Name: ______________________

MATH 3150 **Real Analysis** **Fall 2022**

Instructions: Write your name in the space provided. Calculators are permitted. You are also allowed a one-sided A4 sized note sheet of only definitions and theorems (no examples allowed) from classes and the textbook. Make sure your name is on the sheet, and hand in your note sheet along with your exam. Books, other notes, and laptops are **not** allowed.

---

1. Define a sequence (x_n) in $\mathbb{R}$ recursively by setting $x_1 = 7$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$.

(a) (10 pts) Show that the sequence converges.

(b) (10 pts) Find the limit of the sequence.

2. (15 pts) Let $a_n = (-1)^n + 1/n$. Use the definition of $\liminf$ and $\limsup$ to find $\limsup a_n$ and $\liminf a_n$.

3. (10 pts) Show that a subsequence of a subsequence of a sequence is a subsequence of the original sequence.

4. (15 pts) Let (x_n) and (y_n) be Cauchy sequences of real numbers, and let $z_n = x_n - y_n$. Use the definition of Cauchy sequence to show that (z_n) is also a Cauchy sequence.

5. Let $f\colon [1,2] \to \mathbb{R}$ be a function such that $f(1) > 1$ and $f(2) < 4$.

(a) (10 pts) Assuming f is continuous, show that there is a number $c \in [1,2]$ such that $f(c) = c^2$.

(b) (10 pts) Give an example of a (discontinuous) function $f\colon [1,2] \to \mathbb{R}$ for which the equation $f(c) = c^2$ has no solution $c \in [1,2]$.

6. (10 pts) Let f be a function defined on a domain $D \subset \mathbb{R}$. Given elements $x, y \in D$ with $x \neq y$ set $s(x, y) = (f(x) - f(y))/(x - y)$ and then let

$$S = \{s(x, y) \colon x, y \in D, x \neq y\}$$

Show directly from the definition of uniform continuity that if the set S is bounded, then f is uniformly continuous on D.

7. (10 pts) Let X, Y and Z be subsets of $\mathbb{R}$. Suppose $f\colon X \to Y$ is a uniformly continuous function on X, and $g\colon Y \to Z$ is a uniformly continuous function on Y. Show that the composition $g \circ f\colon X \to Z$ is also uniformly continuous.