Be sure to *fully justify your response* to each problem.

1. (15 pts) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Suppose the series $\sum_{n=1}^{\infty} |a_n|$ converges. Show that the power series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on the interval $[-1,1]$ to a continuous function.

2. (15 pts) Show that $|\sin(x) - \sin(y)| \leq |x - y|$ for all x, y in $\mathbb{R}$.

3. (15 pts) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Assume that f is differentiable on $(0,1)$ and that $f(x) + xf'(x) \geq 0$, for all $x \in (0,1)$. Show that $f(x) \geq 0$, for all $x \in [0,1]$.

4. (15 pts) A *fixed point* of a function f is a value x such that $f(x) = x$. Let $f\colon (a,b) \to \mathbb{R}$ be a differentiable function. Assume that $f'(x) \neq 1$, for all $x \in (a,b)$. Show that f has at most one fixed point.

5. Let $f(x) = \ln(x)$, let $\displaystyle\sum_{n=0}^{\infty} a_n(x-1)^n$ be the Taylor series for f around $x = 1$, and let $R_n(x)$ be the remainder $R_n(x) = \ln(x) - \displaystyle\sum_{k=0}^{n-1} a_k(x-1)^k$ for $x > 0$.

   (a) (10 pts) Using the formula for $R_n(x)$ in §31.3 Taylor's Theorem (p. 250), find an upper bound for $|R_n(x)|$.

   (b) (10 pts) Find all values of $x > 0$ for which it follows from your result in part (a) that $\lim_{n\to+\infty} R_n(x) = 0$.

6. Let f be the function defined by

$$f(t) = \begin{cases} 2 & \text{for } t \leq 0 \\ \sin(t) & \text{for } 0 < t \leq \pi/2 \\ t - \pi/2 + 1 & \text{for } t > \pi/2. \end{cases}$$

   (a) (4 pts) Determine $F(x) = \displaystyle\int_0^x f(t)\,dt$.

   (b) (4 pts) Sketch the graph of F.

   (c) (4 pts) At which points, if any, is F not continuous?

   (d) (4 pts) At which points, if any, is F not differentiable?

   (e) (4 pts) Calculate $F'(x)$ at those points where F is differentiable.