.1), and $W_i, i = 1, \cdots, a$, are the regions of $(G, J_n)$, cf. No. 4.

**Lemma** (22.3). *If $D_{n,j}^\nu$ contains a triple point of $D_n$, then $t(D_{n,j}^*) < t(D_0)$.*

**Proof.** Let $r_n$ be such a triple point. Then\textsuperscript{13} $r = p_1 \cdots p_n(r_n)$ is a triple point of $D_0$, it lies on $D_j^\nu$, but it is not a triple point of $D_j^*$. We now observe that any triple point of $D_j^*$ is a triple point of $D_0$, because of the very constructions involved. From the above it follows that $t(D_j^*) < t(D_0)$. This proves (22.3).

**Lemma** (22.4). *On $f_n(W_\alpha), i = 1, \cdots, a$, there is a triple point of $D_n$.*

**Proof.** Let us suppose that on $f_n(W_\alpha)$ there is no triple point of $D_n (1 \leq \alpha \leq a)$. Then on $\text{bd } W_\alpha$ there is no crossing-point of the $J_n$-curves. Thus each component of $\text{bd } W_\alpha$ is one of the curves of a triple $(J', J'', \psi) \in J_n \subset J_0$ such that $J', J''$ are simple closed and disjoint. This contradicts (20.1). Hence (22.4) is proved.

**Lemma** (22.5). *The map $p_1 \cdots p_n : D_{n,j} \to M_0$ is $\simeq 0$, $n \geq 0$, $j = 1, \cdots, e$, where $p_1 \cdots p_n = \text{identity for } n = 0$.*

**Proof.** $D_j^* \simeq 0$ in $M_0$, because otherwise it would follow that $D_j^* \in \Delta$, by (22.3) and (22.4), it would then follow that $D_j^*$ is simpler than $D_0$, which contradicts (19.3). Therefore
\[ p_1 \cdots p_n(D_{n,j}) \simeq D_j^* \simeq 0 \quad \text{in } M_0, \]
because $D_{n,j} \simeq D_{n,j}^*$ in $V_n$. This proves (22.5).

Let us now consider the simply connected closed 3-manifold $P$, considered in the proof of (22.2). There is a map $p : P \to M_0$ such that $p | V_n = p_1 \cdots p_n$, by (22.5). By Poincaré duality and standard Hurewicz theorems $\pi_2(P) = 0$. Therefore $D_n \simeq 0$ in $P$. Thus from
\[ D_0 = p_1 \cdots p_n(D_n) \subset p(P) \subset M_0 \]
follows that $D_0 \simeq 0$ in $M_0$, which contradicts $D_0 \in \Delta$. Hence case (21.1) leads to a contradiction.

**23. Case (21.2).** $V_n$ is simply connected and $D_n$ is a 2-sphere, $n \geq 0$. We now shall make use of the Nos. 11–12.

**Lemma** (23.1). *There is a triple of $T_n$, whose curves are simple closed and disjoint.*

\textsuperscript{14} $\sim$ means homologous to.
Proof. Let us first suppose that, either \( n = 1 \) and (19.1) holds, or that \( n > 1 \). Then \( d(D_{n-1}) > 0 \), by No. 10, and at least one of the components of \( \operatorname{bd} V_{n-1} \) is not a 2-sphere; because otherwise it would follow from (6.1) that \( \pi_1(V_{n-1}) = 1 \), which contradicts (10.1). By (11.1), \( \Gamma_n \neq \Theta_n \). Thus by (11.2), there exists a \( \tau \in \Theta_n \) such that order \( \tau = \infty \) and \( \tau(D_n) \cap D_n \neq \emptyset \). The last relations imply the existence of a triple \((T', T'', \tau) \in T_n\). By (12.6), (23.1) is proved in the cases under consideration.

Let us now suppose that \( n = 1 \) and (19.2) holds. We shall prove that:

\[
\Gamma_1 \neq \Theta_1.
\]

If at least one of the components of \( \operatorname{bd} V_0 \) is not a 2-sphere, then (23.2) holds, by (11.1). If all the components of \( \operatorname{bd} V_0 \) are 2-spheres, then by (6.1), \( \pi_1(V_0) \) is isomorphic to a subgroup of \( \pi_1(N_0) \), and \( \pi_1(V_0) \neq 1 \), by (10.1). Thus by (19.2), \( \Lambda_1 \neq H_1(V_0) \), cf. No. 11. So (23.2) holds in the case under consideration. Hence (23.2) is proved. By (11.2), there exists a \( \tau \in \Theta_1 \) such that, order \( \tau = \infty \), and \( \tau(D_1) \cap D_1 \neq \emptyset \). The last relations imply the existence of a triple \((T', T'', \tau) \in T_1\). By (12.6), (23.1) is proved in the case under consideration. This completes the proof of (23.1).

The triple provided by (23.1) is carried by \( g_n \) of (12.3) into a triple \((J', J'', \psi) \in J_{n-1} \subseteq J_0 \) such that \( J' \) and \( J'' \) are disjoint simple closed curves. This contradicts (20.1). Hence case (21.2) leads to a contradiction.

24. The assumption \( d(D_0) > 0 \) stated in No. 21 leads to the cases (21.1) and (21.2), each one of which leads to a contradiction. Therefore \( d(D_0) = 0 \), which proves the sphere theorem.

§6. Asphericity of knots

25. A lemma. Let \( B \) be a non-empty proper closed subset of \( S^3 \). We say that \( B \) is \textit{geometrically splittable}, if there is a 2-sphere \( S^2 \subseteq S^3 - B \) such that both components of \( S^3 - S^2 \) contain points of \( B \), cf. Alexander’s theorem below. We say that \( B \) is \textit{algebraically splittable}, if \( \pi_1(S^3 - B) \) is the free product of two groups, each one of which is non-trivial.

\begin{lemma} \textit{(25.1).} Let \( Q \) be a 3-manifold either non-compact or with boundary. Then \( Q \) is aspherical if and only if \( \pi_2(Q) = 0 \).
\end{lemma}

\begin{proof}
Let \( p: \tilde{Q} \to Q \) be the universal covering of \( Q \), where \( \tilde{Q} \) has the induced triangulation. Then \( H_r(\tilde{Q}) = 0 \), for \( r > 3 \), because \( \dim \tilde{Q} = 3 \), and \( H_3(\tilde{Q}) = 0 \), because \( \tilde{Q} \) is either non-compact or has boundary. Thus, by standard Hurewicz theorems, \( \tilde{Q} \) is aspherical if and only if \( \pi_2(\tilde{Q}) = 0 \). This proves (25.1).
\end{proof}

\begin{alexander_theorem}
Let \( S^2 \) be a 2-sphere (semi-linearly) imbedded in the 3-sphere \( S^3 \). Then the closure of each one of the two components of \( S^3 - S^2 \) is a 3-cell. See [1] p. 6.
\end{alexander_theorem}


\begin{theorem} \textit{(26.1).} Let \( U \) be a non-empty proper open connected subset of the
3-sphere \( S^3 \). Then \( U \) is aspherical if and only if \( S^3 - U \) is not geometrically splittable.

**Proof.** \( U \) is an orientable non-compact 3-manifold, whose simplexes are rectilinear in \( S^3 \) [2] p. 143, Satz II. Let us first suppose that \( U \) is not aspherical. Then \( \pi_2(U) \neq 0 \), by (25.1). Thus by the sphere theorem, there is in \( U \) a 2-sphere \( S^2 \) \( \cong 0 \) in \( U \). Therefore, by Alexander's theorem, each component of \( S^3 - S^2 \) contains points of \( S^3 - U \). Hence \( S^3 - U \) is geometrically splittable.

Let us now suppose that \( S^3 - U \) is geometrically splittable. Then there exists a 2-sphere \( S^2 \subset U \) such that both components of \( S^3 - S^2 \) contain points of \( S^3 - U \). We emphasize that \( S^2 \) is semi-linearly imbedded in \( U \), i.e. \( S^2 \) is a subcomplex of the 2-skeleton of a subdivision of the triangulation of \( U \). So it is no loss of generality to assume that \( S^2 \) is a subcomplex of \( U \). Let us suppose that \( S^2 \simeq 0 \) in \( U \). Thus \( X \simeq 0 \) in \( U \), where \( X \) is a basic 2-cycle of \( S^2 \). Therefore \( X = \partial Y \), where \( Y \) is a 3-chain formed by 3-simplexes of \( U \). Let the 3-simplex \( \sigma \) of \( Y \) be contained in \( U' \), where \( U' \) is the closure of a component of \( U - S^2 \). Then \( U' \) must contain all 3-simplexes of \( U' \), which are of an infinite number. We arrived at a contradiction. Thus \( S^2 \neq 0 \) in \( U \). Hence \( \pi_2(U) \neq 0 \). This completes the proof of (26.1).

**Corollary (26.2).** If \( F \) is a non-empty proper closed connected subset of \( S^3 \), then each component of \( S^3 - F \) is aspherical.

**Proof.** Let \( U' \) be a component of \( S^3 - F \). Then \( S^3 - U' = F \cup (\text{all components of } S^3 - F \text{ different to } U') \). \( U' \) is a closed connected subset of \( S^3 \). Thus \( S^3 - U' \) is not geometrically splittable. Hence \( U' \) is aspherical, by (26.1).

The above corollary provides us with a solution of a problem of S. Eilenberg [5] p. 241. An immediate consequence of (26.2) is the following

**Corollary (26.3).** If \( F \) is a knot or connected graph, then \( S^3 - F \) is aspherical.

In the special case where \( F \) is an “alternating” knot in \( S^3 \), the asphericity of \( S^3 - F \) has already been proved by R. J. Aumann [3].

27. Higman's problem. The following theorem solves a problem proposed and partly solved by G. Higman [8].

**Theorem (27.1).** Let \( K \) be a link in \( S^3 \). The following three statements are equivalent:

(i) \( S^3 - K \) is not aspherical.

(ii) \( K \) is geometrically splittable.

(iii) \( K \) is algebraically splittable.

**Proof.** Statements (i) and (ii) are equivalent, by (26.1). Evidently (ii) implies (iii). Let us now suppose that (iii) holds. Then, by [8] p. 122, Theorem 2, \( S^3 - K \) is not aspherical, i.e. (iii) implies (i). This proves (27.1).


**Theorem (28.1).** (i) \( K \) is unknotted if and only if \( \pi_1(S^3 - K) \) is free cyclic. (ii) The number of ends of \( \pi_1(S^3 - K) \) is either 1 or 2. (iii) \( \pi_1(S^3 - K) \) has 1 end if \( K \) is knotted, and 2 ends if \( K \) is unknotted.
§7. Hopf’s conjecture


**Lemma (29.1).** Let \( g \in G_1 \ast G_2 \), and let \( g = \prod_{i=1}^{r} g_i \) be a reduced word, where \( g_i \in G_{j(i)} \) (\( j(i) = 1 \) or \( 2 \)). If \( g \) is of order \( s > 1 \) or \( s < \infty \), then there is an \( a(\geq 1, \leq r) \) such that \( g_a \) is of order \( s_a (\geq 1, \leq s) \) in \( G_{j(a)} \).

**Proof.** The proof is given by induction on the length (i.e. the number of factors) \( l(g) \) of \( g \). (29.1) is obviously true if \( l(g) = 1 \). So let us suppose that (29.1) holds if \( l(g') \leq q - 1 \), where \( g' \in G_1 \ast G_2 \), and that \( l(g) = r > 1 \). From \( g' = 1 \) we conclude that \( g_r = g_1^{-1} \). Thus

\[ g = g_1 \cdot g'' \cdot g_1^{-1} = g_1 \cdot \prod_{i=2}^{r-1} g_i \cdot g_1^{-1}. \]

Hence \( g'' \in G_1 \ast G_2 \), \( l(g'') = q - 2 \) and \( (g'')^s = 1 \). This proves (29.1).

**Lemma (29.2).** If \( Q \) is a compact simply connected 3-manifold whose boundary is a 2-sphere, then \( Q \) is aspherical.

For a proof see [19] Lemma (18.1), where \( h = 0 \) in the present case.

30. A definition. Let \( G \) be a finitely presented group. By \( z(G) \) we denote a number such that there is an isomorphism

\[ G \cong G' \ast \prod_{i=1}^{r} Z_i, \]

where \( Z_i \) is a free cyclic group and \( r = z(G) \), but there is no such isomorphism for \( r > z(G) \). This implies that if \( r = z(G) \), then \( G' \) can never be \( \cong G'' \ast Z \) where \( Z \) is a free cyclic group. Obviously, if \( G \cong G_1 \ast G_2 \), then \( z(G) \geq z(G_1) + z(G_2) \).

31. A general theorem. In the proof of the theorem below we make use of Nielsen-Schreier theorem (cf. Introduction) and a result due essentially to P. A. Smith [10] p. 216, l. 31, namely:

31.1 The fundamental group of an aspherical polyhedron of finite dimension has no element of finite order.

**Theorem (31.2).** Let \( U \) be an open connected subset of an orientable 3-manifold \( N \) such that \( \pi_1(N) \) is locally free.⁶ Then \( \pi_1(U) \) has no element of finite order.

**Proof.** Let \( L \) be a loop representing \( e \in \pi_1(U) \) where order \( e = s (> 1, < \infty) \). Then \( L^s = 0 \) in \( U \), and let \( B \) be a singular disc in \( U \), with boundary \( L^s \). Let \( M \subseteq U \) be a compact orientable 3-manifold such that \( B \subseteq \text{int} \ M \). Then the element \( g \) of \( \pi_1(M) \) corresponding to the loop \( L \), has order \( s \). Not all the components of \( \text{bd} \ M \) are 2-spheres, because if they were, \( \pi_1(M) \) would have been a subgroup of \( \pi_1(N) \), by (6.1). Therefore \( \pi_1(M) \) would have been a free group, because \( M \) is finitely presented. Thus \( \pi_1(M) \) would have no element of finite order. Let \( n(M) > 0 \) be the number of the components of \( \text{bd} \ M \) which are not 2-spheres, and let \( S_j, j = 1, \ldots, m (\geq 0) \), be the components of \( \text{bd} \ M \) which are 2-spheres. Let \( q(M) = n(M) + z(M) \), where \( z(M) = z(\pi_1(M)) \), cf. No. 30. Thus \( q(M) > 0 \), because \( n(M) > 0 \). The following property holds:

31.3 \( M \) is a compact orientable 3-manifold with boundary, semi-linearly imbedded in \( N \) such that \( \pi_1(M) \) has an element of order \( s \), whence \( q(M) > 0 \).
Let us now cut $N$ along $S_j$, $j = 1, \ldots, m \geq 0$, in pieces and let $N''$ be the piece containing $M$. If $S_j$ separates $N$ into two 3-manifolds $N_{j1}$ and $N_{j2}$ then
\[ \pi_1(N) \approx \pi_1(N_{j1}) \ast \pi_1(N_{j2}), \]
and if $S_j$ does not separate $N$ then
\[ \pi_1(N) \approx \pi_1(N_{j''}) \ast Z, \]
where $Z$ is a free cyclic group and $N_{j''}$ is the 3-manifold we obtain cutting $N$ along $S_j$. From this we easily conclude that $\pi_1(N'')$ is a locally free group.\(^\text{15}\) Let $N'$ and $M' \subset N'$ be the orientable 3-manifolds we obtain from $N''$ and $M \subset N''$ attaching 3-cells $E_j$ to $S_j$. We emphasize that $S_j$ means the boundary surface of $M$. It may happen that, for a certain $S_j$, both copies of $S_j$, obtained cutting $N$ along $S_j$, belong to $N''$. In this case, the copy not belonging to $M$ has no special interest for us, and therefore we do not consider it. We also emphasize that if $m = 0$, then $N = N'' = N'$. Thus $\pi_1(N') \approx \pi_1(N'')$ is a locally free group,\(^\text{16}\) $M'$ is compact with boundary, because $n(M) > 0$, and $\pi_1(M') \approx \pi_1(M)$. Therefore $M'$ is not aspherical, by (31.1), because the element of $\pi_1(M')$ corresponding to $g \in \pi_1(M)$ is of order $s$. Hence $\pi_2(M') \neq 0$, by (25.1).

There is in $M'$ a 2-sphere $S \not\subset 0$ in $M'$, by the sphere theorem, because $M' \subset N'$ where $\pi_1(N')$ is a locally free group. Without loss of generality we may suppose that $S \cap S_j$ consists of a finite number $k_j \geq 0$ of disjoint simple closed curves. Let
\[ k(S) = \sum_{j=1}^m k_j, \]
and let us suppose that $k(S) > 0$. Let $A$ be a curve of $\bigcup_{j=1}^m S \cap S_j$, lying, say on $S_b$, and such that one of the components of $S_b - A$, say $E$, does not have points in common with $S$. Let $S'$ and $S''$ be the closure of the components of $S - A$. Then $E \cup S'$ and $E \cup S''$ are 2-spheres in $M'$, at least one of which is $\not\subset 0$ in $M'$. By a small modification of $E \cup S'$ and $E \cup S''$, we obtain two 2-spheres $S_01$ and $S_02$ in $M'$, at least one of which is $\not\subset 0$ in $M'$, and such that $k(S_01)$ and $k(S_02) < k(S)$. By induction on $k$ we conclude that there is in $M'$ a 2-sphere $S_0 \not\subset 0$ in $M'$, and such that $k(S_0) = 0$, i.e. $S_0$ does not meet any $S_j$, $j = 1, \ldots, m \geq 0$.

Thus $S_0 \not\subset E_j$, because $S_0 \not\subset 0$ in $M'$, i.e. $S_0 \subset M$. The following two cases are possible:

(1) $M - S_0$ is connected.
(2) $M - S_0$ is not connected.

Case (1): We cut $M$ and $N''$ along $S_0$, and let us denote by $M_1$ and $N_1$ the orientable 3-manifolds with boundary we obtain. Then $M_1$ is a compact 3-manifold with boundary, semi-linearly imbedded in $N_1$ and
\[ \pi_1(N'') \approx \pi_1(N) \ast Z, \quad \pi_1(M) \approx \pi_1(M) \ast Z \]
\[^{15}\text{This cannot occur if } \pi_1(N) = 1.\]
\[^{16}\text{Especially } \pi_1(N'') = 1, \text{ if } \pi_1(N) = 1.\]
\[^{17}\text{Especially } \pi_1(N') = 1, \text{ if } \pi_1(N) = 1.\]
\[^{18}\text{This suggests that case (1) cannot occur if } \pi_1(N'') = 1, \text{ cf. footnote 16.}\]
where $Z$ is free cyclic. Thus $\pi_1(N_1)$ is locally free, because $\pi_1(N''')$ is locally free. Moreover $\pi_1(M_1)$ has an element of order $> 1$ and $\leq s$, by (29.1), because $g$ is of order $s$ in $\pi_1(M)$. Finally $n(M_1) = n(M) > 0$, but $z(M_1) + 1 \leq z(M)$. Thus $0 < q(M_1) < q(M)$.

Case (2): $S_0$ separates $M$ into two compact orientable 3-manifolds $M_1$ and $M_{0i}$ with boundary, such that $M_1 \cap M_{0i} = S_0$. Thus

$$\pi_1(M) \approx \pi_1(M_1) \ast \pi_1(M_{0i}).$$

(31.4)

Obviously

$$n(M) = n(M_1) + n(M_{0i}), \quad z(M) \geq z(M_1) + z(M_{0i}).$$

(31.5)

Let us now suppose that $n(M_1) + z(M_1) = 0$. By (6.1), $\pi_1(M_1)$ is a subgroup of $\pi_1(N)$, because $M_1 \subset M \subset N$ and $n(M_1) = 0$, i.e. all boundary surfaces of $M_1$ are 2-spheres. Thus $\pi_1(M_1)$ is a free group, because $\pi_1(N)$ is locally free and $M_1$ is compact. Thus $\pi_1(M_1) = 1$, because $z(M_1) = 0$. Let $M_1 \subset M'$ be the compact orientable 3-manifold we obtain from $M_1$ attaching 3-cells to the boundary 2-spheres of $M_1$ different from $S_0$. Thus $\pi_1(M_1') \approx \pi_1(M_1) = 1$. Hence $M_1'$ is a simply connected compact 3-manifold with boundary the 2-sphere $S_0$, and therefore $\pi_1(M_1') = 0$, by (29.2). Thus $S_0 \cong 0$ in $M_1' \subset M'$, which contradicts $S_0 \not\cong 0$ in $M'$. Hence $q(M_1') = n(M_1) + z(M_1) > 0$. In the same way we prove that $q(M_{0i}) = n(M_{0i}) + z(M_{0i}) > 0$. Thus by (31.5),

$$0 < q(M_1') < q(M_1) + q(M_{0i}) \leq q(M).$$

For the sake of uniformity we write in this second case $N = N_1$.

Let now $g = \prod_{i=1}^{r} g_i g_{0i}$ be a reduced word, where $g_i \in \pi_1(M_1)$ and $g_{0i} \in \pi_1(M_{0i})$, cf. (31.4). By (29.1), there is an $a (\geq 1, \leq r)$ such that $g_a$ is of order $s_1 (> 1, \leq s)$ in $\pi_1(M_1)$, because $g$ is of order $s$ in $\pi_1(M)$.

In both cases (1) and (2) the following is proved:

(31.6) If $M$ is a compact 3-manifold with boundary, semi-linearly imbedded in an orientable 3-manifold $N$ whose fundamental group is locally free, and if $\pi_1(M)$ has an element of order $s (> 1, < \infty)$, whence $q(M) > 0$, then there is a compact 3-manifold $M_1$ with boundary, semi-linearly imbedded in an orientable 3-manifold $N_1$ whose fundamental group is locally free, $\pi_1(M_1)$ has an element of order $s_1 (> 1, \leq s)$, and $0 < q(M_1) < q(M)$.

From (31.6) we conclude that there is an infinite sequence $M, M_1, M_2, \ldots$ of compact 3-manifolds with boundary, semi-linearly imbedded in the orientable 3-manifolds $N, N_1, N_2, \ldots$ respectively, where $\pi_1(N_i)$ is locally free, $\pi_1(M_i)$ has an element of order $s_i (> 1, \leq s_{i-1})$, $i = 1, 2, \ldots$ (where $M_0 = M, N_0 = N, s_0 = s$), and such that

$$q(M) > q(M_1) > q(M_2) > \cdots > 0.$$  

This is impossible. This proves (31.2).

**Corollary (31.7).** Let $U$ be an open connected subset of an orientable 3-manifold $N$ such that $\pi_1(N)$ is a free group. Then $\pi_1(U)$ has no element of finite order.

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19 The hypothesis that $g_a$ and not $g_{0a}$ has this property does not diminish the generality.
Proof. According to Nielsen-Schreier theorem, cf. Introduction, \( \pi_1(N) \) is locally free. Thus (31.7) holds, by (31.2).

The following corollary is an immediate consequence of (31.7), and provides us with a solution of a conjecture due to H. Hopf.

**Corollary (31.8).** If \( U \) is an open connected subset of the 3-sphere, then \( \pi_1(U) \) has no element of finite order.

**Corollary (31.9).** Knot groups and link groups have no element of finite order.

We emphasize that (26.3) and (31.1) provide us with an immediate proof that knot groups have no element of finite order.

### §8. 3-spheres with handles

32. Let \( M \) be a 3-manifold, and let \( E' \) and \( E'' \) be two disjoint 3-cells in \( M \), with boundaries \( S' \) and \( S'' \) respectively. Let \( M_0 = M - \text{int} E' - \text{int} E'' \), and and let \( f: S' \to S'' \) be a homeomorphism. Identifying the points \( p \in S' \) and \( f(p) \in S'' \), we obtain a new 3-manifold \( M_* \). We say that \( M_* \) is obtained putting a handle to \( M \).

Let us now suppose that \( M \) is orientable. Let us give an orientation to \( M \), and let us consider the induced orientations on \( S' \) and \( S'' \). Then \( M_* \) is orientable or non-orientable as far as the orientation of \( f(S') \) does not coincide or coincides with the orientation of \( S'' \), and we say that \( M_* \) is obtained putting an orientable or non-orientable handle to \( M \).

We now define a 3-sphere with \( h \) handles by induction as follows: A 3-sphere with 0 handles is a 3-sphere, and a 3-sphere with \( h \) (\( \geq 1 \)) handles is obtained putting an orientable handle to a 3-sphere with \( h - 1 \) handles.

The following theorem gives us an algebraic characterization of the 3-spheres with handles, modulo Poincaré conjecture.

**Theorem (32.1).** If Poincaré conjecture is true, then any orientable closed 3-manifold, whose fundamental group is a free group on \( h \) free generators, is a 3-sphere with \( h \) handles.

Proof. For \( h = 0 \) the theorem holds if Poincaré conjecture is true. Let us now suppose that \( h > 0 \). The number of ends of \( \pi_1(M) \) is 2 or \( \infty \), as far as \( h = \) or \( > 1 \), by [9] p. 100. Thus \( \pi_2(M) \) is a free abelian group of rank 1 or \( \infty \), by [23] p. 325, Satz VI. Therefore \( \pi_3(M) \neq 0 \). By the sphere theorem, there is in \( M \) a 2-sphere \( S \not\subset 0 \) in \( M \). The following two cases are possible:

1. \( S \) separates \( M \).
2. \( S \) does not separate \( M \).

**Case (1):** Let us cut \( M \) along \( S \). So we obtain two compact orientable 3-manifolds \( M' \) and \( M'' \) with boundaries the 2-spheres \( S' \) and \( S'' \) respectively, such that

\[
\pi_1(M) \cong \pi_1(M') * \pi_1(M'').
\]

By Nielsen-Schreier theorem, cf. Introduction, \( \pi_1(M') \) and \( \pi_1(M'') \) are free groups. Let us suppose that they are generated by \( h' \) and \( h'' \) free generators respectively. Then \( h = h' + h'' \), where \( h' \) and \( h'' \) > 0, by (29.2), because \( S \not\subset 0 \) in \( M \). Let us now denote by \( M_1 \) and \( M_2 \) the orientable closed 3-manifolds we obtain from \( M' \) and \( M'' \), attaching 3-cells to \( S' \) and \( S'' \) respectively. Then...
\(\pi_1(M_1)\) and \(\pi_1(M_2)\) are free groups on \(h'\) and \(h'' (< h)\) free generators respectively. By induction it follows easily that \(M\) is a 3-sphere with \(h\) handles.

**Case (2):** Let us cut \(M\) along \(S\). So we obtain a compact orientable 3-manifold \(M'\) with boundary composed of two 2-spheres \(S'\) and \(S''\), such that

\[\pi_1(M) \cong \pi_1(M') \ast Z\]

where \(Z\) is a free cyclic group. By Nielsen-Schreier theorem, \(\pi_1(M')\) is a free group, and is generated by \(h' = h - 1\) free generators. Let us now denote by \(M_1\) the orientable closed 3-manifold we obtain from \(M'\), attaching 3-cells to \(S'\) and \(S''\). Then \(\pi_1(M_1)\) is a free group on \(h'\) free generators. By induction it follows easily that \(M\) is a 3-sphere with \(h\) handles.

This completes the proof of (32.1).

### §9. Some problems

In this § some problems arising from §§3–8 will be discussed.

1. The first and most obvious problem is: *Does the sphere theorem hold for all 3-manifolds?* If it does, then find a method to prove it. If it does not, then find a counter-example, and also find the largest family of 3-manifolds for which the sphere theorem holds.

2. Another problem arises if we try to generalize §§3–4 to surfaces in a 3-manifold \(M\), precisely: *Let \(D\) be a Dehn surface of type \((p, r)\) in \(M\), where \(r > 0\). Does there exist in \(M\) an orientable surface \(D_0\) of type \((p_0, r)\) such that \(\partial D_0 = \partial D\) and \(p_0 \leq p\)?* The case \(r = 1\) is of special interest, because it seems to be related to the problem of the genus of a knot in the 3-sphere [21] p. 571.

3. Another problem arises if we try to generalize §§3 and 5 to surfaces in a 3-manifold \(M\). To formulate the problem we need a definition. Let \(f: S \to M\) be a map such that \(f(S)\) is a singular orientable surface of type \((p, 0)\), i.e. it is closed, and \(p \geq 0\).\(^{20}\) Let us suppose that \(S\) is the boundary of a solid torus\(^{21}\) \(T\). If there exists a map \(F: T \to M\) (n.b. in this case \(F\) need only be continuous) such that \(F\mid S = f\), we then write\(^{22}\) \(S \cong (p)\) in \(M\). The problem arising is: *Let \(D\) be a Dehn surface of type \((p, 0)\) in \(M\), such that \(p > 0\) and \(D \not\cong (p)\) in \(M\). Does there exist in \(M\) an orientable surface \(D_0\) of type \((p_0, 0)\) such that \(D_0 \not\cong (p_0)\) in \(M\) and \(p_0 \leq p\)?* The solution of this problem may help us to obtain some information about the construction of 3-manifolds, as the sphere theorem helped us to prove Theorem (32.1). See also problem 6 below.

4. Another problem arising is: *To what extent do these results and problems generalize to the non-orientable case?*

5. It was suggested by R. H. Fox to consider also other coverings, instead of the universal coverings we considered in §3. Arnold Shapiro recommends es-

\(^{20}\) Cf. Nos. 2 and 3.

\(^{21}\) *Solid torus of genus* \(h\) = Henkelkörper vom Geschlechte \(h\) [22] p. 219. If \(h = 0\), then it is a 3-cell.

\(^{22}\) N.b. \(\cong\) is different from \(\sim\), which means homotopic to. However \(S \cong (0)\) in \(M\) is equivalent to \(S \sim 0\) in \(M\).
pecially the 2-sheeted coverings. Possibly non-universal coverings may not only
generalize or simplify our method, but may also be useful for the solution of
other problems, especially those mentioned above.

6. Let $S^3$ be the 3-sphere and let $T'_i$, $T''_i$ be disjoint solid tori\(^1\) of genus
$h_i \geq 0, i = 1, \ldots, n (\geq 0)$ in $S^3$. Let $f_i: \partial T'_i \rightarrow \partial T''_i$ be a homeomorphism.
Let us identify the points $x \in \partial T'_i$ and $f_i(x) \in \partial T''_i$ of

$$S^3 = \bigcup_{i=1}^n \text{int } T'_i \cup \text{int } T''_i.$$

Thus we obtain a closed 3-manifold $M$ which is orientable or non-orientable,
depending on the maps $f_i$. The problem arising is: Do we obtain in this way all
closed 3-manifolds? In the special case where $M$ is orientable and $\pi_1(M)$ is the
free group on $h \geq 0$ free generators, Theorem (32.1) provides us with a solution
of this problem. Possibly the complete solution depends on the solution of the
problem 3 above, as the proof of (32.1) depends on the sphere theorem.

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