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MATH 3150 **Real Analysis** **Fall 2022**

Handout: Pointwise and uniform convergence

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined on a set $S \subset \mathbb{R}$. We have the following notions pertaining to such a sequence.

Pointwise convergence. The sequence (f_n) converges *pointwise* to a function f (written $f_n \to f$, or $\lim_{n\to\infty} f_n = f$) if $f_n(x)$ converges to $f(x)$, for all $x \in S$. That is:

$$\boxed{\forall \epsilon > 0, \quad \forall x \in S, \quad \exists N = N(\epsilon, x) \in \mathbb{N} \quad \text{such that} \quad |f_n(x) - f(x)| < \epsilon, \quad \forall n > N.}$$

By properties of limits, if $f_n \to f$ and $g_n \to g$, then $f_n + g_n \to f + g$ and $f_n g_n \to fg$. If all the functions f_n are continuous, and $f_n \to f$, there is no guarantee that f is continuous.

Uniform convergence. The sequence (f_n) converges *uniformly* to f on S (written $f_n \xrightarrow{u} f$) if

$$\boxed{\forall \epsilon > 0, \quad \exists N = N(\epsilon) \in \mathbb{N} \quad \text{such that} \quad |f_n(x) - f(x)| < \epsilon, \quad \forall x \in S \text{ and } \forall n > N.}$$

This condition is equivalent to

$$\boxed{\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} = 0.}$$

By logical negation, the sequence (f_n) does not converges uniformly to f on S if

$$\boxed{\exists \epsilon > 0 \text{ such that} \quad \forall N \in \mathbb{N}, \quad \exists n > N \text{ and } \exists x \in S \quad \text{such that} \quad |f_n(x) - f(x)| \geq \epsilon,}$$

or, equivalently,

$$\boxed{\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in S\} > 0.}$$

It is readily seen that $f_n \xrightarrow{u} f$ and $g_n \xrightarrow{u} g$, then $f_n + g_n \xrightarrow{u} f + g$; in general, though, $f_n g_n$ does not converge uniformly to fg. On the other hand, if $f_n \colon [a, b] \to \mathbb{R}$ are continuous, and $f_n \xrightarrow{u} f$, then f is continuous.

Example 1. Let $f_n(x) = x/n$, for $x \in \mathbb{R}$.

(a) $f_n \to 0$ pointwise, since, for all $x \in \mathbb{R}$, $\lim_{n\to\infty} x/n = 0$ (by the Archimedean property of the reals).

(b) f_n does *not* converge uniformly to 0 on $\mathbb{R}$. Indeed, take $\epsilon = 1$, and let $N \in \mathbb{N}$. Then, for any $n > N$, we may take $x \geq n$ (again by the Archimedean property), so that $|f_n(x) - 0| = x/n \geq 1$. Alternatively,

$$\sup\{|f_n(x) - 0| : x \in \mathbb{R}\} = \sup\{x/n : x \in \mathbb{R}\} = \infty.$$

Example 2. Let $f_n(x) = x + 1/n$ and $f(x) = x$, for $x \in \mathbb{R}$.

(a) $f_n \to f$ uniformly on $\mathbb{R}$, since

$$\sup\{|f_n(x) - f(x)| : x \in \mathbb{R}\} = 1/n \to 0.$$

(b) f_n^2 does *not* converge uniformly to f^2 on $\mathbb{R}$ (although $f_n^2 \to f^2$ pointwise). Indeed,

$$\sup\{\left|f_n^2(x) - f^2(x)\right| : x \in \mathbb{R}\} = \sup\{2x + 1/n^2 : x \in \mathbb{R}\} = \infty.$$

Example 3. Let $f_n(x) = nxe^{-nx^2}$ for $x \in \mathbb{R}$.

(a) $f_n \to 0$ pointwise on $\mathbb{R}$. This is clear for $x = 0$ (since $f_n(0) = 0$), while for $x \neq 0$, l'Hospital's rule gives

$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{(e^{x^2})^n} = 0 \quad (\text{since } e^{x^2} > 1).$$

(b) f_n does *not* converge uniformly to 0 on any interval containing 0. Indeed,

$$f_n(1/\sqrt{n}) = \sqrt{n}/e \xrightarrow{n\to\infty} \infty.$$

Since any interval containing 0 must contain $1/\sqrt{n}$ for some large enough n, we conclude that $|f_n| \to \infty$.

(c) On the other hand, f_n does converge uniformly to 0 on any interval of the form $[a, \infty)$ with $a > 0$. Indeed, since $e^{nx^2} = 1 + nx^2 + \frac{1}{2}n^2x^4 + \cdots$,

$$|f_n(x)| = \frac{nx}{e^{nx^2}} \leq \frac{2nx}{n^2x^4} = \frac{2}{nx^3} \leq \frac{2}{na^3},$$

and thus

$$\lim_{n\to\infty} \sup\{|f_n(x)| : x \in [a, \infty)\} = 0.$$

Example 4. Let $f_n \colon [0, 2] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0 & \text{if } 2/n < x \leq 2, \\ n^2x^2 - 2nx & \text{if } 0 \leq x \leq 2/n. \end{cases}$$

(a) $f_n \to 0$ pointwise. Indeed, $f_n(0) = 0$ for all n, and, for all $x \in (0, 2]$, there is an $n \in \mathbb{N}$ such that $x > 2/n$, and thus $f_m(x) = 0$ for all $m \geq n$.

(b) f_n does not converge uniformly to 0. Indeed, $f_n(1/n) = -1$ for all $n \in \mathbb{N}$, and thus

$$\sup\{|f_n(x)| : x \in [0, 2]\} \geq 1.$$