ON PRODUCT IDENTITIES AND THE CHOW RINGS OF HOLOMORPHIC SYMPLECTIC VARIETIES

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Abstract. For a moduli space M of stable sheaves over a $K3$ surface X, we propose a series of conjectural identities in the Chow rings $CH_\star(\mathsf{M} \times X^\ell)$, $\ell \geq 1$, generalizing the classic Beauville-Voisin identity for a $K3$ surface. We emphasize consequences of the conjecture for the structure of the tautological subring $R_\star(\mathsf{M}) \subset CH_\star(\mathsf{M})$. We prove the proposed identities when M is the Hilbert scheme of points on a $K3$ surface.

1. Introduction

Understanding the Chow ring of irreducible holomorphic symplectic varieties is a problem of considerable interest. In the case of a smooth projective $K3$ surface X, an essential role in approaching the cycle structure is played by a distinguished zero-cycle c_X, first noted and studied in [BV]. The cycle c_X has degree one and is the Chow class of any point lying on a rational curve in X. The intersection of any two divisors is a multiple of c_X, while the second Chern class of the tangent bundle satisfies

$$c_2(TX) = 24\, c_X.$$

In higher dimensions, it is natural to consider moduli spaces of stable sheaves on $K3$ surfaces. For a smooth projective $K3$ surface X, we let $v \in H^\star(X, \mathbb{Z})$ be a primitive Mukai vector, and let M be the moduli space of stable sheaves over X with Mukai vector v, relative to a v-generic polarization. We note nevertheless that all statements in this text apply more broadly to moduli spaces of Bridgeland-stable sheaf complexes with respect to a v-generic stability condition σ. The moduli space M is a smooth projective irreducible holomorphic symplectic variety of dimension

$$\dim \mathsf{M} = m = \langle v,\, v\rangle + 2,$$

admitting a quasi-universal sheaf

$$\mathcal{F} \to \mathsf{M} \times X.$$

To keep the exposition simple we assume in fact that M is a fine moduli space, so $\mathcal{F}$ is a universal object. The restriction is not essential, as we later explain. We denote the two projections by $\pi : \mathsf{M} \times X \to \mathsf{M}$ and $\rho : \mathsf{M} \times X \to X$.

In parallel with the $K3$ geometry, there is a distinguished zero-cycle

$$c_{\mathsf{M}} \in CH_0(\mathsf{M})$$

of degree one: this is the class of any stable sheaf F such that

$$c_2(F) = k\, c_X \ \text{ in } \ CH_0(X), \tag{1}$$

where k is the degree of the second Chern class specified by the Mukai vector v. Sheaves satisfying (1) exist in M (cf. [OG2]), and have the same Chow class as shown in [MZ], following a conjecture of [SYZ]. In analogy with the surface situation, one expects ([V1], [SYZ]) that the special cycle corresponds to the largest rational equivalence orbit of points on M. The intersection-theoretic properties of c_{M} are not understood as well as those of its counterpart c_X in the two-dimensional context.

We study the geometry of the universal sheaf and of the special cycles c_X and c_{M} in two strands:

(1) We single out the *tautological subring* $R_\star(\mathsf{M}) \subset CH_\star(\mathsf{M})$, generated by the classes

$$\pi_\star \left(M(c_i(\mathcal{F})) \cdot \rho^\star \beta \right),$$

with M any monomial in the Chern classes of $\mathcal{F}$, and β any class in the Beauville-Voisin subring

$$R_\star(X) = CH_2(X) + CH_1(X) + \mathbb{Z}\, c_X \subset CH_\star(X).$$

(2) We emphasize the rank zero virtual sheaf

$$\overline{\mathcal{F}} = \mathcal{F} - \rho^\star F, \ \text{ with } \ F \in \mathsf{M} \text{ such that } [F] = c_{\mathsf{M}} \in CH_0(\mathsf{M}).$$

Intuitively, the second Chern class of $\overline{\mathcal{F}}$ reflects to some extent the variation of rational equivalence classes across points in M, relative to the special class.

These two strands come together naturally within the framework of the following, which is the main conjecture of the paper.

Conjecture 1. *Let $\alpha \in R_\star(\mathsf{M})$ be a tautological class of codimension d. Consider the product $\mathsf{M} \times X^\ell$, where $d + \ell > \dim \mathsf{M}$. Let $\overline{\mathcal{F}}_i$ denote the pullback to $\mathsf{M} \times X^\ell$ of the virtual universal sheaf on M and the ith factor of X. Then for every $i_1, \ldots, i_\ell \geq 0$,*

$$\alpha \cdot ch_{i_1}(\overline{\mathcal{F}}_1) \cdots ch_{i_\ell}(\overline{\mathcal{F}}_\ell) = 0 \ \in \ CH_\star(\mathsf{M} \times X^\ell). \tag{2}$$

The Künneth components along M of the Chern classes $c_i(\overline{\mathcal{F}})$ on $\mathsf{M} \times X$ have positive cohomological degrees for $i > 0$. Since M has no odd cohomology, the products (2) are thus homologically trivial for dimension reasons due to the inequality $d + \ell > \dim \mathsf{M}$.

Conjecture 1 yields a rich collection of interesting Chow identities and we highlight some of them now. In case $\mathsf{M} = X$, viewed trivially as the Hilbert scheme of one point on itself, we have

$$\overline{\mathcal{F}} = \overline{\mathcal{I}}, \text{ where } \overline{\mathcal{I}} = \mathcal{I}_{\Delta} - \mathcal{I}_{X\times c} \text{ on } X \times X,$$

with $c \in X$ a point of Chow class c_X. Therefore

$$ch_2(\overline{\mathcal{F}}) = ch_2(\overline{\mathcal{I}}) = -\overline{\Delta},$$

where we have set

$$\overline{\Delta} = \Delta - X \times c_X \text{ in } CH_2(X \times X).$$

Thus when $\mathsf{M} = X$, for the tautological class $\alpha = 1$, the identity

$$ch_2(\overline{\mathcal{F}}_1) \cdot ch_2(\overline{\mathcal{F}}_2) \cdot ch_2(\overline{\mathcal{F}}_3) = 0 \text{ in } CH_2(X \times X^3)$$

predicted under (2) takes the form

$$\overline{\Delta}_{01} \cdot \overline{\Delta}_{02} \cdot \overline{\Delta}_{03} = 0 \text{ in } CH_2(X \times X^3), \tag{3}$$

while the K-theoretic identity

$$\overline{\mathcal{I}}_{01} \cdot \overline{\mathcal{I}}_{02} \cdot \overline{\mathcal{I}}_{03} = 0 \text{ in } K(X \times X^3) \tag{4}$$

also holds. Here the index 0 is used to keep track of the first distinguished factor X in the quadruple product X^4, and $\overline{\Delta}_{0i}$ and $\overline{\mathcal{I}}_{0i}$ indicate pullbacks from the 0th and ith factors.

Pushing forward to the product of the last three factors, equation (3) is easily seen to be equivalent to the fundamental Beauville-Voisin identity [BV]

$$\Delta - \Delta_c + \Delta_{c,c} = 0 \text{ in } CH_2(X \times X \times X). \tag{5}$$

Here c again denotes a fixed point of Chow class c_X; Δ is the small diagonal of points (x, x, x); Δ_c consists of triples of the form (x, x, c), (c, x, x), (x, c, x); $\Delta_{c,c}$ is the set of triples of the form (c, c, x), (c, x, c), (x, c, c) for $x \in X$.

If we now take $\alpha = D$, a divisor class on X, the vanishing

$$\alpha \cdot ch_2(\overline{\mathcal{F}}_1) \cdot ch_2(\overline{\mathcal{F}}_2) = 0 \text{ in } CH_3(X \times X^2)$$

predicted by Conjecture 1 becomes

$$D^{(0)} \cdot \overline{\Delta}_{01} \cdot \overline{\Delta}_{02} = 0 \text{ in } CH_3(X \times X^2), \tag{6}$$

(with the divisor D pulled back from the 0th factor) which is known to hold. Indeed, any divisor class D on X is a linear combination of classes of rational curves on X, and the cycles $\overline{\Delta}_{01}$, $\overline{\Delta}_{02}$ restrict to zero for every point on a rational curve in the 0th factor X. The spreading principle (cf. [V2, Theorem 3.1]) implies (6). We also note the trivial identity

$$c_X^{(0)} \cdot \overline{\Delta}_{01} = 0 \text{ in } CH_0(X \times X). \tag{7}$$

Returning now to the case of a general moduli space M, we see that for $\alpha = 1$, the expected vanishing

$$ch_2(\overline{\mathcal{F}}_1) \cdot ch_2(\overline{\mathcal{F}}_2) \cdots ch_2(\overline{\mathcal{F}}_{m+1}) = 0 \text{ in } CH_m(\mathsf{M} \times X^{m+1}) \tag{8}$$

predicted by Conjecture 1 is the natural generalization of the Beauville-Voisin fundamental identity (5) in the triple product of a $K3$ surface. The beautiful identity

$$\overline{\mathcal{F}}_1 \cdot \overline{\mathcal{F}}_2 \cdots \overline{\mathcal{F}}_{m+1} = 0 \text{ in } K(\mathsf{M} \times X^{m+1}) \tag{9}$$

is also predicted by Conjecture 1.

For tautological classes $\alpha \in R_\star(\mathsf{M})$ of positive codimension, the series of identities predicted by Conjecture 1 should be viewed as generalizing (6) and (7) from the $K3$ context to a general moduli setup.

The identities of Conjecture 1 lead in turn to a large collection of conjectural Chow vanishings in the self-products $\mathsf{M} \times \mathsf{M} \times \cdots \times \mathsf{M}$. We set

$$\overline{\Delta} = \Delta - \mathsf{M} \times c_{\mathsf{M}} \text{ in } CH_m(\mathsf{M} \times \mathsf{M}),$$

and observe

Theorem 1. *The system of identities* (2) *of Conjecture 1 is equivalent to the vanishings*

$$\alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\ell} = 0 \text{ in } CH_\star(\mathsf{M} \times \mathsf{M}^\ell). \tag{10}$$

for any tautological class $\alpha \in R_\star(\mathsf{M})$ of codimension d and integer ℓ satisfying

$$d + \ell > \dim \mathsf{M}.$$

Here the first factor of M is labeled by 0, and α is pulled back from this factor.

Theorem 1 is immediately seen to have a few interesting consequences. At one end, if we take $\alpha \in R_\star(\mathsf{M})$ to be a tautological zero-cycle, and pick $\ell = 1$, we obtain the vanishing

$$\alpha \cdot \overline{\Delta}_{01} = 0 \text{ in } CH_0(\mathsf{M} \times \mathsf{M}).$$

Pushing forward to the second factor of M, this gives

$$\alpha = n\, c_{\mathsf{M}},$$

where n is the degree of α, and allows us to conclude

Corollary 1. *Assuming Conjecture 1 holds, the tautological ring $R_\star(\mathsf{M})$ has rank one in dimension zero:*

$$R_0(\mathsf{M}) = \mathbb{Q} \cdot c_{\mathsf{M}}.$$

At the other end, taking $\alpha = 1$ yields the vanishing

$$\overline{\Delta}_{01} \cdots \overline{\Delta}_{0,m+1} = 0 \text{ in } CH_m(\mathsf{M} \times \mathsf{M}^{m+1}). \tag{11}$$

It is easy to see that the pushforward of this product cycle, via the projection $\mathsf{M} \times \mathsf{M}^{m+1} \to \mathsf{M}^{m+1}$ forgetting the first factor, is the modified diagonal cycle studied in [OG3]. We thus obtain

Corollary 2. *Assuming Conjecture 1 holds, the modified diagonal cycle*

$$\Gamma^{m+1}(\mathsf{M}, c_{\mathsf{M}}) = \Delta - \Delta_c + \Delta_{c,c} - \cdots + \Delta_{c,c,\ldots,c}$$

vanishes in $CH_m(\mathsf{M}^{m+1})$.

We also note

Corollary 3. *Assuming Conjecture 1, for every codimension* $0 \leq k \leq m$ *there is a filtration*

$$F^0 \subset F^1 \subset \cdots \subset F^{k-1} \subset F^k = CH^k(\mathsf{M}).$$

For a fixed codimension k *and* $0 \leq i \leq k$ *we set*

$$F^i(CH^k(\mathsf{M})) = \{\alpha \textit{ with } \alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\,m-k+i+1} = 0 \subset CH_\star(\mathsf{M} \times \mathsf{M}^{m-k+i+1})\}.$$

Here α *is pulled back to the product from the first factor.*

Clearly, in view of the conjectured vanishing (11), we would have

$$F^k = \{\alpha \text{ with } \alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\,m+1} = 0 \subset CH_\star(\mathsf{M} \times \mathsf{M}^{m+1})\} = CH^k(\mathsf{M}).$$

Notice further that

$$F^0 = \{\alpha \text{ with } \alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\,m-k+1} = 0 \subset CH_\star(\mathsf{M} \times \mathsf{M}^{m-k+1})\}.$$

Thus for every k, Conjecture 1 and Theorem 1 would place all codimension k tautological classes in $F^0(CH^k(\mathsf{M}))$. It is also known [SYZ] that M admits Lagrangian constant cycle subvarieties for the special cycle c_{M}. The spreading principle then places these subvarieties in $F^0(CH^n(\mathsf{M}))$ for $n = m/2$. It should be interesting to investigate this filtration in all codimensions and compare it in top codimension with the filtrations on $CH_0(\mathsf{M})$ studied in [V1], [SYZ].

As evidence for Conjecture 1, we show:

Theorem 2. *Let* X *be a smooth projective* $K3$ *surface. Conjecture 1 holds for* $\mathsf{M} = X^{[n]}$*, the Hilbert scheme of* n *points on* X.

A richer version of Theorem 1 is proven in Section 2 as Theorem 1*. Section 2 also discusses the tautological subring of the Chow ring in a broader context, for the products $\mathsf{M} \times X^\ell$, $\ell \geq 0$. This leads to generalizations of Conjecture 1 and Theorem 2 to the setting of a product $\mathsf{M} \times X^\ell$ which are needed in the inductive argument of Section 3.

Theorem 2 is argued in Section 3 inductively on the number of points, using a Chow refinement of the classic inductive mechanism [EGL]. Tautological classes and the natural products (2) are well-behaved through the induction. The base case is shown to come down to the three fundamental identities (3), (6) and (7) in the surface context.

Together, Theorems 1 and 2 prove in particular, among many other identities, the vanishing of the modified diagonal cycle for the Hilbert scheme $X^{[n]}$, first established in [V3].

We note that in the context of the tautological ring of the product $\mathsf{M} \times X^{\ell}$ it is natural to formulate the stronger

Conjecture 2. *The restriction of the cycle class map to the tautological subring,*

$$\tau : R_{\star}(\mathsf{M} \times X^{\ell}) \to H_{\star}(\mathsf{M} \times X^{\ell}),$$

is injective.

This statement follows the line of conjectures on the injectivity of the cycle class map on suitable subrings of Chow initiated in [B2], [V4]. It completely subsumes Conjecture 1, our main object in this paper. Indeed the identities (2) are among tautological classes and hold in homology. The advantage of Conjecture 1 is that it proposes a *concrete* set of relations in the Chow ring, and is thus easier to come to grips with than the elusive Conjecture 2.

Finally, the interesting problem of understanding corrections of the identities (2) in a relative setting over the moduli space of polarized $K3$ surfaces is left for future explorations.

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2. Tautological rings and product cycles

2.1. **Tautological rings.** Let X be a smooth projective $K3$ surface, $v \in H^{\star}(X, \mathbb{Z})$ a primitive Mukai vector, and let M be the moduli space of stable sheaves with

Mukai vector v on X relative to a v-generic polarization. To ease the exposition, we make the nonrestrictive assumption that M admits a universal family

$$\mathcal{F} \to \mathsf{M} \times X.$$

If this is not the case, we use instead the universal *Chern character* $\mathrm{ch}\,\mathcal{F} \in CH_\star(\mathsf{M} \times X)$ discussed in [Ma1], [Ma2] to define the tautological ring, as well as to formulate the identities (2) of Conjecture 1. The construction in Section 3 of [Ma1] and Section 3.1 of [Ma2] follows the original explanation of [Mu2], Appendix A.5. Specifically, a universal sheaf can be glued together over $\mathbb{P} \times X$, where $\mathbb{P} \to \mathsf{M}$ is a suitable projective bundle over M. After appropriate twisting, the universal Chern character (but not the sheaf) is seen to descend as a rational class to the product $\mathsf{M} \times X$. When available, the universal sheaf is defined up to tensoring by line bundles from M. If a universal sheaf is not available, the universal Chern character is correspondingly (cf. [Ma1], Section 3) defined up to multiplication by the Chern character of a line bundle over the moduli space. We now recall from the introduction:

Definition 1. The tautological ring

$$R_\star(\mathsf{M}) \subset CH_\star(\mathsf{M})$$

is the subring of Chow generated by the following classes

- $\pi_\star(M(c_i(\mathcal{F})))$, M any monomial in the Chern classes of $\mathcal{F}$;
- $\pi_\star\left(M(c_i(\mathcal{F})) \cdot \rho^\star D\right)$, $D \in CH_1(X)$, M any monomial in the Chern classes of $\mathcal{F}$;
- $\pi_\star\left(M(c_i(\mathcal{F})) \cdot \rho^\star c_X\right)$, M any monomial in the Chern classes of $\mathcal{F}$.

Thus $R_\star(\mathsf{M})$ is generated by all classes of the form

$$\pi_\star(M(c_i(\mathcal{F})) \cdot \rho^\star \beta),$$

for an arbitrary monomial M in the Chern classes of $\mathcal{F}$, and $\beta \in R_\star(X)$ any class in the Beauville-Voisin ring

$$R_\star(X) = \mathbb{Z}\, c_X + CH_1(X) + CH_2(X) \subset CH_\star(X).$$

As we repeatedly consider Chern characters, we will work throughout with $\mathbb{Q}$ coefficients.

It is useful to extend Definition 1 to the arbitrary products

$$\mathsf{M} \times X^k, \text{ for } k \geq 0.$$

For $1 \leq s \leq k$, let $\rho_s : \mathsf{M} \times X^k \to X$ be the map to the factor indexed by s, with the accompanying projection $\overline{\rho}_s : \mathsf{M} \times X^k \to \mathsf{M} \times X$. Denote by

$$\mathcal{F}_s = \overline{\rho}_s^\star \mathcal{F}$$

the universal sheaf on $\mathsf{M} \times X^k$ pulled back from $\mathsf{M} \times X$ via the sth projection.

Definition 2. The tautological system of rings $R_\star(\mathsf{M}\times X^k) \subset CH_\star(\mathsf{M}\times X^k),\ k \geq 0$ is the smallest system of $\mathbb{Q}$-algebras satisfying the following three properties:

(i) $R_\star(\mathsf{M} \times X^k)$ contains the Chern classes $c_i(\mathcal{F}_s),\ 1 \leq s \leq k$, as well as the classes $\rho_s^* D$, for $D \in CH_1(X)$.

(ii) The system is closed under pushforward via the natural projections $\pi : \mathsf{M} \times X^n \to \mathsf{M} \times X^k$ forgetting factors of X, where $n \geq k$.

(iii) The system is closed under pushforward via the natural inclusions $\iota : \mathsf{M} \times X^n \to \mathsf{M} \times X^k$ for $n \leq k$ through diagonal embeddings of factors of X or embeddings using the special cycle c_X.

Concretely, this means that for each $k \geq 1$, the subring $R_\star(\mathsf{M} \times X^k) \subset CH_\star(\mathsf{M} \times X^k)$ is generated by the following classes:

- the pullbacks $\pi^\star\alpha$ from M to the product $\mathsf{M} \times X^k$, where $\alpha \in CH_\star(\mathsf{M})$ is tautological in the sense of Definition 1.
- the Chern classes $c_i(\mathcal{F}_s),\ 1 \leq s \leq k$;
- the pullback classes $\rho_s^\star D,\ \ \rho_s^\star c_X,\ 1 \leq s \leq k$;
- the diagonal classes $\rho_{rs}^\star \Delta,\ 1 \leq r, s \leq k$.

(It is straightforward to check that each of the classes above is in $R_\star(\mathsf{M} \times X^k)$, and that any polynomial in these classes satisfies the three properties in Definition 2. Thus they generate $R_\star(\mathsf{M} \times X^k)$.)

Remark 1. We note that the tautological ring $R_\star(\mathsf{M})$ is independent of the modular interpretation of the holomorphic symplectic manifold M. Suppose that

$$\mathsf{M} = \mathsf{M}_v \simeq \mathsf{M}_{v'},$$

where $\mathsf{M}_{v'}$ is a moduli space of stable sheaves with Mukai vector v' relative to a polarization H' over a $K3$ surface X'. There is then a derived (anti-)equivalence

$$\Phi : D^b(X) \simeq D^b(X')$$

with kernel $\mathcal{E} \in D^b(X \times X')$ inducing the isomorphism

$$\overline{\Phi} : \mathsf{M}_v \to \mathsf{M}_{v'}.$$

Let $\mathcal{F} \to \mathsf{M}_v \times X$, $\mathcal{F}' \to \mathsf{M}_{v'} \times X'$ be the universal objects, and $\pi : \mathsf{M} \times X \to \mathsf{M}$, $\pi' : \mathsf{M} \times X' \to \mathsf{M}$ the projections. Considering the extended equivalence induced by $\mathcal{E}$,

$$\Phi_{\mathsf{M}} : D^b(\mathsf{M}_v \times X) \simeq D^b(\mathsf{M}_v \times X'),$$

we have

$$\mathcal{F}_0 =_{\text{def}} \Phi_{\mathsf{M}}(\mathcal{F}) = (\overline{\Phi} \times \mathrm{id}_{X'})^\star(\mathcal{F}').$$

Let $\pi'_\star\left(P(c_i(\mathcal{F}')\cdot\beta'\right)$ be a class on M tautological in the sense of v'. Here P is a polynomial in the Chern classes of the universal sheaf $\mathcal{F}'\to\mathsf{M}_{v'}\times X'$ and $\beta'\in R_\star(X')$ is in the Beauville-Voisin ring of X'.

Under the identification $\overline{\Phi}$, we have

$$\pi'_\star\left(P(c_i(\mathcal{F}')\cdot\beta'\right)=\pi'_\star\left(P(c_i(\mathcal{F}_0)\cdot\beta'\right).$$

As

$$\mathrm{ch}\,\mathcal{F}_0=\mathrm{ch}\,\Phi_{\mathsf{M}}(\mathcal{F})=(\pi\times\mathrm{id}_{X'})_\star\left(\mathrm{ch}\,\mathcal{F}\cdot\mathrm{ch}\,\mathcal{E}\cdot\mathrm{td}\,X\right),$$

it is standard to write the pushforward $\pi'_\star\left(P(c_i(\mathcal{F}_0)\cdot\beta'\right)$ from $\mathsf{M}\times X'$ as a pushforward $\pi_\star\left(Q(c_i(\mathcal{F})\cdot\beta\right)$ from $\mathsf{M}\times X$, for some polynomial Q in the Chern classes of the universal object $\mathcal{F}$ and a class $\beta\in CH_\star(X)$. Importantly, β is in fact in the Beauville-Voisin subring $R_\star(X)\subset CH_\star(X)$: as discussed in [H], the derived equivalence preserves Beauville-Voisin rings. Thus a tautological class in the sense of $\mathsf{M}_{v'}$ is also tautological in the sense of M_v. □

2.2. **Examples of tautological classes.**

2.2.1. *Divisors.* It is well-known (cf. [Mu1] , [Mu2], [OG1], [Y]) that the determinant line bundle homomorphism

$$\Theta_v: v^\perp\to NS(\mathsf{M})$$

is an isomorphism for $\langle v,\,v\rangle>0$, and is in all cases surjective. Here

$$v^\perp\subset H^\star_{alg}(X,\,\mathbb{Z})$$

denotes the orthogonal complement of the Mukai vector v in the algebraic Mukai lattice. Divisors on M are thus tautological, making Definition 1 independent of the choice of universal family/universal Chern character.

2.2.2. *Chern classes of the tangent bundle.* We have

$$\mathrm{ch}\,(T\mathsf{M})=2-\mathrm{ch}\,\mathcal{E}xt^\bullet_\pi(\mathcal{F},\,\mathcal{F})=2-\pi_\star(\mathrm{ch}\,\mathcal{F}^\vee\cdot\mathrm{ch}\,\mathcal{F}\cdot\rho^\star\mathrm{td}\,X),$$

therefore the Chern classes $c_i(T\mathsf{M})$ are tautological.

2.2.3. *The special cycle* c_{M}. We show now that the distinguished zero-cycle c_{M} is tautological. Consider the product $\mathsf{M}\times\mathsf{M}\times X$, equipped with the universal sheaves $\mathcal{E},\,\mathcal{F}$ which correspond to the two copies of M, and are pulled back to the product. Let $\pi:\mathsf{M}\times\mathsf{M}\times X\to\mathsf{M}\times\mathsf{M}$ be the projection. We form the natural relative Ext complex (shifted for convenience),

$$\mathsf{W}(\mathcal{E},\mathcal{F})=\mathcal{E}xt^\bullet_\pi(\mathcal{E},\,\mathcal{F})[1]\ \text{ on }\ \mathsf{M}\times\mathsf{M}.$$

We further fix $F_0\to X$ a sheaf parameterized by M, and denote by

$$\mathsf{W}(\mathcal{E},\,\rho^\star F_0)=\mathcal{E}xt^\bullet_\pi(\mathcal{E},\,\rho^\star F_0)[1]\ \text{ on }\ \mathsf{M},$$

the pullback of $\mathsf{W}(\mathcal{E},\,\mathcal{F})$ under the inclusion $\mathsf{M}\times[F_0]\hookrightarrow\mathsf{M}\times\mathsf{M}$.

As observed in [MZ], the complex W plays a role in understanding Chow classes of points on M, since the middle Chern class of W is the class of the diagonal in the product $\mathsf{M}\times\mathsf{M}$. The formula

$$c_m\left(\mathsf{W}(\mathcal{E},\mathcal{F})\right)=[\Delta]\ \text{ in }\ CH_m(\mathsf{M}\times\mathsf{M}) \tag{12}$$

was established in [Ma1], and is aligned with earlier work of Beauville [B1] and Ellingsrud-Strømme [ES] on representing the diagonal in terms of the universal Chern classes, in the context of moduli spaces of Gieseker-stable sheaves. By pull-back, the diagonal formula (12) gives

$$c_m\left(\mathsf{W}(\mathcal{E},\,\rho^\star F_0)\right)=[F_0]\ \text{ in }\ CH_0(\mathsf{M}), \tag{13}$$

and by Grothendieck-Riemann-Roch we have

$$\mathrm{ch}\,\left(\mathsf{W}(\mathcal{E},\,\rho^\star F_0)\right)=-\pi_\star\left[\mathrm{ch}\,\mathcal{E}^\vee\cdot\rho^\star(\mathrm{ch}\,F_0\cdot(1+2c_X))\right].$$

In particular, if $F\in\mathsf{M}$ is any sheaf such that $c_2(F)\in CH_0(X)$ is a multiple of c_X, then

$$c_{\mathsf{M}}=[F]=c_m\left(\mathsf{W}(\mathcal{E},\,\rho^\star F)\right) \tag{14}$$

is manifestly tautological.

Remark 2. As shown in [SYZ], there exist Lagrangian constant cycle subvarieties for the special cycle c_{M}. For any such subvariety $V\subset\mathsf{M}$ of dimension $n=m/2$, the vanishing

$$[V]\cdot\overline{\Delta}_{01}\cdot\overline{\Delta}_{02}\cdots\overline{\Delta}_{0,n+1}=0\text{ in }CH_\star(\mathsf{M}\times\mathsf{M}^{n+1}),$$

holds by the spreading principle ([V2], Theorem 3.1). It would be very interesting to understand whether the class of such a subvariety V is in the tautological ring.

2.3. **Vanishing of product cycles.** We now show that the system of product identities of Conjecture 1 leads in turn to a large collection of conjectural Chow vanishings in the self-products $\mathsf{M}\times\mathsf{M}\times\cdots\times\mathsf{M}$. Among them is the vanishing of O'Grady's modified diagonal cycle. Aligned with our point of view, the modified diagonal cycle is also cast here in product form.

To start, let us single out the complex

$$\mathsf{W}(\mathcal{E},\overline{\mathcal{F}})=\mathcal{E}xt^\bullet_\pi(\mathcal{E},\,\mathcal{F}-\rho^\star F)[1]\ \text{ on }\ \mathsf{M}\times\mathsf{M}, \tag{15}$$

where F represents the special zero-cycle, $[F]=c_{\mathsf{M}}$, and $\mathcal{F},\,\mathcal{E}$ are the universal objects on the first and second factors respectively.
We also set

$$\overline{\Delta}=\Delta-\mathsf{M}\times c_{\mathsf{M}}=c_m\left(\mathsf{W}(\mathcal{E},\mathcal{F})\right)-c_m\left(\mathsf{W}(\mathcal{E},\,\rho^\star F)\right)\ \text{ in }\ CH_m(\mathsf{M}\times\mathsf{M}).$$

Further, in the context of a product $\mathsf{M}\times \mathsf{M}^\ell\times X$, we let $\mathcal{E}_1,\dots,\mathcal{E}_\ell$, $\mathcal{F}$ be the universal sheaves corresponding to the last ℓ factors of M, respectively the first distinguished factor. We label this factor by 0, and show

Theorem 1*. *For any class $\alpha\in CH_\star(\mathsf{M})$ of codimension d satisfying the inequality*

$$d+\ell>\dim\mathsf{M},$$

the following three vanishing statements are equivalent.

(i) $\alpha\cdot ch_{i_1}(\overline{\mathcal{F}}_1)\cdots ch_{i_\ell}(\overline{\mathcal{F}}_\ell)=0\in CH_\star(\mathsf{M}\times X^\ell)$, *for all* $i_1,\dots,i_\ell\geq 0$. *Here* $\overline{\mathcal{F}}_s$, $1\leq s\leq \ell$, *is the normalized universal sheaf pulled back from* M *and the* s*th factor in the product* X^ℓ.

(ii) $\alpha\cdot c_{i_1}(\mathsf{W}(\mathcal{E}_1,\overline{\mathcal{F}}))\cdot c_{i_2}(\mathsf{W}(\mathcal{E}_2,\overline{\mathcal{F}}))\cdot\dots\cdot c_{i_\ell}(\mathsf{W}(\mathcal{E}_\ell,\overline{\mathcal{F}}))=0\in CH_\star\left(\mathsf{M}\times\mathsf{M}^\ell\right)$, *for all* $i_1,\dots,i_\ell>0$. *Here the complex* $\mathsf{W}(\mathcal{E}_s,\overline{\mathcal{F}})$, $1\leq s\leq\ell$, *is pulled back from the distinguished factor* M *and the* s*th factor in the product* M^ℓ.

(iii) $\alpha\cdot\overline{\Delta}_{01}\cdots\overline{\Delta}_{0,\ell}=0\in CH_\star\left(\mathsf{M}\times\mathsf{M}^\ell\right)$,

In all three cases, the class α is pulled back to the product from the first distinguished factor M.

Remark 3. Note that (i) is the vanishing predicted by Conjecture 1 in case α is tautological. Theorem 1 is therefore the equivalence of (i) and (iii) for $\alpha\in R_\star(\mathsf{M})$ and is completely subsumed by the statement above. The vanishing of the modified diagonal cycle, corresponding to $\alpha=1$, is thus implied by Conjecture 1.

Proof. We show first that (i) implies (ii). To start, we note that for the complex

$$\mathsf{W}(\mathcal{E},\overline{\mathcal{F}})\ \text{ on }\ \mathsf{M}\times\mathsf{M},$$

each Chern class $c_k\left(\mathsf{W}(\mathcal{E},\overline{\mathcal{F}})\right)$ for $k>0$ is expressible in terms of pure-degree pieces of the Chern character, and is therefore a sum of products of factors of type

$$\alpha_{ij}=\pi_\star\left[\mathrm{ch}_i\mathcal{E}^\vee\cdot\mathrm{ch}_j\overline{\mathcal{F}}\cdot\mathrm{td}X\right]\ \text{ and }\ \beta_{ij}=\pi_\star\left[\mathrm{ch}_i\mathcal{E}^\vee\cdot\mathrm{ch}_j\overline{\mathcal{F}}\right].$$

We consider now the larger product

$$\mathsf{M}\times\mathsf{M}^\ell\times X^\ell$$

along with a class $\alpha\in CH_\star(\mathsf{M})$ pulled back from the distinguished first factor M, satisfying

$$\operatorname{codim}\alpha+\ell>\dim\mathsf{M}.$$

We let $\mathcal{F}_1,\dots\mathcal{F}_\ell$ be the universal sheaves pulled back from $\mathsf{M}\times X^\ell$, where M is the distinguished first factor. We also consider the universal sheaves $\mathcal{E}_1,\dots\mathcal{E}_\ell$ on the new factors of M each paired with a factor of X.

By (i), the vanishing

$$\alpha \cdot \mathrm{ch}_{j_1}\overline{\mathcal{F}}_1 \cdots \mathrm{ch}_{j_\ell}\overline{\mathcal{F}}_\ell = 0$$

holds in $CH_\star(\mathsf{M} \times \mathsf{M}^\ell \times X^\ell)$, pulled back from $\mathsf{M} \times X^\ell$. This trivially implies the vanishing of the larger product

$$\alpha \cdot \mathrm{ch}_{i_1}\mathcal{E}_1^\vee \cdot \mathrm{ch}_{j_1}\overline{\mathcal{F}}_1 \cdot (\mathrm{td}\, X_1)^{a_1} \cdots \mathrm{ch}_{i_\ell}\mathcal{E}_\ell^\vee \cdot \mathrm{ch}_{j_\ell}\overline{\mathcal{F}}_\ell \cdot (\mathrm{td}\, X_\ell)^{a_\ell} = 0$$

in $CH_\star(\mathsf{M} \times \mathsf{M}^\ell \times X^\ell)$, for $i_1, j_1, \ldots i_\ell, j_\ell \geq 0$. Here the exponents $a_1, \ldots, a_\ell$ are either 0 or 1. Pushing forward via the projection $\mathsf{M} \times \mathsf{M}^\ell \times X^\ell \to \mathsf{M} \times \mathsf{M}^\ell$ gives

Lemma 1. *Consider the product* $\mathsf{M} \times \mathsf{M}^\ell \times X$ *and a cycle* α *on the distinguished factor* M*, satisfying codim* $\alpha + \ell > \dim \mathsf{M}$*. Denote by* $\mathcal{E}_1, \ldots \mathcal{E}_\ell, \mathcal{F}$ *the universal sheaves associated with the last* ℓ *copies of* M*, respectively the first one, pulled back to the product* $\mathsf{M} \times \mathsf{M}^\ell \times X$*. Let* $\pi : \mathsf{M} \times \mathsf{M}^\ell \times X \to \mathsf{M} \times \mathsf{M}^\ell$ *be the projection. Then*

$$\alpha \cdot \prod_{k=1}^{\ell} \pi_\star \left[ch_{i_k}\mathcal{E}_k^\vee \cdot ch_{j_k}\overline{\mathcal{F}} \cdot (td\, X)^{a_k} \right] = 0 \text{ in } CH_\star(\mathsf{M} \times \mathsf{M}^\ell), \tag{16}$$

for any $i_k, j_k \geq 0$*, and* a_k *taken either* 0 *or* 1*.*

As observed earlier, each factor $c_{i_k}\left(\mathsf{W}(\mathcal{E}_k, \overline{\mathcal{F}})\right)$ in the products (ii) of Theorem 1* is a sum of terms each containing a factor of type appearing in (16) of the lemma, so the vanishings (ii) follow.

Notice next that (ii) implies (iii). Indeed, we have in K-theory,

$$\mathsf{W}(\mathcal{E}, \mathcal{F}) = \mathsf{W}(\mathcal{E}, \overline{\mathcal{F}}) + \mathsf{W}(\mathcal{E}, \rho^\star F) \text{ in } K(\mathsf{M} \times \mathsf{M}),$$

therefore

$$c_m(\mathsf{W}(\mathcal{E}, \mathcal{F})) = \sum_{i=0}^{m} c_i(\mathsf{W}(\mathcal{E}, \overline{\mathcal{F}})) \cdot c_{m-i}(\mathsf{W}(\mathcal{E}, \rho^\star F)),$$

and

$$\overline{\Delta} = c_m(\mathsf{W}(\mathcal{E}, \mathcal{F})) - c_m(\mathsf{W}(\mathcal{E}, \rho^\star F)) = \sum_{i=1}^{m} c_i(\mathsf{W}(\mathcal{E}, \overline{\mathcal{F}})) \cdot c_{m-i}(\mathsf{W}(\mathcal{E}, \rho^\star F)).$$

It is thus clear that any term in the expansion of the product $\overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\ell}$ contains a product $c_{i_1}(\mathsf{W}(\mathcal{E}_1, \overline{\mathcal{F}})) \cdot c_{i_2}(\mathsf{W}(\mathcal{E}_2, \overline{\mathcal{F}})) \cdot \cdots \cdot c_{i_\ell}(\mathsf{W}(\mathcal{E}_\ell, \overline{\mathcal{F}}))$ for some $i_1, \ldots, i_\ell > 0$. Accordingly, the vanishing (ii) implies (iii).

Finally, it is easy to see that (iii) implies (i). We start with the vanishing

$$\alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\ell} = 0 \in CH_\star\left(\mathsf{M} \times \mathsf{M}^\ell\right),$$

pulled back from $\mathsf{M} \times \mathsf{M}^\ell$ to the larger product $\mathsf{M} \times \mathsf{M}^\ell \times X^\ell$. Trivally, we also have

$$\alpha \cdot \overline{\Delta}_{01} \cdots \overline{\Delta}_{0,\ell} \cdot \mathrm{ch}_{i_1}(\mathcal{F}_1) \cdots \mathrm{ch}_{i_\ell}(\mathcal{F}_\ell) = 0 \in CH_\star\left(\mathsf{M} \times \mathsf{M}^\ell \times X^\ell\right),$$

for any $i_1, \ldots, i_\ell \geq 0$. Here each $\mathcal{F}_s$ is pulled back from a factor $\mathsf{M} \times X$ in the product $\mathsf{M}^\ell \times X^\ell$. Pushing forward under the projection $\mathsf{M} \times \mathsf{M}^\ell \times X^\ell \to \mathsf{M} \times X^\ell$ gives

$$\alpha \cdot \mathrm{ch}_{i_1}(\overline{\mathcal{F}}_1) \cdots \mathrm{ch}_{i_\ell}(\overline{\mathcal{F}}_\ell) = 0 \in CH_\star\left(\mathsf{M} \times X^\ell\right),$$

for any $i_1, \ldots, i_\ell \geq 0$. This concludes the proof of the theorem. □

2.4. **Extension of Conjecture 1.** We end this section by formulating the following natural extension of our main conjecture. In the context of the product $\mathsf{M} \times X^k \times X^\ell$, let us index by $\{1, \ldots, \ell\}$ the individual factors in the product X^ℓ and by $\{\hat{1}, \ldots, \hat{k}\}$ the factors in the product X^k. As usual, $\overline{\mathcal{F}}_t$ denotes the normalized universal sheaf from the t-th factor.

Conjecture 1*. *For any integers $k \geq 0$ and $\ell > 0$ consider the product $\mathsf{M} \times X^k \times X^\ell$ and a tautological class $\alpha \in R^d(\mathsf{M} \times X^k)$ satisfying*

$$d + \ell > \dim(\mathsf{M} \times X^k).$$

For any indices $i_1, \ldots, i_\ell \geq 0$, partition $\Omega \sqcup \Theta = \{1, \ldots, \ell\}$, assignment $s : \Theta \to \{\hat{1}, \ldots, \hat{k}\}$ we have

$$\alpha \cdot \prod_{t \in \Omega} \mathrm{ch}_{i_t}\left(\overline{\mathcal{F}}_t\right) \cdot \prod_{t \in \Theta} \mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}) = 0 \;\; \text{in} \;\; CH_\star\left(\mathsf{M} \times X^k \times X^\ell\right).$$

Observe that Conjecture 1 is a special case of Conjecture 1*, specifically the case $k = 0$ and (necessarily) $\Theta = \varnothing$. In K-theory, Conjecture 1* predicts the natural generalization of (9), namely that for any $0 \leq a \leq \ell = \dim \mathsf{M} + 2k + 1$ and assignment $s : \{a+1, \ldots, \ell\} \to \{\hat{1}, \ldots, \hat{k}\}$, we have:

$$\overline{\mathcal{F}}_1 \cdot \overline{\mathcal{F}}_2 \cdot \cdots \cdot \overline{\mathcal{F}}_a \cdot \mathcal{O}_{\overline{\Delta}_{s_{a+1},a+1}} \cdot \mathcal{O}_{\overline{\Delta}_{s_{a+2},a+2}} \cdot \cdots \cdot \mathcal{O}_{\overline{\Delta}_{s_\ell,\ell}} = 0.$$

In Section 3 we will prove the following, which implies Theorem 2 in the introduction.

Theorem 2*. *Conjecture 1* holds when $\mathsf{M} = X^{[n]}$ is the Hilbert scheme of n points on X.*

3. The product identities for $\mathsf{M} = X^{[n]}$

The aim of this section is to prove Theorem 2*. We let $\mathcal{I}_n$ denote the ideal sheaf of the universal subscheme

$$\mathcal{Z}_n \subset X^{[n]} \times X$$

and set

$$\overline{\mathcal{I}}_n := \mathcal{I}_n - \rho^* \mathsf{I}_n,$$

the rank zero virtual universal sheaf, where I_n is the ideal sheaf on X of any subscheme of length n with $c_2(\mathsf{I}_n) = n\, c_X$. We state the theorem explicitly. In the context of the product $X^{[n]} \times X^k \times X^\ell$, let us index by $\{1, \ldots, \ell\}$ the individual factors in the product X^ℓ and by $\{\hat{1}, \ldots, \hat{k}\}$ the factors in the product X^k. Further, $\overline{\mathcal{I}}_n^{(t)}$ denotes the normalized universal sheaf from the t-th factor. We then restate:

Theorem 2*. *For any integers $n \geq 1$, $\ell \geq 1$, $k \geq 0$, consider the product $X^{[n]} \times X^k \times X^\ell$ and a tautological class $\alpha \in R^d(X^{[n]} \times X^k)$ satisfying*

$$d + \ell > 2n + 2k.$$

For any indices $i_1, \ldots, i_\ell \geq 0$, partition $\Omega \sqcup \Theta = \{1, \ldots, \ell\}$, assignment $s : \Theta \to \{\hat{1}, \ldots, \hat{k}\}$ we have

$$\alpha \cdot \prod_{t \in \Omega} \mathrm{ch}_{i_t}\left(\overline{\mathcal{I}}_n^{(t)}\right) \cdot \prod_{t \in \Theta} \mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t, t}}) = 0 \ \text{ in } CH_\star\left(X^{[n]} \times X^k \times X^\ell\right).$$

3.1. **Induction preliminaries.** We argue inductively on the number of points using the geometry of the nested Hilbert scheme

$$X^{[n,n+1]} \subset X^{[n]} \times X^{[n+1]}$$

parametrizing pairs $(\xi, \xi') \in X^{[n]} \times X^{[n+1]}$ such that $\xi \subset \xi'$. The inductive technique was first used in [EGL] to relate top intersections on $X^{[n+1]}$ and $X^{[n]} \times X$; we now recall its main features. Each point $(\xi, \xi') \in X^{[n,n+1]}$ corresponds to an exact sequence

$$0 \to I_{\xi'} \to I_\xi \to \mathcal{O}_x \to 0, \tag{17}$$

leading to projection maps

$$\begin{array}{ccccc} & & X^{[n,n+1]} & & \\ & \overset{\phi}{\swarrow} & \downarrow p & \overset{\psi}{\searrow} & \\ X^{[n]} & & X & & X^{[n+1]} \end{array}, \tag{18}$$

and globally to an isomorphism

$$X^{[n,n+1]} \cong \mathbb{P}(\mathcal{I}_n)$$

of smooth projective varieties.

An important role for the geometry of $X^{[n,n+1]}$ is played by the hyperplane line bundle

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{I}_n)}(1)$$

with first Chern class $c_1(\mathcal{L}) = \lambda$. Letting

$$\sigma = \phi \times p : X^{[n,n+1]} \to X^{[n]} \times X,$$

we have (cf. [EGL])

$$\sigma_\star\left(\lambda^i\right) = (-1)^i c_i(-\mathcal{I}_n). \tag{19}$$

The following fundamental exact sequence on $X^{[n,n+1]} \times X$ relates the universal ideal sheaves and the exceptional line bundle $\mathcal{L}$:

$$0 \to \psi_X^\star \mathcal{I}_{n+1} \to \phi_X^\star \mathcal{I}_n \to \pi^\star \mathcal{L} \otimes \sigma_X^\star \mathcal{O}_\Delta \to 0. \tag{20}$$

Here (and elsewhere in the paper) we use the notation $f_X = f \times \mathrm{id}_X$; the map $\pi : \mathbb{P}(\mathcal{I}_n) \times X \to \mathbb{P}(\mathcal{I}_n)$ is the projection; Δ denotes the diagonal in $X \times X$, pulled back in (20) via $\sigma_X : X^{[n,n+1]} \times X \to X^{[n]} \times X \times X$.

Furthermore, for a vector bundle $F \to X$, let $F^{[n]}$ denote the associated tautological vector bundle on $X^{[n]}$,

$$F^{[n]} = \pi_\star \left(\mathcal{O}_{\mathcal{Z}_n} \otimes \rho^\star(F) \right).$$

As usual in this text, π and ρ are the projections from $X^{[n]} \times X$ to the first and second factors respectively. The K-theoretic equality

$$\psi^\star F^{[n+1]} = \phi^\star F^{[n]} + \mathcal{L} \cdot p^\star F \ \text{ in } \ K(X^{[n,n+1]})$$

follows from (20). In particular,

$$\mathcal{L} = \psi^\star \mathcal{O}^{[n+1]} - \phi^\star \mathcal{O}^{[n]} \ \text{ in } \ K(X^{[n,n+1]}). \tag{21}$$

The induction in [EGL] only tracks degrees of top codimension classes on the Hilbert scheme $X^{[n]}$. Since our argument involves Chow classes of arbitrary codimension, we note the following

Lemma 2. *Let α be a class in $CH_\star(X^{[n+1]} \times X^k)$. Then*

$$\alpha = 0 \iff \sigma_\star \psi^\star \alpha = \sigma_\star(\lambda \cdot \psi^\star \alpha) = 0 \text{ in } CH_\star(X^{[n]} \times X \times X^k).$$

In the statement of the lemma and also onwards, we abuse notation and denote

$$\psi = \psi \times \mathrm{id}_{X^{[k]}} : X^{[n,n+1]} \times X^k \to X^{[n+1]} \times X^k$$
$$\sigma = \sigma \times \mathrm{id}_{X^{[k]}} : X^{[n,n+1]} \times X^k \to X^{[n]} \times X \times X^k,$$

Proof. We use the description of the Chow groups of

$$\mathrm{Hilb} = \amalg_{n \geq 0} X^{[n]}$$

in [dCM], as well as an aspect of Lehn's formulas in Chow recently established in [MN]. Recall first the definition of the Nakajima operators $\mathfrak{q}_i, \mathfrak{q}_{-i}, i > 0$. For every $i > 0$ consider the subscheme

$$X^{[n,n+i]} = \{(I \supset I') \,\text{s.t.}\, I/I' \,\text{is supported at a single point}\, x \in X\} \subset X^{[n]} \times X^{[n+i]},$$

with accompanying maps

$$\begin{array}{ccccc} & & X^{[n,n+i]} & & \\ & \overset{\phi}{\swarrow} & \downarrow p & \overset{\psi}{\searrow} & \\ X^{[n]} & & X & & X^{[n+i]} \end{array}$$

This correspondence defines the operators:

$$\mathfrak{q}_{\pm i} :\ CH_\star(\mathrm{Hilb}) \to CH_\star(\mathrm{Hilb} \times X),$$

$$\mathfrak{q}_i = (\psi \times p)_\star \circ \phi^\star,$$

$$\mathfrak{q}_{-i} = (\phi \times p)_\star \circ \psi^\star,$$

which satisfy the commutation relations in the Heisenberg algebra. By composition one can form any string operator

$$\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_\ell} : CH_\star(\mathrm{Hilb}) \to CH_\star(\mathrm{Hilb} \times X^\ell).$$

Any class $\Gamma \in CH_\star(X^\ell)$ then defines an endomorphism of $CH_\star(\mathrm{Hilb})$ via

$$\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_\ell}(\Gamma) = \pi_{1*}(\mathfrak{q}_{i_1} \cdots \mathfrak{q}_{i_\ell} \cdot \pi_2^*(\Gamma)),$$

where $\pi_1, \pi_2 \colon \mathrm{Hilb} \times X^\ell \to \mathrm{Hilb}, X^\ell$ are the standard projections.

The main theorem of [dCM] establishes that all Chow classes of the Hilbert scheme arise from Chow classes of symmetric products X^ℓ through the Nakajima correspondences. To state this precisely, let the vacuum vector v be the generator of $CH_\star(X^{[0]}) \simeq \mathbb{Q}$. Then

$$CH_\star(\mathrm{Hilb}) = \bigoplus_{\substack{n_1 \geq \ldots \geq n_\ell > 0 \\ \Gamma \in CH_\star(X^\ell)^{\mathrm{sym}}}} \mathbb{Q} \cdot \mathfrak{q}_{n_1} \cdots \mathfrak{q}_{n_\ell}(\Gamma) \cdot v, \tag{22}$$

where $CH_\star(X^\ell)^{\mathrm{sym}} \subset CH_\star(X^\ell)$ denotes the subring of classes invariant under transpositions (ij) for which $n_i = n_j$. The isomorphism (22) is induced by a correspondence whose transpose gives the inverse map. It follows that for each n,

$$\bigoplus_{\substack{n_1 + \cdots + n_\ell = n \\ n_1 \geq \ldots \geq n_\ell > 0}} \mathfrak{q}_{-\mathfrak{n}_1} \cdots \mathfrak{q}_{-n_\ell} :\ CH_\star(X^{[n]}) \longrightarrow \bigoplus_{\substack{n_1 + \cdots + n_\ell = n \\ n_1 \geq \ldots \geq n_\ell > 0}} CH_\star(X^\ell)$$

is injective.

The Chow groups of the products $\mathrm{Hilb} \times X^k$, $k > 0$, admit a parallel description, since the inverse of the isomorphism (22) is induced by the transpose correspondence. We thus have

$$CH_\star(\mathrm{Hilb} \times X^k) = \bigoplus_{\substack{n_1 \geq \ldots \geq n_\ell > 0 \\ \Gamma \in CH_\star(X^{\ell+k})^{\mathrm{sym}}}} \mathbb{Q} \cdot \mathfrak{q}_{n_1} \cdots \mathfrak{q}_{n_l}(\Gamma) \cdot v. \tag{23}$$

(Here $CH_\star(X^{\ell+k})^{\text{sym}} \subset CH_\star(X^{\ell+k})$ is the subring of classes invariant under transpositions on the first ℓ factors for which $n_i = n_j$.) Correspondingly, for every n and k, the map

$$\bigoplus_{\substack{n_1+\cdots+n_\ell=n\\ n_1\geq\ldots\geq n_\ell>0}} \mathfrak{q}_{-\mathfrak{n}_1}\cdots\mathfrak{q}_{-n_\ell} : CH_\star(X^{[n]}\times X^k) \longrightarrow \bigoplus_{\substack{n_1+\cdots+n_\ell=n\\ n_1\geq\ldots\geq n_\ell>0}} CH_\star(X^{\ell+k})$$

is injective.

For a class $\alpha \in CH_\star(X^{[n+1]}\times X^k)$ we thus have, relevant to the lemma:

$$\mathfrak{q}_{-i}(\alpha) = 0 \text{ for all } i>0 \implies \alpha = 0.$$

Assume now that $\sigma_\star\psi^\star\alpha = \sigma_\star(\lambda\cdot\psi^\star\alpha) = 0$. We note right away that

$$\sigma_\star\psi^\star = \mathfrak{q}_{-1} : CH_\star\left(X^{[n+1]}\times X^k\right) \to CH_\star\left(X^{[n]}\times X\times X^k\right).$$

Furthermore, recall from [L, Definition 3.8] that the boundary operator

$$\delta :\, CH_\star(X^{[n]}) \to CH_\star(X^{[n]})$$

represents multiplication by the divisor class $c_1(\mathcal{O}^{[n]})$ on $X^{[n]}$. Since via (21) we have

$$\lambda = \psi^\star c_1(\mathcal{O}^{[n+1]}) - \phi^\star c_1(\mathcal{O}^{[n]}),$$

the operation of pulling back via ψ, intersecting with the hyperplane λ, and pushing forward by σ, is the commutator $\mathfrak{q}_{-1}^{(1)}$ of $\mathfrak{q}_{-1}$ with the boundary δ,

$$\sigma_\star(\lambda\cdot\psi^\star) = [\delta,\, \mathfrak{q}_{-1}] = \mathfrak{q}_{-1}^{(1)} :\, CH_\star(X^{[n+1]}\times X^k) \to CH_\star(X^{[n]}\times X\times X^k).$$

It is known that the operators $\mathfrak{q}_{-1}$ and $\mathfrak{q}_{-1}^{(1)}$ generate all Nakajima lowering operators on the level of Chow, as explained in [MN]. To be precise (cf. [MN] equation (1.12) in Theorem 1.7), we have

$$[\mathfrak{q}_{-1}^{(1)},\, \mathfrak{q}_{-i}] = i\,\Delta_\star(\mathfrak{q}_{-i-1}).$$

The left and right side are homomorphisms $CH_\star(\text{Hilb}\times X^k) \to CH_\star(\text{Hilb}\times X^2\times X^k)$ where the last k factors of X are inert for the Chow action. Thus

$$\mathfrak{q}_{-1}(\alpha) = \mathfrak{q}_{-1}^{(1)}(\alpha) = 0 \implies \mathfrak{q}_{-i}(\alpha) = 0 \text{ for all } i>0 \implies \alpha = 0.$$

This ends the proof of the lemma. □

3.2. **Proof of Theorem 2*.** We now complete the inductive argument giving the theorem.

3.2.1. *The base case* $n = 1$. The proof of Theorem 2* follows in this case from the three identities stated in the introduction,

$$\overline{\Delta}_{01} \cdot \overline{\Delta}_{02} \cdot \overline{\Delta}_{03} = 0 \text{ in } CH_2(X \times X^3) \tag{24}$$

$$D^{(0)} \cdot \overline{\Delta}_{01} \cdot \overline{\Delta}_{02} = 0 \text{ in } CH_3(X \times X^2), \tag{25}$$

$$c_X^{(0)} \cdot \overline{\Delta}_{01} = 0 \text{ in } CH_0(X \times X), \tag{26}$$

of which the first one is the Beauville–Voisin identity.

We have $X^{[1]} = X$ and the universal ideal sheaf is

$$\mathcal{I}_1 = \mathcal{I}_\Delta \text{ on } X \times X.$$

A tautological class $\alpha \in R^d(X \times X^k)$ is a polynomial in pullbacks of diagonals, divisors, and special cycles c_X from various factors of the product X^{k+1}. Thus α is necessarily a linear combination of subvarieties of X^{k+1} of type

$$c_X^{(1)} \times \cdots \times c_X^{(a)} \times D_1^{(a+1)} \times \cdots \times D_b^{(a+b)} \times X^{(a+b+1)} \times \cdots \times X^{(a+b+c)} \subset X^{k+1} \tag{27}$$

where the embedding in X^{k+1} is by diagonals (and up to ordering of the factors). Here $a + b + c \leq k + 1$ and

$$\dim \alpha = b + 2c.$$

We now assume α is of the form (27). Indexing the first copy of X by 0, we have

$$\overline{\mathcal{I}}_1^{(t)} = -\mathcal{O}_{\overline{\Delta}_{0,t}}$$

in K-theory. We seek to establish the vanishing

$$\alpha \cdot \prod_{t \in \Omega} \mathrm{ch}_{i_t}\left(\mathcal{O}_{\overline{\Delta}_{0,t}}\right) \cdot \prod_{t \in \Theta} \mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}) = 0 \text{ in } CH_\star\left(X \times X^k \times X^\ell\right),$$

whenever $\ell > \dim \alpha$. Here t runs through $\{1, \ldots, \ell\}$ and $\Omega \sqcup \Theta = \{1, \ldots, \ell\}$. The Chern character degrees i_t are arbitrary.

Noting now that in $X \times X$ we have

$$\mathrm{ch}(\mathcal{O}_{\overline{\Delta}}) = \overline{\Delta} - 2\, c_X \times c_X$$

and the cycle $c_X \times c_X$ is a rational multiple of $\overline{\Delta}^2$, it is enough to show the vanishing

$$\alpha \cdot \prod_{t \in \Omega} \overline{\Delta}_{0,t} \cdot \prod_{t \in \Theta} \overline{\Delta}_{s_t,t} = 0 \text{ in } CH_\star\left(X \times X^k \times X^\ell\right), \tag{28}$$

whenever $\ell > \dim \alpha = b + 2c$. Since α is of the form (27), this inequality guarantees that a factor of c_X receives a matching normalized diagonal, or a factor of D receives two matching normalized diagonals, or a factor of X receives three matching normalized diagonals. (Here "matching" means that the normalized diagonal shares an index with the class in question.) The fundamental identities (24), (25), (26) therefore ensure that the product (28) vanishes. □

3.2.2. *The induction step.* Let $\alpha \in R^d(X^{[n+1]} \times X^k)$. We want to show that for every ℓ satisfying

$$\ell > \dim \alpha,$$

and any indices $i_1, \ldots, i_\ell \geq 0$, the class

$$\gamma := \alpha \cdot \prod_{t \in \Omega} \mathrm{ch}_{i_t}(\overline{\mathcal{I}}^{(t)}_{n+1}) \cdot \prod_{t \in \Theta} \mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}) = 0 \text{ in } CH_\star(X^{[n+1]} \times X^k \times X^\ell). \tag{29}$$

According to Lemma 2 it suffices to show

$$\sigma_\star \psi^\star \gamma = \sigma_\star(\lambda \cdot \psi^\star \gamma) = 0 \text{ in } CH_\star(X^{[n]} \times X^{k+1} \times X^\ell).$$

To start, note that as a tautological class, α is a polynomial in classes

$$\beta_j, \quad \Delta_{s,s'}, \quad D^{(s)}, \quad c_X^{(s)}, \quad \text{and} \quad \mathrm{ch}_i(\mathcal{I}^{(s)}_{n+1}),$$

with $\beta_j \in R^\star(X^{[n+1]})$ and $s \in \{\hat{1}, \ldots, \hat{k}\}$. Recalling the exact sequence (20),

$$0 \to \psi_X^\star \mathcal{I}_{n+1} \to \phi_X^\star \mathcal{I}_n \to \pi^\star \mathcal{L} \otimes \sigma_X^\star \mathcal{O}_\Delta \to 0 \text{ on } X^{[n,n+1]} \times X$$

it follows that the pullback $\psi^\star \alpha$ is of the form

$$\psi^\star \alpha = \sum_{j=0}^{d} \alpha_{d-j} \cdot \lambda^j \in CH_\star(X^{[n,n+1]} \times X^k), \tag{30}$$

where

$$\alpha_{d-j} \in R^{d-j}(X^{[n]} \times X^{k+1})$$

are pulled back under $\sigma \times \mathrm{id}_{X^k} : X^{[n,n+1]} \times X^k \to X^{[n]} \times X^{k+1}$. (Here we suppressed the pullback from the notation.)

Furthermore, the fundamental exact sequence (20) gives immediately the K-theoretic equality

$$\overline{\mathcal{I}}^{(t)}_{n+1} = \overline{\mathcal{I}}^{(t)}_n - \mathcal{L} \cdot \mathcal{O}_{\overline{\Delta}_{0,t}} - (\mathcal{L} - 1) \cdot \mathcal{O}_{c_X^{(t)}} \text{ in } K(X^{[n,n+1]} \times X^\ell).$$

Here 0 denotes the factor of X which $X^{[n,n+1]}$ maps to under $p : X^{[n,n+1]} \to X$; we suppressed the pullbacks from the notation; as usual t indexes the factors in X^ℓ.

Thus the class $\psi^\star \gamma \in CH_\star(X^{[n,n+1]} \times X^k \times X^\ell)$ is a linear combination of terms of the form

$$\psi^\star \alpha \cdot \prod_{t \in \Omega_1} \mathrm{ch}_{i_t}(\overline{\mathcal{I}}^{(t)}_n) \cdot \prod_{t \in \Omega_2} \mathrm{ch}_{i_t}(\mathcal{L} \otimes \mathcal{O}_{\overline{\Delta}_{0,t}}) \cdot \prod_{t \in \Omega_3} \mathrm{ch}_{i_t}((\mathcal{L} - 1) \otimes \mathcal{O}_{c_X^{(t)}}) \cdot \prod_{t \in \Theta} \mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}) \tag{31}$$

Here $\Omega_1 \sqcup \Omega_2 \sqcup \Omega_3 \sqcup \Theta = \{1, \ldots, \ell\}$, the indexing set for factors in the product X^ℓ, and $\psi^\star \alpha$ is the codimension d class given by (30). The expression (31) is a polynomial in λ with coefficients pulled back from $R_\star(X^{[n]} \times X^{k+1} \times X^\ell)$. Noting now that

$$\mathrm{ch}((\mathcal{L} - 1) \otimes \mathcal{O}_{c_X^{(t)}}) = \lambda \cdot p(\lambda) \cdot c_X^{(t)} \text{ for } p \in \mathbb{Q}[\lambda],$$

we conclude from (31) that the classes $\psi^\star\gamma$, $\lambda\cdot\psi^\star\gamma \in CH_\star(X^{[n,n+1]}\times X^k\times X^\ell)$ are linear combinations of terms of type

$$\widetilde{\alpha}\cdot\prod_{t\in\Omega_1}\mathrm{ch}_{i_t}(\overline{\mathcal{I}}_n^{(t)})\cdot\prod_{t\in\Omega_2}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{0,t}})\cdot\prod_{t\in\Omega_3}c_X^{(t)}\cdot\prod_{t\in\Theta}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}),$$

where the quadruple product is now pulled back from $R_\star(X^{[n]}\times X^{k+1}\times X^\ell)$. The class $\widetilde{\alpha}$ is a polynomial in λ with coefficients in $R_\star(X^{[n]}\times X^{k+1})$, and

$$\operatorname{codim}\widetilde{\alpha}\geq d+\omega,\ \text{ where } \omega=|\Omega_3|.$$

Correspondingly, since powers of λ push forward to tautological classes (cf. (19)), the pushforwards $\sigma_\star\psi^\star\gamma$ and $\sigma_\star(\lambda\cdot\psi^\star\gamma)$ are linear combinations of terms of the form

$$\beta\cdot\prod_{t\in\Omega_1}\mathrm{ch}_{i_t}(\overline{\mathcal{I}}_n^{(t)})\cdot\prod_{t\in\Omega_2}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{0,t}})\cdot\prod_{t\in\Omega_3}c_X^{(t)}\cdot\prod_{t\in\Theta}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}}),$$

for $\beta\in R_\star(X^{[n]}\times X^{k+1})$ satisfying

$$d':=\operatorname{codim}\beta\geq d+\omega.$$

Setting $\ell'=\ell-\omega$ and omitting the X factors which carry a class c_X, we note that the product

$$\beta\cdot\prod_{t\in\Omega_1}\mathrm{ch}_{i_t}(\overline{\mathcal{I}}_n^{(t)})\cdot\prod_{t\in\Omega_2}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{0,t}})\cdot\prod_{t\in\Theta}\mathrm{ch}_{i_t}(\mathcal{O}_{\overline{\Delta}_{s_t,t}})=0\in R_\star(X^{[n]}\times X^{k+1}\times X^{\ell'})$$

by the induction hypothesis, since $d'+\ell'\geq d+\ell>2n+2k+2$. □

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