Lecture 4: Knot Complements

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1 Introduction

Here we discuss properties of the knot complement, $S^3 \setminus K$, for a knot K.

Definition 1.1. A *tubular neighborhood* $V_k \subset S^3$ is an embedding

$$t : K \times B^2 \to S^3$$

such that $t(x, 0) = x$, x in K and B^2 is the open unit disk.

We of course recall that an embedding is nothing more than a homeomorphism onto the image of the target.

Definition 1.2. The *knot exterior*, $X_K = \overline{S^3 \setminus V_K}$, is a compact 3-manifold with boundary, $T^2 = S^1 \times S^1$, which is a deformation retract of the open *knot complement*, $S^3 \setminus K$.

The knot exterior itself supplies a very important class of 3-manifolds. In upcoming lectures, we will study homology and homotopy of these manifolds. Unfortunately, the knot complement doesn't care about the orientation of the knot, so it cannot be used to distinguish the knot and its inverse.

The following theorem has a simple motivation. If we consider two knots, K_1 and K_2, and say they are isotopic, then we see that their complements are also isotopic. This is not a challenge. However, we immediately ask the converse. Given two knot complements that are isotopic, are the knots themselves isotopic? The big theorem of Gorden-Leuche in 1989 provides a positive answer to that question. We will not provide a proof of this theorem.

Theorem 1.3 (Gorden-Leuche, 1989)**.** *If K_1 and K_2 are unoriented knots in S^3 and there is an orientation-preserving homeomorphism*

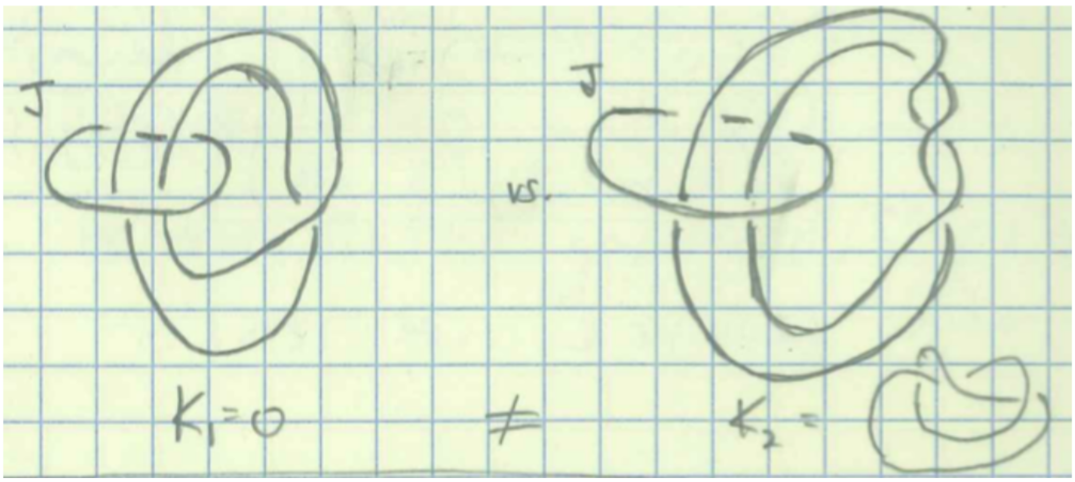
$$S^3 \setminus K_1 \xrightarrow{\sim} S^3 \setminus K_2$$

,

then K_1 and K_2 are equivalent as unoriented knots.

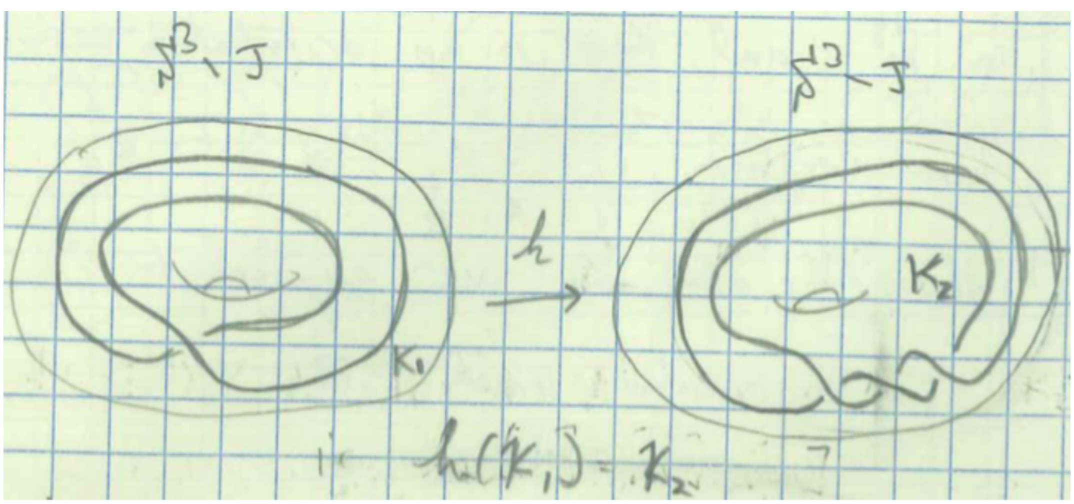
Remark 1.4. This theorem is very false for links in S^3.

Take two links as shown blow, the first created by linking two copies of the unknot (J and K_1), and the second created by linking an unknot, J, with the trefoil, K_2.



Both links have linking number ± 2, depending on orientation. Thus they are nontrivial. But they cannot be equivalent, because the trefoil is nontrivial as shown using 3-colorability in the last lecture.

Here have have that $L_1 \neq L_2$ and we wish to show that we have a homeomorphism $S^3 \setminus L_1 \simeq S^3 \setminus L_2$. You can see these two knot complements below.



And of course, the unknot's complement in the 3-sphere is a solid torus, so we can view both K_1 and K_2 as curves in the solid torus. We make note of the fact that

$S^3 \setminus J \simeq S^1 \times B^2$. Next, we make an appeal to Complex Analysis to provide us with the "twist" homeomorphism:

$$h : S^1 \times B^2 \to S^1 \times B^2$$

$$h(z, w) = (z, zw)$$

where (z, w) is in $\mathbb{C}^2$. The amazing thing of this twist for the solid torus to itself is that it takes K_1 to K_2 in the torus. You can see that twisting happening in the picture above.

We note in particular that this acts as expected on longitude and meridian homology classes with winding occurring for the longitude on the boundary of the torus.

$$h[m] = [m]$$

$$h[l] = [l] + [m]$$

The homology of the knot complement will be very useful going forward, but it will require being clever. One clever tactic is to consider the *reduced* homology. This will allow us to dispense with dealing with special cases each time and lead to more simply stated theorems.

Reduced Homology We relate the normal homology H with the reduced homology $\widetilde{H}$ as follows:

$$H_i(X) \simeq \widetilde{H}_i(X), i > 0$$

$$H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$$

This provides a uniform description of the n-sphere.

That is, $\widetilde{H}_i(S^n) = \mathbb{Z}$ for $i = n$ and 0 otherwise!

Proposition 1.5. *a) For an embedding $h : D^k \to S^n$,*

$$\widetilde{H}_i(S^n \setminus h(D^k)) = 0 \ \text{ for all } i.$$

b) For an embedding $h : S^k \to S^n$ with $k < n$,

$$H_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \textit{otherwise} \end{cases}$$

Before we prove this proposition, let's look at an immediate corollary.

Corollary 1.6. *(For n=3, k=1)*

The knot complement $S^3 \setminus K$ has homology

$$H_i(S^3 \setminus K) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i \geq 2 \end{cases}$$

In particular, H_1 is generated by the class of a meridian m (which is what gives $\mathbb{Z}$).

As this corollary has a little less complexity than the main proposition above, we will use its proof to recall the tools necessary to prove the proposition itself.

We'd like to get at the generator specifically. Let's identity it using the Mayer-Vietoris sequence. The sequence allows us to look at homologies of parts of the space X to get at the homology for the entire space. It also looks the same for regular or reduced homology.

If $A, B \subset X$ are an open cover of X, then we know we have a long exact sequence of homologies, the Mayer-Vietoris sequence:

$$\cdots \to H_n(A \cap B) \xrightarrow{i_k} H_n(A) \oplus H_n(B) \to H_n(A \cup B) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots H_0(X) \to 0$$

where i_k is an inclusion map.

Now, we specify the required components as follows

$$A = S^3 \setminus K \text{ and } B = V_K.$$

This tells us that

$$A \cap B \simeq \partial V_K = T^2 \text{ and } A \cup B = S^3.$$

We combine this information and place it into the Mayer-Vietoris sequence to yield after simplification

$$H_1(T^2) \simeq H_1(S^3 \setminus K) \oplus H_1(V_K).$$

We know that $H_1(T^2) \simeq \mathbb{Z}[l] \oplus \mathbb{Z}[m]$ and $H_1(V_K) \simeq \mathbb{Z}[l]$.

So, since $i_k[m] = 0$ in $H_1(V_K)$ as seen in the MV sequence, it must be that $i_k[m]$ generates $H_1(S^3 \setminus K)$ to create the proper isomorphism above.

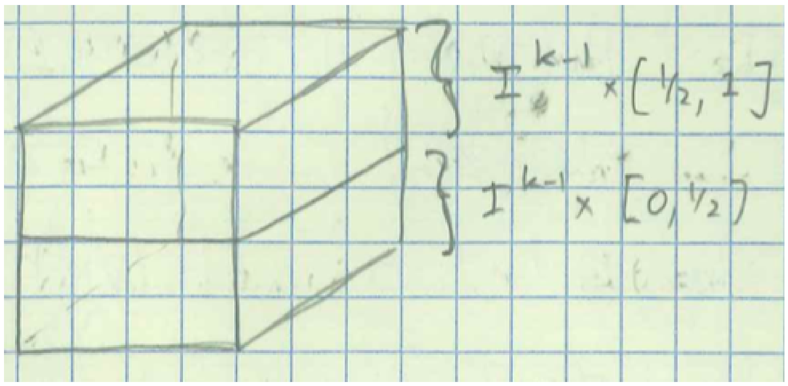
To prove this proposition, we will employ the Mayer-Vietoris sequence again along with induction on k.

Proof of Proposition 1.5.

For k=0, D^0 is a point, and $S^n \setminus h(D^0) \simeq \mathbb{R}^n$, so of course $\widetilde{H}_i = 0$ for all i, trivially!

For the induction step, replace the disk D^k by the unit cube, I^k and divide I^k in half.

Observe this for the case k=3



Now, we need A and B for the MV sequence. So,

$A = S^n \setminus h(I^{k-1} \times [0, \frac{1}{2}])$ and $B = S^n \setminus h(I^{k-1} \times [\frac{1}{2}, 1])$.

which leads us to

$A \cap B = S^n \setminus h(I^k)$ and $A \cup B = S^n \setminus h(I^{k-1} \times \{\frac{1}{2}\})$.

By induction, $\widetilde{H}_k(A \cup B) = 0$, so the MV sequence implies that we have the isomorphism

$\widetilde{H}_i(S^n \setminus h(I^k)) \xrightarrow{\sim} \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$.

This isomorphism can be viewed geometrically as being induced by the inclusions $S^n \setminus h(I^k) \hookrightarrow A, B$.

So if α is a cycle in $S^n \setminus h(I^k)$ which is NOT a boundary in $S^n \setminus h(I^k)$, then α is also NOT a boundary in at least one of A and B. (Remember, we have a cycle if and only if $\partial\alpha = 0$.)

Now we iterate this process by subdividing the last factor of I^k into a nested sequence of closed subintervals, namely $I_1 \supset I_2 \supset \cdots$, converging to some point p in I, so that α is NOT a boundary in the spaces $S^n \setminus h(I^{k-1} \times I_m)$ for any m.

But we know by induction that $\alpha = \partial\beta$ must be a boundary of some chain β in $S^n \setminus h(I^{k-1} \times \{p\})$. The chain β is a finite $\mathbb{Z}$ linear combination of (singular) i-simplices $\sigma(\delta_i)$ with a compact image in $S^n \setminus h(I^{k-1} \times \{p\})$.

The chain β is also covered by the nested sequence of open sets $\left\{S^n \setminus h(I^{k-1} \times \{p\})\right\}_{m\in\mathbb{N}}$.

Thus by compactness, β must be contained in $S^n \setminus h(I^{k-1} \times I_m)$ for some m.

However, this is a contradiction our original β. Thus $\alpha = \partial\beta$ is a boundary in $S^n \setminus h(I^k)$, completing the induction step.

b) Thankfully, this part of the proposition moves along faster. It is also done by induction k.

For k=0, $S^n \setminus h(S^0) \simeq \mathbb{R}^n \setminus \{p\} \simeq S^{n-1} \times \mathbb{R}$, where p is a point.

Hence, $\widetilde{H}_i(S^n \setminus h(S^0)) = \mathbb{Z}$ for $i = n-1$ and 0 otherwise.

For the induction step, we decompose our spheres so that we can apply the MV sequence as follows

$$S^k = D^k_+ \cup D^k_-$$

which gives us automatically $D^k_+ \cap D^k_- \simeq S^{k-1}$.

We apply the MV sequence for the cover by using the assignments

$$A = S^n \setminus h(D^k_+) \text{ and } B = S^n \setminus h(D^k_-)$$

giving us

$$A \cap B = S^n \setminus h(S^k)$$

$$A \cup B = S^n \setminus h(S^{k-1}).$$

Now we can use a) to see that $\widetilde{H}_k(A) = \widetilde{H}_k(B) = 0$. So by the MV sequence we get

$$\widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \simeq \widetilde{H}_i(S^n \setminus h(S^1))$$

which gives us the induction as desired.

□

Having gone through all that work on that proposition, we immediately get some corollaries which we shall not prove here.

Corollary 1.7. *(i) Brouwer Separation*

If X is a topological S^{n-1} embedded in S^n, then $S^n \setminus X$ has two path-components, each with boundary X. Moreover each component has the homology of a point.

(ii) If $U \subset \mathbb{R}^n$ is open and $h : U \to \mathbb{R}^n$ is continuous and injective, then $h(U)$ is also open in R^n.

When we've been considering the knot complement, it has been sitting inside more than two dimensional space. We need a theorem for this space that is analogous to the Jordan Curve Theorem to be able to speak about separation in the right way. Thankfully, we have that theorem, and as one might expect it is somewhat more complicated.

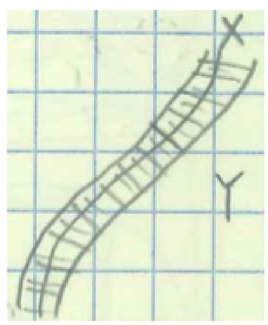
Theorem 1.8 (Schoenflies Theorem)**.** *Let J be a simple closed curve in $\mathbb{R}^2$ or S^2. Then the closure of one (or both) components of $\mathbb{R}^2 \setminus J$ is homeomorphic to the unit disk D^2.*

We'll need Schoenflies's theorem in higher dimensions, most especially for the embeddings of the 2-sphere in $\mathbb{R}^2 \setminus S^3$. The proof goes quite similarly to the proof we'll provide in a more visually tractable case. Before we can state the generalized version, we need the following notion

Definition 1.9. A subset $X \subset Y$ is *bicollared* in Y if there exists an embedding

$$b : X \times [-1, 1] \to Y$$

such that $b(x, 0) = x$, x in X.



When $dim(X) = n$ and $dim(Y) = n + 1$, the bicollar is an example of a tubular neighborhood.

Remark 1.10. Bicollars and tubular neighborhoods always exist for *smooth* submanifolds $X \subset Y$. Requiring bicollars helps to eliminate wild behavior that we wish to avoid.

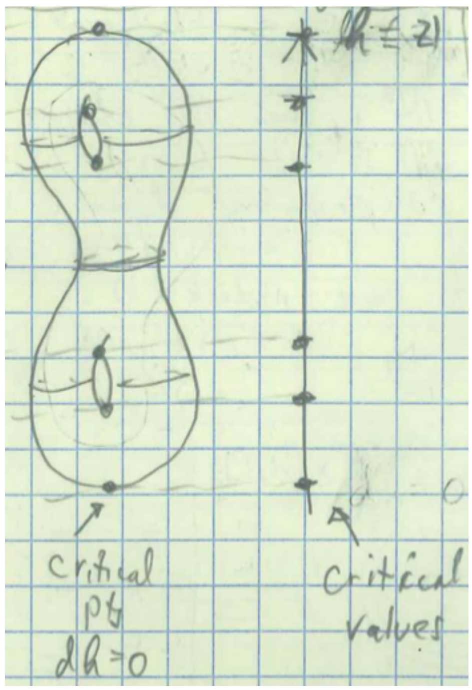
Theorem 1.11 (Generalized Schoenflies, Brown 1960)**.** *Suppose X is a bicollared (n-1)-sphere in $\mathbb{R}^n$, then the closure of its bounded complementary domain is homeomorphic to an n-disk, D^n. This is similarly true for S^n.*

The proof we provide is for the special case $S \setminus \mathbb{R}^3$ (smooth), originally due to J. W. Alexander.

Proof:

Our main approach will be to use "baby" Morse Theory which will allow us to decompose the embedded sphere into pieces which we know bound smooth balls. Then we can reassemble these into one big 3-ball.

Let $S \setminus \mathbb{R}^3$ be an embedded smooth surface with height function $h : S \to \mathbb{R}$. In the figure below, you can see an example of this for a genus 2 surface in 3-space.



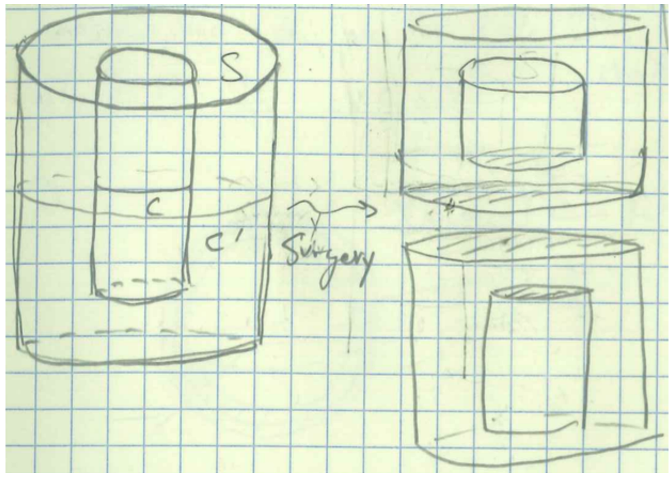
Without loss of generality, assume h is *Morse* with finitely many non-degenerate critical points each with distinct critical values.

Let $z_1 < \cdots < z_n$ be regular (i.e. not critical) values of h such that each open interval $(-\infty, z_1), (z_1, z_2), \cdots (z_n, \infty)$ contains EXACTLY one critical value.

Then for each z_j we see that $h^{-1}(z_j)$ is a disjoint union of embedded circles in the xy-plane. By the 2D Schoenflies, each circle bounds a smooth disk.

Let $C \subset h^{-1}(z_j)$ be an invariant circle. This means that the disk bounded by C is disjoint from the other circles in $h^{-1}(z_j)$.

Now surger S along C as in the figure below. Repeat successively on all circle in $h^{-1}(z_j)$ for each j.

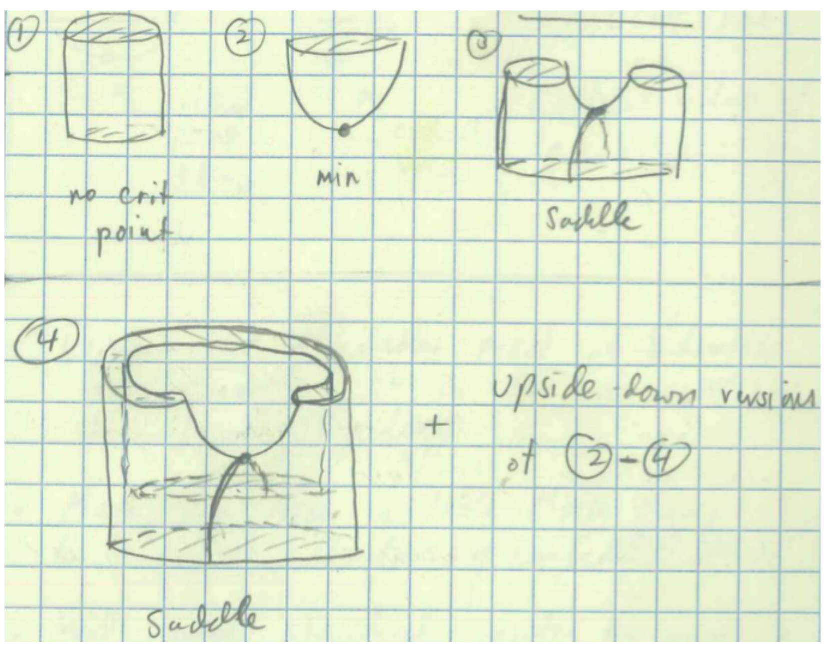


After surgeries, S decomposes into a disjoint, that is

$$S = \bigsqcup S_k$$

of closed surfaces S_k composed of horizontal caps glued onto connected subsurfaces of S with at most one critical point.

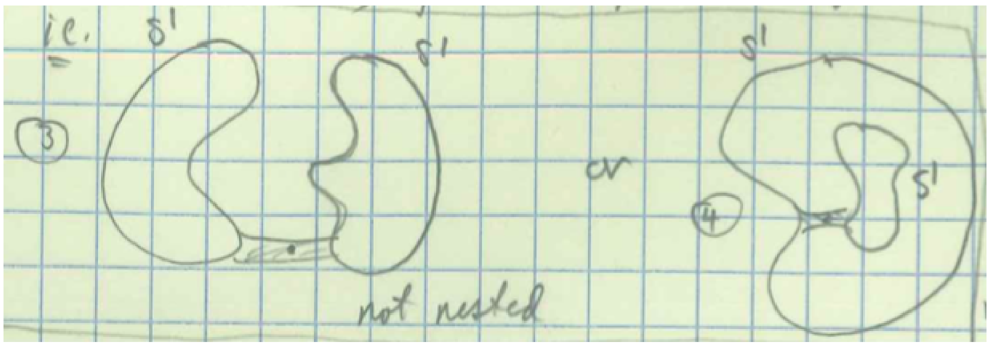
We need a lemma to assist us here.



Lemma 1.12. *Each S_k is isotopic to one of seven models as shown above (with the upside down versions of 2-4 noted). Hence, each S_k bounds a smooth ball.*

Proof:

We isotope S_k so that the surface is composed of vertical segments outside of the neighborhood of the critical point which is either a max/min or saddle. Say it's a saddle. Then S_k looks from above like two smooth disjoint circles joined by a neighborhood of an arc, as seen in the picture below.



Isotope this viewpoint to the standard model.

$\square$

And now that we've covered all of the cases, we are done. $\square$