The Chern character of the Verlinde bundle over $\overline{\mathcal{M}}_{g,n}$

A. Marian [*]   D. Oprea [†]   R. Pandharipande [‡]   A. Pixton [§]

D. Zvonkine [¶]

November 2013

Abstract

We prove an explicit formula for the total Chern character of the Verlinde bundle over $\overline{\mathcal{M}}_{g,n}$ in terms of tautological classes. The Chern characters of the Verlinde bundles define a semisimple CohFT (the ranks, given by the Verlinde formula, determine a semisimple fusion algebra). According to Teleman's classification of semisimple CohFTs, there exists an element of Givental's group transforming the fusion algebra into the CohFT. We determine the element using the first Chern class of the Verlinde bundle on the interior $\mathcal{M}_{g,n}$ and the projective flatness of the Hitchin connection.

1 Introduction

1.1 The Verlinde bundle

Let G be a complex, simple, simply connected Lie group. For any choice of genus g, and n irreducible representations $\mu_1, \ldots, \mu_n$ of the Lie algebra $\mathfrak{g}$ at level ℓ, the Verlinde bundle

$$\mathbb{E}_g(\mu_1, \ldots, \mu_n) \to \overline{\mathcal{M}}_{g,n}$$

over the moduli space of stable pointed curves is constructed via the theory of conformal blocks [TUY], [Ts]. Geometrically, over smooth pointed curves, the fibers of the Verlinde bundle are the spaces of non-abelian theta functions – spaces of global sections of the determinant line bundles over the moduli of parabolic G-bundles.

Two basic invariants of the Verlinde bundles have been studied so far. First,

$$\text{rank } \mathbb{E}_g(\mu_1, \ldots, \mu_n) = d_g(\mu_1, \ldots, \mu_n)$$

---

[*]Department of Mathematics, Northeastern University, a.marian@neu.edu

[†]Department of Mathematics, University of California, San Diego, doprea@math.ucsd.edu

[‡]Department of Mathematics, ETH Zürich, rahul@math.ethz.ch

[§]Department of Mathematics, Harvard University, apixton@math.harvard.edu

[¶]CNRS, Institut de Mathématiques de Jussieu, zvonkine@math.jussieu.fr

is given by the Verlinde formula; see for instance [B]. Second, an explicit expression for the first Chern class was obtained in genus 0 in [Fa], [Mu], and in higher genus in [MOP]. We calculate here the total Chern character. As a consequence of our formula, the higher Chern classes lie in the tautological ring [FP] in cohomology,

$$RH^\star(\overline{\mathcal{M}}_{g,n}) \subset H^\star(\overline{\mathcal{M}}_{g,n}),$$

answering a question raised in [Fa].

The expression for the Chern character is derived as follows. It is well known that the restriction of the Verlinde bundle to each boundary divisor of $\overline{\mathcal{M}}_{g,n}$ decomposes into a direct sum of tensor products of analogous Verlinde bundles [TUY]. These factorization rules imply that the total Chern characters

$$\Omega_g(\mu_1, \ldots, \mu_n) = \operatorname{ch} \mathbb{E}_g(\mu_1, \ldots, \mu_n)$$

define a cohomological field theory (CohFT). Furthermore, the CohFT obtained is semisimple. To explicitly solve the theory, three ingredients are needed:

(i) the Givental-Teleman classification of semisimple CohFTs [Tel];

(ii) the existence of a projectively flat connection in the Verlinde bundle over the moduli of smooth pointed curves $\mathcal{M}_{g,n}$ [TUY];

(iii) the determination of the first Chern class over $\mathcal{M}_{g,n}$, which, in the form we need, is a consequence of [Ts].

1.2 The main result

We recall the following representation theoretic quantities. First, for a simple Lie algebra $\mathfrak{g}$ and for each level ℓ, the conformal anomaly is given by

$$c = c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\check{h} + \ell}$$

where $\check{h}$ is the dual Coxeter number. Next, for each representation with highest weight μ of level ℓ, we set

$$\mathsf{w}(\mu) = \frac{(\mu, \mu + 2\rho)}{2(\check{h} + \ell)}.$$

Here ρ is half of the sum of the positive roots, and the Cartan-Killing form $(,)$ is normalized so that the longest root θ has

$$(\theta, \theta) = 2.$$

2

Example. For $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$, the highest weight of a representation of level ℓ is given by an r-tuple of integers

$$\mu = (\mu^1, \ldots, \mu^r), \ \ell \geq \mu^1 \geq \cdots \geq \mu^r \geq 0,$$

defined up to shifting the vector components by the same integer. Furthermore, we have

$$c(\mathfrak{g}, \ell) = \frac{\ell(r^2 - 1)}{\ell + r},$$

$$\mathsf{w}(\mu) = \frac{1}{2(\ell + r)} \left(\sum_{i=1}^{r} (\mu^i)^2 - \frac{1}{r} \left(\sum_{i=1}^{r} \mu^i \right)^2 + \sum_{i=1}^{r} (r - 2i + 1) \mu^i \right).$$

$\Diamond$

The Verlinde Chern character will be expressed as a sum over stable graphs Γ of genus g with n legs. The contribution of each graph is a product of several factors coming from legs, edges and vertices, which we now describe.

(i) **Leg factors.** Each leg l determines a marking, and is assigned

— a cotangent class ψ_l;

— a representation μ_l of the Lie algebra $\mathfrak{g}$.

The corresponding factor is

$$\mathsf{Cont}(l) = \exp\left(\mathsf{w}(\mu_l) \cdot \psi_l\right).$$

(ii) **Vertex factors.** Every vertex v in a stable graph comes with

— a genus assignment g_v;

— n_v adjacent half-edges including the legs labeled with representations $\mu_1, \ldots, \mu_{n_v}$.

The vertex factor is defined as the rank of the Verlinde bundle

$$\mathsf{Cont}(v) = d_{g_v}(\mu_1, \ldots, \mu_{n_v}).$$

(iii) **Edge factors.** To each edge e we assign:

— a pair of conjugate representations of level ℓ: μ at one half-edge and the dual representation μ^* at the other half-edge;

— each edge corresponds to a node of the domain curve, thus determining two cotangent classes ψ'_e and ψ''_e.

3

The edge factor is then defined as

$$\mathsf{Cont}(e) = \frac{1 - \exp(\mathsf{w} \cdot (\psi'_e + \psi''_e))}{\psi'_e + \psi''_e},$$

where $\mathsf{w} = \mathsf{w}(\mu) = \mathsf{w}(\mu^*)$. The edge factor depends on the representation assignment μ, but this will not be indicated explicitly in the notation.

Next, for each stable graph Γ of type (g, n), we consider all possible assignments

$$\mu : E \to P_\ell$$

of conjugate representations of level ℓ to the edges E of the graph Γ. For each such graph Γ and representation assignment μ, we form the cohomology class

$$(\iota_\Gamma)_\star \left(\prod_l \mathsf{Cont}(l) \prod_v \mathsf{Cont}(v) \prod_e \mathsf{Cont}(e) \right) \in H^\star(\overline{\mathcal{M}}_{g,n}),$$

where ι_Γ denotes the inclusion of the boundary stratum of curves of type Γ.

Let $\lambda \in H^2(\overline{\mathcal{M}}_{g,n})$ denote the first Chern class of the Hodge bundle

$$\mathbb{H}_g \to \overline{\mathcal{M}}_{g,n}$$

with fiber $H^0(C, \omega_C)$ over the moduli point $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$.

Theorem 1 *The Chern character of the Verlinde bundle is given by*

$$\mathrm{ch}\,\mathbb{E}_g(\mu_1, \ldots, \mu_n) =$$
$$\exp\left(-\frac{c(\mathfrak{g}, \ell)}{2} \cdot \lambda \right) \sum_{\Gamma, \mu} \frac{1}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_\star \left(\prod_l \mathsf{Cont}(l) \prod_v \mathsf{Cont}(v) \prod_e \mathsf{Cont}(e) \right).$$

1.3 Acknowledgements

2 Semisimple CohFTs

2.1 Definitions

We succinctly recall terminology related to cohomological field theories. Fix a finite dimensional vector space V, endowed with a non-degenerate pairing η and a distinguished element $1 \in V$. A *CohFT* is the data of cohomology classes

$$\Omega = (\Omega_{g,n}), \quad \Omega_{g,n} \in H^\star(\overline{\mathcal{M}}_{g,n}) \otimes (V^\star)^{\otimes n}$$

for $2g - 2 + n > 0$, subject to the requirements:

(i) each $\Omega_{g,n}$ is invariant under the action of the symmetric group S_n;

(ii) Ω is compatible with the gluing maps. Explicitly, for the gluing map

$$gl : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

the pullback $gl^\star \Omega_{g,n}$ equals the contraction of $\Omega_{g-1,n+2}$ with η^{-1} at the two extra-markings. The same requirement is enforced for the second gluing map

$$gl : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}.$$

(iii) for all $v_1, \ldots, v_n \in V$, we have

$$\Omega_{g,n+1}(v_1 \otimes \ldots \otimes v_n \otimes 1) = p^\star \Omega_{g,n}(v_1 \otimes \ldots \otimes v_n),$$

where $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map, and

$$\Omega_{0,3}(v_1 \otimes v_2 \otimes 1) = \eta(v_1, v_2).$$

When $\Omega_{g,n}$ are cohomology classes of degree 0, the definition above recovers the axioms of topological quantum field theory (TQFT).

Each CohFT defines on V the structure of an associative algebra with unit via the quantum product

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

We are concerned with theories for which the algebra V is semisimple.

Such theories are classified in [Tel]. Specifically, Ω is obtained from the algebra V via the action of an R-matrix

$$R \in \mathbf{1} + z \cdot \mathrm{End} V[[z]],$$

satisfying the symplectic condition

$$R(z)R^\star(-z) = \mathbf{1}.$$

Here $R^\star$ denotes the adjoint with respect to η and $\mathbf{1}$ is the identity matrix. The explicit reconstruction of the semisimple CohFT from the R-matrix action will be explained below, following [PPZ].

2.2 Actions on CohFTs

We begin by describing two basic actions on cohomological field theories.

Assume that $\Omega = (\Omega_{g,n})$ is a CohFT with underlying vector space $(V, 1, \eta)$. Fix a symplectic matrix $R \in \mathbf{1} + z \cdot \mathrm{End}\,(V)[[z]]$ as above. A new CohFT on the space $(V, 1, \eta)$ is obtained via the cohomology elements

$$R\Omega = (R\Omega)_{g,n},$$

defined as sums over stable graphs Γ of genus g with n legs, with contributions coming from vertices, edges and legs. Specifically,

$$(R\Omega)_{g,n} = \sum_\Gamma \frac{1}{|\mathrm{Aut}\,(\Gamma)|} (\iota_\Gamma)_\star \left(\prod_l \mathsf{Cont}(l) \prod_v \mathsf{Cont}(v) \prod_e \mathsf{Cont}(e) \right) \tag{1}$$

where:

(i) the vertex contribution is
$$\mathsf{Cont}(v) = \Omega_{g(v),n(v)},$$

with $g(v)$ and $n(v)$ denoting the genus and number of half-edges and legs of the vertex;

(ii) the leg contribution is
$$\mathsf{Cont}(l) = R(\psi_l)$$

where ψ_l is the cotangent class at the marking corresponding to the leg;

(iii) the edge contribution is

$$\mathsf{Cont}(e) = \frac{\eta^{-1} - R(\psi_e')\eta^{-1}R(\psi_e'')^\top}{\psi_e' + \psi_e''}.$$

Here ψ_e' and ψ_e'' are the cotangent classes at the node which represents the edge e. The symplectic condition guarantees that the edge contribution is well-defined.

6

Remark 2.1 To simplify our formulas, we have changed Givental's and Teleman's conventions by replacing R with R^{-1}. In particular, equation (1) above determines a right group action on CohFTs, rather than a left group action as in Givental's and Teleman's papers. This will play no role here.

A second action on CohFTs is given by translations. As before, let $(\Omega, V, 1, \eta)$ be a CohFT, and consider a power series $T \in V[[z]]$ with no terms of degrees 0 and 1:

$$T(z) = T_2 z^2 + T_3 z^3 + \ldots, \quad T_k \in V.$$

A new CohFT based on $(V, 1, \eta)$, denoted $T\Omega$, is defined by setting

$$(T\Omega)_{g,n}(v_1 \otimes \ldots \otimes v_n) = \sum_{m=0}^{\infty} \frac{1}{m!} (p_m)_\star \Omega_{g,n+m}(v_1 \otimes \ldots \otimes v_n \otimes T(\psi_{n+1}) \otimes \ldots \otimes T(\psi_{n+m})) \quad (2)$$

where

$$p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$$

is the forgetful morphism.

2.3 Reconstruction

With the above terminology understood, we can state the Givental-Teleman classification theorem [Tel]. Fix Ω a semisimple CohFT on $(V, 1, \eta)$, and write ω for the degree 0 topological part of the theory. Given any symplectic matrix

$$R \in \mathbf{1} + z \cdot \mathrm{End}(V)[[z]]$$

as above, we form a power series T by plugging $1 \in V$ into R, removing the free term, and multiplying by z:

$$T(z) = z(1 - R(1)) \in V[[z]].$$

The equality

$$\Omega = RT\omega$$

then holds for a suitable symplectic matrix R. This will be used crucially below.

Lemma 2.2 *The R-matrix of a semisimple CohFT Ω is uniquely determined by the restrictions of the elements $\Omega_{g,n}$ to $\mathcal{M}_{g,n}$.*

Proof. We establish the lemma using the reconstruction statement

$$\Omega = RT\omega,$$

and working in an idempotent basis for V, afforded by semisimplicity. Explicitly, let v_α be so that

$$v_\alpha \bullet v_\beta = \delta_{\alpha,\beta} v_\alpha.$$

Denote by $c_\alpha = \eta(v_\alpha, v_\alpha)$ the scalar square of v_α. Note that $c_\alpha \neq 0$. The TQFT axioms imply

$$\omega_{g,n}(v_\alpha, v_\beta, \dots, v_\beta) = c_\beta^{1-g} \delta_{\alpha,\beta}$$

as is easily checked by decomposing the surface into pairs of pants. We expand

$$R(z) \cdot v_\alpha = \sum_{\beta,k} (R_k)_\alpha^\beta \cdot v_\beta \cdot z^k.$$

Clearly, it suffices to explain the uniqueness of $(R_k)_\alpha^\beta$. Consider the cohomology class

$$\Omega_{g,n}(v_\alpha, v_\beta, \dots, v_\beta) \in H^*(\mathcal{M}_{g,n})$$

over the open part of the moduli space. If $g > 3k$ the coefficient of ψ_1^k in this cohomology class is well-defined, because the tautological ring $R^\star(\mathcal{M}_{g,n})$ is generated by the classes

$$\kappa_1, \dots, \kappa_{[g/3]}, \psi_1, \dots, \psi_n$$

with no relations up to degree $g/3$. The freeness up to degree $g/3$ is a consequence of the stability results of [Bo, L], see also [BiFo]. The exact expression for the Givental group action

$$\Omega = RT\omega$$

given in (1) and (2) is used to find the coefficient of ψ_1^k in the restriction of Ω to $\mathcal{M}_{g,n}$. We claim this coefficient equals $c_\beta^{1-g} \cdot (R_k)_\alpha^\beta$. This follows from the following observations:

- in expression (1), the only graph Γ contributing to the restriction of

$$\Omega = RT\omega$$

  to $\mathcal{M}_{g,n}$ is the single vertex graph;

- in the translation action (2), the terms $m \geq 1$ contribute monomials in κ-classes;

- finally, for the terms corresponding to the single vertex graph and $m = 0$, we extract the coefficient of ψ_1^k with the aid of the identity

$$\omega_{g,n}(v_\alpha, v_\beta, \dots, v_\beta) = c_\beta^{1-g} \delta_{\alpha,\beta}.$$

As a consequence, every $(R_k)_\alpha^\beta$ is uniquely determined by the restriction of the classes $\Omega_{g,n}$ to $\mathcal{M}_{g,n}$ for g large enough, as claimed. $\diamondsuit$

3 Proof of the Theorem

3.1 The CohFT obtained from conformal blocks

The total Chern character of the bundle of conformal blocks defines a CohFT. Explicitly:

(a) the vector space V has as basis the irreducible representations of $\mathfrak{g}$ at level ℓ. The distinguished element 1 corresponds to the trivial representation. The pairing η is given by

$$\eta(\mu, \nu) = \delta_{\mu, \nu^\star}$$

where $\nu^\star$ denotes the dual representation.

(b) The cohomology elements defining the theory are

$$\Omega_{g,n}(\mu_1, \ldots, \mu_n) = \mathrm{ch}_t\left(\mathbb{E}_g(\mu_1, \ldots, \mu_n)\right) \in H^\star(\overline{\mathcal{M}}_{g,n}).$$

Here, for a vector bundle $\mathbb{E}$ with Chern roots $r_1, \ldots, r_k$ we write

$$\mathrm{ch}_t(\mathbb{E}) = \sum_{j=1}^{k} \exp(t \cdot r_j).$$

Axiom (i) in the definition of CohFT is obvious. Axiom (ii) follows from the fusion rules of [TUY]. For instance, for the irreducible boundary divisor we have

$$gl^\star \mathbb{E}_g(\mu_1, \ldots, \mu_n) = \bigoplus_{\nu \in P_\ell} \mathbb{E}_{g-1}(\mu_1, \ldots, \mu_n, \nu, \nu^\star)$$

and taking Chern characters we find

$$\Omega_{g,n}(\mu_1 \otimes \ldots \otimes \mu_n) = \sum_{\nu \in P_\ell} \Omega_{g-1,n+2}(\mu_1 \otimes \ldots \otimes \mu_n \otimes \nu \otimes \nu^\star),$$

as required by (ii). The existence of a unit as required by axiom (iii) is *propagation of vacua*, and is proved in the form needed here in [Fa], Proposition 2.4(i). Indeed, under the marking-forgetting map

$$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n},$$

we have

$$p^\star \mathbb{E}_g(\mu_1, \ldots, \mu_n) = \mathbb{E}_g(\mu_1, \ldots, \mu_n, 1).$$

Finally, the requirement that

$$\Omega_{0,3}(\mu \otimes \nu \otimes 1) = \delta_{\mu, \nu^\star}$$

9

is Corollary 4.4 of [B].

The CohFT Ω thus constructed is semisimple. Indeed, when $t = 0$, the resulting TQFT

$$\omega_{g,n} = \Omega_{g,n}|_{t=0}$$

is given by

$$\omega_{g,n} \in (V^\star)^n, \quad \omega_{g,n}(\mu_1, \ldots, \mu_n) = \mathrm{rk}\ \mathbb{E}_g(\mu_1, \ldots, \mu_n) = d_g(\mu_1, \ldots, \mu_n).$$

The associated Frobenius algebra is the Verlinde fusion algebra, and is known to be semisimple. An account can be found in Proposition 6.1 of [B].

3.2   The R-matrix of the CohFT of conformal blocks

Teleman's classification [Tel] ensures that Ω is obtained from the Verlinde fusion algebra ω by Givental's group action of an R-matrix $R \in \mathrm{End}\,V[[z]]$. The R-matrix of the theory will be found below, and shown to be diagonal in the natural basis of V consisting of irreducible representations at level ℓ.

It will be more convenient to consider a slightly modified CohFT Ω', given by

$$\Omega'_{g,n} = \Omega_{g,n} \exp\left(t \cdot \frac{c(\mathfrak{g}, \ell)}{2} \cdot \lambda\right).$$

The fact that Ω' still satisfies the requisite axioms follows from the fact that the Hodge bundle splits compatibly over the boundary divisors in $\overline{\mathcal{M}}_{g,n}$.

We proceed to find the restrictions of the theory Ω' to $\mathcal{M}_{g,n}$. The Verlinde bundle is projectively flat over $\mathcal{M}_{g,n}$ by results of [TUY], [Ts]. Therefore over $\mathcal{M}_{g,n}$ its Chern character is given by

$$\mathrm{ch}_t(\mathbb{E}_g(\mu_1, \ldots, \mu_n)) = d_g(\mu_1, \ldots, \mu_n) \cdot \exp\left(t \cdot \frac{c_1(\mathbb{E}_g(\mu_1, \ldots, \mu_n))}{d_g(\mu_1, \ldots, \mu_n)}\right). \tag{3}$$

Over the open part of the moduli space $\mathcal{M}_{g,n}$, the slope of the Verlinde bundle was written in [MOP] as a consequence of [Ts]:

$$\frac{c_1(\mathbb{E}_g(\mu_1, \ldots, \mu_n))}{d_g(\mu_1, \ldots, \mu_n)} = -\frac{c(\mathfrak{g}, \ell)}{2} \cdot \lambda + \sum_{i=1}^n \mathsf{w}(\mu_i)\psi_i.$$

The signs differ from [MOP], where the dual bundle of covacua was used. Therefore,

$$\Omega'_{g,n}(\mu_1 \otimes \ldots \otimes \mu_n) = d_g(\mu_1, \ldots, \mu_n) \cdot \exp(t \cdot \sum_{i=1}^n \mathsf{w}(\mu_i)\psi_i) \tag{4}$$

10

over $\mathcal{M}_{g,n}$.

We must have $\Omega' = RT\omega$ for a symplectic matrix R and the translation T introduced previously. Let $W \in \operatorname{End} V[[z]]$ be the diagonal matrix in the basis of level ℓ representations, whose diagonal elements are given by

$$W(z)_\mu^\mu = \exp(tz \cdot \mathsf{w}(\mu)).$$

Since $\mathsf{w}(0) = 0$, we see that $W(z) \cdot 1 = 1$. Hence, the associated $T(z)$ vanishes,

$$T(z) = z(1 - W(z) \cdot 1) = 0.$$

By the above discussion, the identification of the two CohFT's

$$\Omega' = W\omega$$

holds over $\mathcal{M}_{g,n}$ for all g and n. This is precisely the content of equation (4) for the left-hand side, and equation (1) for the right-hand side. By the unique reconstruction of the R-matrix from restrictions, proved in Lemma 2.2, we conclude that $R = W$. Hence the equality

$$\Omega' = W\omega$$

also holds over the compact moduli space $\overline{\mathcal{M}}_{g,n}$. Formula (1) applied to the matrix W and to the theory ω yields the expression of the Theorem.

Remark 3.1 It is, of course, possible to find the R-matrix that takes the Verlinde fusion algebra directly to the CohFT Ω rather than Ω'. This R-matrix is given by

$$R(z)_\mu^\mu = \exp\left(tz \cdot \left(\mathsf{w}(\mu) + \frac{c(\mathfrak{g}, \ell)}{24}\right)\right).$$

Remark 3.2 The Verlinde slope formula over $\mathcal{M}_{g,n}$, written in [MOP], is used as input for our derivation. The full slope formula in [MOP], on the compactification $\overline{\mathcal{M}}_{g,n}$, is then recovered by the result proven here. The matching of the formula here with [MOP] provides a nontrivial check.

Remark 3.3 Since the projective flatness of the Hitchin connection is used as an input, our formula for the Chern character of the Verlinde bundle yields no nontrivial relations in the tautological ring $RH^*(\mathcal{M}_{g,n})$ by imposing projective flatness.

References

[B]   A. Beauville, *Conformal blocks, fusion rules, and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry, Israel Math. Conf. Proc, vol. 9, 1996, 75–96.

[BiFo] G. Bini, C. Fontanari, *On the tautological ring of $\mathcal{M}_{g,n}$*, `arXiv:1307.7266`.

[Bo]   S. K. Boldsen, *Improved homological stability for the mapping class group with integral or twisted coefficients*, Math. Z. 270 (2012), 297–329.

[FP]   C. Faber and R. Pandharipande, *Tautological and non-tautological cohomology of the moduli space of curves* in *Handbook of moduli*, 293–330, G. Farkas and I. Morrison eds., International Press, 2013.

[Fa]   N. Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, 145 – 176, Contemp. Math., 564, Amer. Math. Soc., Providence, RI, 2012.

[L]   E. Looijenga, *Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map*, J. Alg. Geom. 5 (1996) 135 – 150.

[MOP]   A. Marian, D. Oprea, R. Pandharipande, *The Chern classes of the Verlinde bundles,* `arXiv:1308.4425`.

[Mu]   S. Mukhopadhyay, *Rank-level duality and conformal block divisors*, preprint, `arXiv:1308.0854`.

[Tel]   C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. 188 (2012), 525–588.

[PPZ]   R. Pandharipande, A. Pixton, D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures,* `arXiv:1303.1043`.

[Ts]   Y. Tsuchimoto, *On the coordinate-free description of the conformal blocks*, J . Math. Kyoto Univ. 33 (1993), 29–49.

[TUY]   A. Tsuchiya, K. Ueno, Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, 459–566.