On the group of zero-cycles of holomorphic symplectic varieties

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A Chow-theoretic study of holomorphic symplectic varieties of K3 type was undertaken in [SYZ]. The article considered moduli spaces of Bridgeland-stable sheaf complexes on a smooth projective K3 surface, and proposed to understand rational equivalence of zero-cycles on the moduli space in terms of the cycle structure of the underlying surface. We prove the following result conjectured in [SYZ].

Theorem. *Let X be a smooth projective K3 surface. For a primitive $v \in H^\star(X, \mathbb{Z})$, and a v-generic stability condition σ, let $\mathsf{M}_\sigma(v)$ be the moduli space of σ-stable complexes on X of Mukai vector v. Let F_1 and F_2 be two points in $\mathsf{M}_\sigma(v)$. Then*

$$[F_1] = [F_2] \in CH_0(\mathsf{M}_\sigma(v)) \iff c_2(F_1) = c_2(F_2) \in CH_0(X).$$

As noted in [SYZ], if F_1 and F_2 have the same Chow class on $\mathsf{M}_\sigma(v)$, then by restricting a quasi-universal family on $\mathsf{M}_\sigma(v) \times X$ to the two points one concludes that the Chern characters, hence the second Chern classes of F_1 and F_2 are equal in the Chow ring of X.

We show now that the existence of a quasi-universal family also establishes the other direction. Assume in fact for simplicity that $\mathsf{M}_\sigma(v)$ admits a universal family

$$\mathcal{E} \to \mathsf{M}_\sigma(v) \times X.$$

This restriction is certainly not essential, as we will discuss shortly. Denote the two projections by

$$\pi : \mathsf{M}_\sigma(v) \times X \to \mathsf{M}_\sigma(v) \ \text{ and } \ \rho : \mathsf{M}_\sigma(v) \times X \to X.$$

The main role in the argument is played by the π-relative extension complex

$$\mathsf{W}_F = \mathcal{E}xt_\pi^\bullet(\mathcal{E}, \rho^\star F)[1] \ \text{ on } \ \mathsf{M}_\sigma(v),$$

which is well-suited to detect in its Chern classes the Chow class of the point $F \in \mathsf{M}_\sigma(v)$. For any $E \in \mathsf{M}_\sigma(v)$ not isomorphic to F, by stability and duality we have

$$\mathrm{Ext}^0(E, F) = 0 \text{ and } \mathrm{Ext}^2(E, F) = 0 \text{ on } X.$$

In this situation, the middle group has constant dimension

$$\dim \mathrm{Ext}^1(E, F) = \langle v, v \rangle,$$

where the last quantity is the Mukai self-pairing of v. Note that the dimension of the moduli space is

$$m = \langle v, v \rangle + 2 = \dim \mathsf{M}_\sigma(v).$$

On $M_\sigma(v) \setminus \{F\}$, we see therefore that W_F is a locally free sheaf of rank $m - 2$. This implies that on $M_\sigma(v)$ we have:

$$
\begin{aligned}
c_{m-1}(W_F) &= 0, \\
c_m(W_F) &= \text{const} \cdot [F].
\end{aligned}
$$

The constant can be determined by a Thom-Porteous calculation using a locally free complex isomorphic to W_F. We simply have, in fact:

Proposition. *The top Chern class of the complex W_F is*

$$c_m(W_F) = [F].$$

The proposition gives the theorem via Grothendieck-Riemann-Roch. As

$$\text{ch}\, W_F = -\pi_\star \left(\text{ch}\, \mathcal{E}^\vee \cdot \rho^\star \text{ch}\, F \cdot \rho^\star \text{td} X \right),$$

we have in Chow,

$$\text{ch}\, F_1 = \text{ch}\, F_2 \text{ on } X \implies \text{ch}\, W_{F_1} = \text{ch}\, W_{F_2} \text{ on } M_\sigma(v) \implies c_m(W_{F_1}) = c_m(W_{F_2}) \text{ on } M_\sigma(v).$$

The Chern characters on $M_\sigma(v)$ require rational coefficients. As the moduli space has trivial first cohomology group, Roitman's theorem [Bl], [R] ensures that $CH_0(M_\sigma(v))$ is torsion-free, hence the equality of the top Chern classes holds over $\mathbb{Z}$.

The proposition is not a new result, since the extension complex W_F repeatedly played a role in the development of moduli theory of sheaves on K3 surfaces. Allowing F to vary in moduli, one obtains the complex

$$W = \mathcal{E}xt_\pi^\bullet(\mathcal{E}, \mathcal{F})[1] \text{ on } M_\sigma(v) \times M_\sigma(v),$$

where $\pi : M_\sigma(v) \times M_\sigma(v) \times X \to M_\sigma(v) \times M_\sigma(v)$ is the projection, and $\mathcal{E}$, $\mathcal{F}$ are the universal objects pulled back to the triple product using the first and second factors respectively. This setting was considered in [Ma] when $M_\sigma(v)$ parametrizes stable sheaves. In this situation, the main theorem in [Ma] asserts

$$
(1) \qquad\qquad c_m(W) = \Delta \in CH_m(M_\sigma(v) \times M_\sigma(v)).
$$

This diagonal formula holds unsurprisingly for any stability condition σ, as we now indicate. The first step in [Ma] toward establishing the formula in the sheaf case is to write an explicit geometric three-term complex of locally free sheaves which is isomorphic to the extension complex W. It is clear that for any generic stability condition σ, W is isomorphic to a three-term complex of locally free sheaves in $D^b(M_\sigma(v) \times M_\sigma(v))$,

$$0 \to A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\beta} A_2 \to 0,$$

where A_1 is placed in homological degree 0. This is true on general grounds [BM, Proposition 5.4] as long as for all points $x = (E, F) \in \mathsf{M}_\sigma(v) \times \mathsf{M}_\sigma(v)$, the vanishings

$$H_i(\mathsf{W} \overset{L}{\otimes} \mathcal{O}_x) = 0 \text{ for } i \neq -1, 0, 1$$

hold. As we have $\mathsf{W} \overset{L}{\otimes} \mathcal{O}_x = \mathrm{Ext}_X^\bullet(E, F)[1]$, and E, F both belong to the heart of the stability condition σ, this is true.

Note next that for the highest extension sheaf $\mathcal{E}xt_\pi^2(\mathcal{E}, \mathcal{F})$, cohomology and base change commute, so this sheaf is a line bundle supported on the diagonal Δ. By contrast, $\mathcal{E}xt_\pi^0(\mathcal{E}, \mathcal{F})$ is torsion-free hence zero. For the above locally free complex, the map α is thus injective, while the cokernel of β is a line bundle supported on the diagonal. Furthermore, Grothendieck-Verdier duality gives

$$\mathsf{W}^\vee = \mathcal{E}xt_\pi^\bullet(\mathcal{F}, \mathcal{E})[1],$$

so the cokernel sheaf of $\alpha^\vee : A_1^\vee \to A_0^\vee$ is also supported as a line bundle on Δ.

The following general lemma proven in [Ma] by a suitable Thom-Porteous calculation then yields the diagonal formula (1) for any generic stability condition σ.

Lemma. *([Ma], Lemma 4) Let*

$$0 \to A_0 \overset{\alpha}{\to} A_1 \overset{\beta}{\to} A_2 \to 0$$

be a complex of locally free sheaves on a smooth projective variety M, such that

(1) The cohomology sheaves satisfy $\mathcal{H}_0 = 0$ and $\mathcal{H}_2$ is a line bundle supported on a smooth subvariety Z of even codimension m.

(2) The cokernel sheaf of $\alpha^\vee : A_1^\vee \to A_0^\vee$ is also supported as a line bundle on Z.

(3) The ranks satisfy $a_1 - a_0 - a_2 = m - 2 \geq 0$.

Then $c_m(A_1 - A_0 - A_2) = [Z]$.

Since for any $F \in \mathsf{M}_\sigma(v)$, the complex W_F is the (derived) pullback

$$\mathsf{W}_F = \iota_F^\star \mathsf{W}$$

under the inclusion

$$\iota_F : \mathsf{M}_\sigma(v) \simeq \mathsf{M}_\sigma(v) \times F \hookrightarrow \mathsf{M}_\sigma(v) \times \mathsf{M}_\sigma(v),$$

the proposition is simply obtained from the diagonal formula (1) by pullback of $c_m(\mathsf{W})$.

The moduli space $\mathsf{M}_\sigma(v)$ does not generally admit a universal family. Nevertheless, as explained in [Mu, Theorem A.5], and later [Ma, Section 3], a universal object can be glued together over $\mathbb{P} \times X$, where $\mathbb{P} \to \mathsf{M}_\sigma(v)$ is a suitable projective bundle of a certain rank N. As further shown in [KLS, Lemma 4.2], for any point $F \in \mathsf{M}_\sigma(v)$, by taking the intersection of $\mathbb{P}$ with N generic hyperplanes, one obtains a smooth projective variety $\widetilde{M}$ parametrizing a family

$$\mathcal{E} \to \widetilde{M} \times X$$

of σ-stable objects in $\mathrm{D}^b(X)$, such that the classifying morphism $f : \widetilde{M} \to \mathsf{M}_\sigma(v)$ is surjective, generically finite, and étale over a neighborhood of F.

Considering the π-relative extension complex

$$\mathsf{W}_F = \mathcal{E}xt_\pi^\bullet(\mathcal{E}, \rho^\star F)[1] \text{ on } \widetilde{M},$$

the argument above gives

$$f_* c_m\left(\mathsf{W}_F\right) = n[F],$$

where n is the generic degree of f. Since $CH_0(\mathsf{M}_\sigma(v))$ is torsion-free, the theorem follows from this formula in conjunction with Grothendieck-Riemann-Roch. $\qquad\square$

Remark 1. In the classic paper [Mu], for isotropic Mukai vectors v, the author used W_F to establish irreducibility of the two-dimensional moduli space $\mathsf{M}_\sigma(v)$. An argument with the same flavor was later made in [KLS] to show irreducibility in a more general setting. In a different direction, the aim in [Ma] is to represent the diagonal in terms of Chern classes of the universal sheaves, which entails that the cohomology of the moduli space is generated by the Künneth components of the universal Chern classes. The origin of this idea can in turn be traced back to the earlier work [Be1], [ES].

The subring of $CH_\star\left(\mathsf{M}_\sigma(v)\right)$ generated in a suitable sense by the universal Chern classes is important for understanding the cycle class map to cohomology along the lines of [Be2], [V1].

Remark 2. The argument above gives the same statement for abelian surfaces. For surfaces with effective anti-canonical divisors (when Ext^2 vanishes by Serre duality), the proposition reads

$$c_m\left(\mathsf{W}_F\right) = (-1)^m[F],$$

and the proof is completely similar.

Remark 3. The consequences of the theorem were investigated in [SYZ]. In particular, the result implies the existence of algebraic coisotropic subvarieties $q_i : Z_i \dashrightarrow B_i$ for every i, where Z_i is a subvariety of codimension i in $\mathsf{M}_\sigma(v)$, and the generic fibers of q_i are constant cycle subvarieties of dimension i. The existence of such varieties was conjectured in [V2] for any holomorphic symplectic variety M, and was used to construct a Beauville-Voisin filtration on $CH_0(M)$. Another filtration on $CH_0(\mathsf{M}_\sigma(v))$ was constructed in [SYZ], and was shown to be related to the one in [V2] assuming the theorem.

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